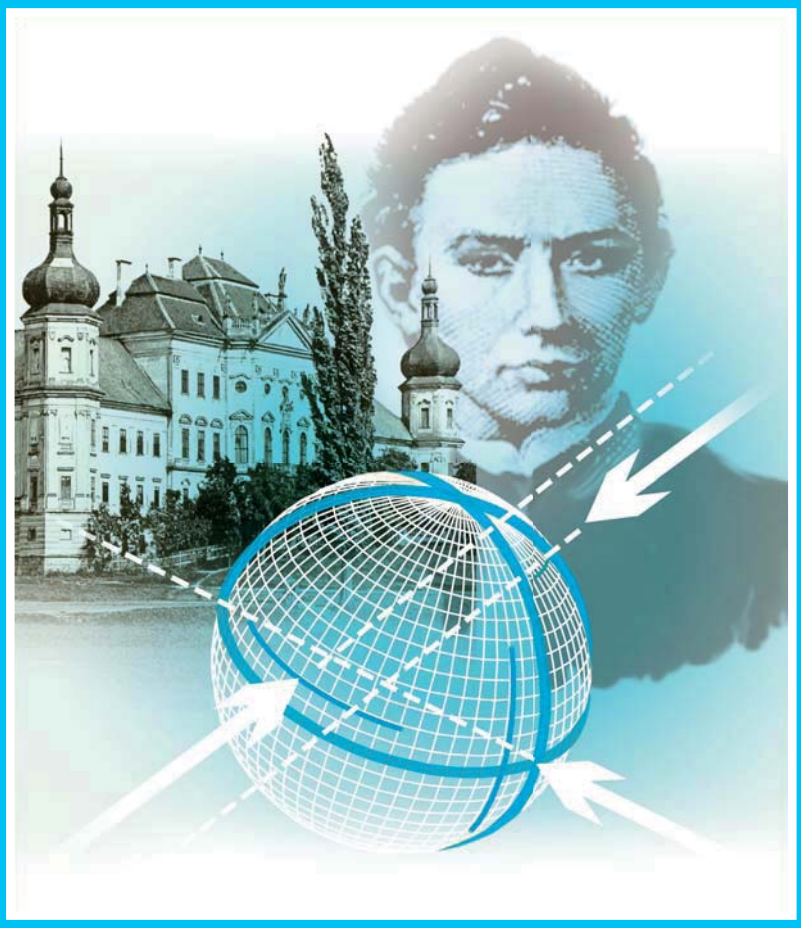

DIFFERENTIAL GEOMETRY

OF SPECIAL MAPPINGS

Josef Mikeš
Elena Stepanova
Alena Vanžurová
et al.

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Palacký University, Olomouc
Faculty of Science

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Reviewed by: prof. RNDr. Miroslav Doupovec, CSc.
doc. RNDr. Miroslav Kureš, CSc.

Authors: Josef Mikeš
Elena S. Stepanova
Alena Vanžurová
Sándor Bácsó
Vladimir E. Berezovski
Olena Chepurna
Marie Chodorová
Hana Chudá
Michail L. Gavrilchenko
Michael Haddad
Irena Hinterleitner
Marek Jukl
Lenka Juklová
Dzhanybek Moldobaev
Patrik Peška
Mohsen Shiha
Igor G. Shandra
Dana Smetanová
Sergey E. Stepanov
Vasilij S. Sobchuk
Irina I. Tsyganok

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Elena Stepanova, Ph.D.
doc. RNDr. Alena Vanžurová, CSc.
et al.

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INTRODUCTION

During the last 50 years, many new and interesting results have appeared in the theory of conformal, geodesic, holomorphically projective, F-planar and others mappings and transformations of manifolds with affine connection, Riemannian, Kähler and Riemann-Finsler manifolds. The authors dedicate the present monograph to the exposition of this topic.

Problems connected with this field were considered in many monographs, surveys (pp. 513–518) and dissertation theses (pp. 519–520).

In the theory of geodesic, conformal and holomorphically projective mappings and some generalizations, three main directions have been specified:

- the investigation of general laws and rules;
- the integration of basic equations, and
- the investigations for special spaces.

Recently, new results that were not reflected in the papers mentioned above have been obtained. On the one hand, some results of a general character, on the other hand, results concerning mappings of special manifolds with affine connection and Riemannian spaces, including spaces of constant curvature, Kählerian, Einsteinian spaces, conformally flat spaces, etc.

Many works have been dedicated to the problem of non-existence of conformal, geodesic and holomorphically projective mappings and transformations, and concircular vector fields in spaces of a special kind. Such problems are often closely related. However, much attention has not been paid to their investigation yet. New results on the integration of basic geodesic mappings equations are considered in the review [9,11] and in the monograph [10] by A.V. Aminova.

We give the basic concepts of the theory of manifolds with affine connection, Riemannian, Kählerian and Riemann-Finsler manifolds, using the notation from [50, 51, 118, 119, 121, 122, 139, 156, 170, 173, 200].

Unless otherwise stated, the investigations are carried out in tensor form, locally, in the class of sufficiently smooth real functions. The dimension n of the spaces under consideration is supposed to be higher than two, as a rule. This fact is not explicitly stipulated in the text. All the spaces are assumed to be connected. Under Riemannian manifolds we mean both positive as well as pseudo-Riemannian manifolds.

The book was edited by J. Mikeš, E. Stepanova, A. Vanžurová. The book consists of 15 chapters. The first four chapters of the book are of introductory character, and include also some historical remarks.

Chapter 1 treats the basic concepts of topological spaces (Vanžurová, Mikeš).

Chapter 2 treats the theory of manifolds with affine connection. Particularly, the problem of semi-geodesic coordinates (Mikeš, Hinterleitner, Vanžurová).

Chapter 3 is devoted to Riemannian and Kähler manifolds. Particularly, reconstruction of a metric (Mikeš, Vanžurová), equidistant spaces (Mikeš, Chepurna, Chodorová, Hinterleitner), variational problems in Riemannian spaces (Mikeš, Hinterleitner, Smetanová, Stepanova, Vanžurová), $SO(3)$-structure as a model of statistical manifolds (Mikeš, Stepanova), decomposition of tensors (Mikeš, Jukl, Juklová).

Chapter 4 is devoted to the theory of differentiable mappings and transformations of manifolds. Among others we mention the problem of metrization of affine connection (Vanžurová), harmonic diffeomorphisms (Stepanov, Shandra).

Chapter 5 treats conformal mappings and transformations. Especially conformal mappings onto Einstein spaces (Mikeš, Gavrilchenko), conformal transformations of Riemannian manifolds (Mikeš, Moldobayev).

Chapter 6 is devoted to geodesic mappings (GM). We stress geodesic equivalence of a manifold with affine connection to an equiaffine manifold (Mikeš, Hinterleitner).

Chapter 7. We examine GM onto Riemannian manifolds (Mikeš, Berezovski, Hinterleitner).

Chapter 8 treats GM between Riemannian manifolds. Among others GM of equidistant spaces, GM of $\mathbb{V}_n(B)$ spaces (Mikeš, Hinterleitner), and its field of symmetric linear endomorphisms (Mikeš, Stepanova, Tsyganok).

Chapter 9 is devoted to GM of special spaces, particularly Einstein, Kähler, pseudo-symmetric manifolds and their generalizations (Mikeš, Hinterleitner, Shiha, Sobchuk).

Chapter 10 treats global geodesic mappings and deformations, GM between Riemannian manifolds of different dimensions (Stepanov), global GM (Mikeš, Chudá, Hinterleitner). Geodesic deformations of hypersurfaces in Riemannian spaces (Mikeš, Gavrilchenko, Hinterleitner).

Chapter 11. We give some applications of GM to general relativity, namely we present three invariant classes of the Einstein equations and geodesic mappings (Stepanov, Jukl, Mikeš).

Chapter 12 treats F-planar mappings of spaces with affine connection (Mikeš, Chudá, Hinterleitner, Peška).

Chapter 13. We examine holomorphically projective mappings (HPM) of Kähler manifolds. Among others fundamental equations of HPM, HPM of special Kähler manifolds(Mikeš, Chudá, Haddad, Hinterleitner), HPM of parabolic Kähler manifolds (Mikeš, Chudá, Peška, Shiha).

Chapter 14 deals with almost geodesic mappings, which generalize geodesic mappings (Berezovski, Mikeš, Vanžurová).

Chapter 15 is devoted to Riemann-Finsler spaces and their geodesic mappings (Bácsó), geodesic mappings of Berwald spaces onto Riemannian spaces (Bácsó, Berezovski, Mikeš).

We would like to stress that we use here the classical definition of geodesics, i.e. with a general parameter, which is widely used in applications in theoretical physics. Further note that the definition of the Ricci tensor was splitted, since 1950' its sign is used with an opposite sign, see [170]. We go back to the original notation, L.P. Eisenhart [50].

Some parts of the text are based on several graduate courses on topology, differential geometry, tensor analysis, Riemannian geometry, geodesic mappings, holomorphically mappings and Lie groups given by N.S. Sinyukov, M.L. Gavrilchenko and J. Mikeš at Odessa State University and topology by A. Vanžurová at Palacky University in Olomouc.

The authors believe that the text might evoke interest and might be helpful for post-graduate students in mathematics, geometry or physics as well as for research-work specialists in these fields.

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May 2015

Josef Mikeš
Palacky University
Olomouc

1 TOPOLOGICAL SPACES

1.1 From metric spaces to abstract topological spaces

We intend to work here with spaces that are more general than the Euclidean space or affine spaces known from algebra and elementary geometry, although they look locally like Euclidean spaces and have various applications in mechanics, theoretical physics etc. Together with a distinguished class of spaces, mathematics is interested also in the family of mappings that preserve the typical properties of spaces from the class under consideration. To describe our field of interest we need also concepts from metric spaces, topology and the theory of continuous functions. Let us recall some basic notions and notation.

As well known from linear algebra, the real n-dimensional vector space $\mathbb{R}^n$ is a family of all n-tuples $x = (x^1, \ldots, x^n)$ of real numbers that are added and multiplied by reals component-wise in a familiar way, which defines on $\mathbb{R}^n$ a linear structure of a finite-dimensional vector space. Besides the linear structure the vector n-space $\mathbb{R}^n$ carries a natural inner product, the *dot product* $x \cdot y$ and the induced *norm* $\|x\|$:

$$x \cdot y = x^1 y^1 + \ldots + x^n y^n \quad \text{and} \quad \|x\| = \sqrt{x \cdot x} = \sqrt{{x^1}^2 + \ldots + {x^n}^2}\,.$$

By the *Euclidean space* $\mathbb{E}^n$ we usually mean just $\mathbb{R}^n$ endowed with this dot product. The dot product defines naturally a *metric* $d(x, y) = \|x - y\|$, and the metric induces the *metric topology* on $\mathbb{R}^n$, or on $\mathbb{E}^n$: a subset $O \subset \mathbb{R}^n$ is open if and only if for any point $x \in O$, there exists an open ball $B(x, r)$ with center x and radius $r > 0$ which lies entirely in O; it can be seen that this metric topology coincides with the product topology of the Cartesian product $\mathbb{R} \times \ldots \times \mathbb{R}$ where the reals $\mathbb{R}$ are taken with the natural norm. Note that this metric is *translation invariant* in the following sense: $d(x + z, y + z) = d(x, y)$ holds for all x, y, z from $\mathbb{R}^n$; note that this property is common to *all* metric spaces arising from norms.

Definition 1.1 A *metric space* is a set X together with a function d from $X \times X$ to the non negative real numbers, such that for each $x, y, z \in X$:

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq d(x, y) + d(y, z)$;

the function d is called a *metric* on X.

A *pseudometric* d satisfies (2) and (3) from the definition of metric but (1) is substituted by a weaker condition, namely $d(x, y) = 0$ if $x = y$ (that is, even distinct points might have zero distance).

Given a point x in a metric space X and a real number $r > 0$, the *open ball (= disc) of radius r about x (with the center x)* is a subset

$$B(x, r) = \{y \in X : \ d(x, y) < r\}.$$

We call a subset $O \subset X$ *open* in metric space X if for each point $x \in O$ there exists an open ball about x in X which is entirely in O.

If $A \subset X$ we introduce a *diameter* of A: $\operatorname{diam} A = \sup\{d(x, y) : \ x, y \in A\}$.

In a metric space X, a subset $A \subset X$ is *bounded* if it has finite diameter.

1.1.1 A couple of examples

Example 1.1 On at least two-element set, the so-called *discrete metric* is defined by $d(x, y) = 1$ for $x \neq y$, $d(x, y) = 0$ for $x = y$.

Example 1.2 A normed real vector space V with the norm $\| \cdot \|$ determines a natural metric d defined by $d(u, v) = \|u - v\|$.

Example 1.3 Let $X = C^0\langle a, b\rangle$ be the set of all real functions continuous on a closed interval $\langle a, b\rangle$. We can define a metric on X directly by

$$d(f, g) = \sup\{|f(x) - g(x)| : \ x \in \langle a, b\rangle\} \quad \text{for } f, g \in X.$$

This metric comes from a norm. On X, the linear operations $f \mapsto cf$ and $(f, g) \mapsto f + g$ are defined pointwise which turn X into an infinite-dimensional vector (linear) space. If we define $\|f\| = \max_{x \in \langle a,b\rangle} |f(x)|$ for $f \in X$ we have a norm on X that gives just the metric d. Similarly for the space of continuous complex functions.

There are metrics on vector spaces that do not arise from any norm.

Example 1.4 The set X of all real sequences $\{a_n\}_{n=1}^{\infty}$ is a real infinite-dimensional vector space. The function $\varrho(a, b) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|a_n - b_n|}{1 + |a_n - b_n|}$ is a metric.

Suppose that the metric ϱ comes from some norm $\| \cdot \|$ in the way described above. Then $\|x\| = \varrho(x, 0)$ for each $x \in X$ and $\|ax\| = |a| \cdot \|x\|$, that is, ϱ have to satisfy

$$\varrho(ax, 0) = |a| \, \varrho(x, 0) \qquad \text{for all } a \in \mathbb{R},\ x \in X.$$

Take $x = (1, 0, 0, \dots)$, then $ax = (a, 0, 0, \dots)$ for any $a \in \mathbb{R}$. Now $\varrho(x, 0) = \frac{1}{4}$, $\varrho(ax, 0) = \dfrac{1}{2} \cdot \dfrac{|a|}{1 + |a|}$. But this equality does not hold in general: if we take $a = 2$ then $\varrho(2x, 0) = \frac{1}{3}$, on the other hand $2\varrho(x, 0) = \frac{1}{2}$, a contradiction. Therefore such a norm cannot exist.

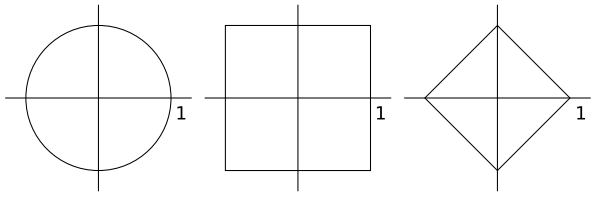
Example 1.5 For two points $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ in $\mathbb{R}^2$, the formulae

$$\begin{aligned} d_1(P, Q) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \\ d_2(P, Q) &= \max\{|x_1 - y_1|, |x_2 - y_2|\}, \\ d_3(P, Q) &= |x_1 - y_1| + |x_2 - y_2| \end{aligned}$$

define three metrics which provide the plane with three distinct structures as a metric space. Yet the family of all open sets is the same in all three cases. We can consider them "equivalent". Not only open sets, but of course also all concepts based on open sets are the same: closed sets, as their complements, etc.; also convergence of sequences is the same. It makes us think what is "behind" this fact.

Note that d_3 is sometimes called a *taxicab distance function (metric)*.

The figure shows the ball of radius 1, central at the origin, for each of these three metrics.



1.1.2 Euclidean space

Similar metrics can be introduced in $\mathbb{R}^n$ for arbitrary $n \in \mathbb{N}$; the first metric space is usually called *Euclidean* or *cartesian space*; the three metrics define the same open sets again.

Let us give first a bit of motivation. Let f be a map of the Euclidean space $\mathbb{E}^m$ to $\mathbb{E}^n$. The classical "ε, δ" definition of continuity for f generalizes continuity of a real-valued function of one real variable well-known from the Calculus and goes as follows: f is *continuous at* $x \in \mathbb{E}^m$ if given any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(y) - f(x)\| < \varepsilon$ whenever $\|y - x\| < \delta$, $y \in \mathbb{E}^m$.

More geometric speaking, f is *continuous* if for any open ball $D' = B(f(x), \varepsilon)$ in $\mathbb{E}^n$ about $f(x)$ with radius $\varepsilon > 0$ there exists an open ball $D = B(x, \delta)$ in $\mathbb{E}^m$ with center x and radius $\delta > 0$ which is mapped into $B(f(x), \varepsilon)$ under f: $f(D) \subset D'$. The function is *continuous* if it is continuous in each point.

Call a subset U of $\mathbb{E}^m$ a *neighbourhood* of the point $x \in \mathbb{E}^m$ if for some real number $r > 0$ the open ball of radius r and center x lies entirely in U. It is easy to rephrase the above definition of continuity as follows: f is continuous if given any $x \in \mathbb{E}^m$ and any neighbourhood U of the image $f(x)$ in the space $\mathbb{E}^n$, then the inverse image $f^{-1}(U)$ is a neighbourhood of the point x in $\mathbb{E}^m$.

More generally, we can proceed similarly in any metric space. A map $f\colon X \to Y$ of a metric space (X, ϱ) to a metric space (Y, σ) is *continuous* if for any $x \in X$ given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma(f(x), f(z)) < \varepsilon$ whenever $\varrho(x, z) < \delta$, $z \in X$. Again, a neighbourhood of the point $x \in X$ is a subset which contains a disc centered at x, and continuity can be rephrased using the concept of neighbourhoods.

1.1.3 Natural topology on metric space

Note that defining neighbourhoods in Euclidean spaces or metric spaces, we use very strongly the distance function. In constructing an "abstract space" we would like to retain the concept of neighbourhood but rid ourselves of *any* dependence, of the definition of the space itself as well as of continuity of maps between abstract spaces, on a distance function. Just this point is crucial: any point of the "space" should be endowed with a family of "neighbourhoods"

settled in such a way that a "good" definition of continuity can be expected. Note that Maurice Fréchet, the French mathematician who created the first definition of an abstract topological space, used just this way, namely generating topology by neighbourhoods.

We ask for a set X and for each point $x \in X$ a nonempty collection $\mathcal{U}(x)$ of subsets of X, called *neighbourhoods* of x, that are required to satisfy the following four conditions (axioms):

(a) x lies in each of its neighbourhoods.
(b) The intersection of two neighbourhoods of x is itself a neighbourhood of the point x.
(c) If V is a subset of X which contains U and U is a neighbourhood of x, then V is a neighbourhood of x.
(d) If U is a neighbourhood of x then there exists a neighbourhood O of x such that $O \subset U$ and O is a neighbourhood of z whenever $z \in O$; O is an *interior* of U.

This whole structure can be called a *topological space*, and we say that the assignment of a collection of neighbourhoods satisfying (a) – (d) to each point $x \in X$ gives a *topology* generated by a neighbourhood system on the set X.

We call a subset O of X *open* in this topology if it is open neighbourhood of each of its points. The union of any collection of open sets is open by (c), and the intersection of any *finite* number of open sets is open by axiom (b) (on the other hand, the intersection of an infinite collection of open sets need not be open). The empty set is open, as is the whole space X. Axiom (d) tells us that given a neighbourhood U of a point x, the interior of U is an open set which contains x and which lies in U. To understand better motivation for the last condition we can take the closed ball $\{z \in \mathbb{E}^m : d(x,z) \leq r\}$ as U, then as O we can take the open ball $\{z \in \mathbb{E}^m : d(x,z) < r\}$.

Although the above concept was formulated quite comprehensible and fits well our idea what a space ought to be, unfortunately such a definition is not so practical to work with. It was found out that an equivalent, more manageable, set of axioms can be given. During the time it was discovered that more convenient, especially in proofs, is to start with the idea of open set, then build up a collection of neighbourhoods for each point, and to show that both approaches are equivalent. Then all concepts build up on open sets will be topological notions.

1.1.4 Isometry of metric spaces

As morphisms between metric spaces, we prefer those maps which preserve distances between points. A map of metric spaces $f : (X, \varrho) \to (Y, \sigma)$ is an *isometry* if $\varrho(x,y) = \sigma(f(x), f(y))$ for every pair $x, y \in X$. Any isometry is a continuous map with continuous inverse (i.e. homeomorphism). All isometries of the given metric space X with map composition operation constitute a group $iso(X)$, the *isometry group* of the space.

Example 1.6 Translations, rotations, symmetries, skew symmetries (and the identity map) are well-known isometries in $\mathbb{E}^2$.

1.1.5 Abstract topological spaces, topology

Let X be a given set. Recall that the family of all subsets in X is called a *potence set* of X and is denoted by $\mathcal{P}(X)$.

Definition 1.2 The system τ of subsets in X, $\tau \subseteq \mathcal{P}(X)$, is called a *topology* on X if the following three axioms hold:
(O1) The empty set $\varnothing$ and the whole space X belong to τ.
(O2) The intersection of two[1)] of sets from τ is in τ.
(O3) The union of any family of sets from τ is in τ.
The pair (X, τ) is called a *topological space*.

The sets from τ are called *open*. Given a point $x \in X$ we shall call a subset U of X a *neighbourhood* of x if we can find an open set O such that $x \in O \subset U$. For a fixed $x \in X$, the set of all neighbourhoods of x in the given topology is denoted by $\mathcal{U}(x)$.

We can verify that this definition of neighbourhood makes X into a topological space according to the above "neighbourhood" definition. For each point, at least X is a neighbourhood of x. If U_1, U_2 are neighbourhoods of x and O_1, O_2 are the corresponding open sets satisfying $x \in O_1 \subset U_1$, $x \in O_2 \subset U_2$, then $x \in O_1 \cap O_2 \subset U_1 \cap U_2$ where $O_1 \cap O_2$ is open. Therefore $U_1 \cap U_2$ is a neighbourhood of x, and we have checked axiom (b). To check (a) and (c) is easy. Similarly the converse implication.

Theorem 1.1 *A subset of a topological space is open if and only if it is a neighbourhood of each of its points.*

1.1.6 Examples of topological spaces

Example 1.7 As a well known example of topological space, recall the set of real numbers $X = \mathbb{R}$ with natural topology: a subset O in $\mathbb{R}$ is open when with each point $r \in O$, an open interval $(r-\varepsilon, r+\varepsilon)$ is in O. Notice that this natural topology is just the metric topology corresponding to $d(a,b) = |a-b|$, $a, b \in \mathbb{R}$.

Example 1.8 For our purpose, particularly the real n-dimensional space $\mathbb{R}^n$ with natural topology is important; this topology arises as the product topology of $\mathbb{R} \times \ldots \times \mathbb{R}$ (n copies), and can be introduced directly as follows. Each point of the space $\mathbb{R}^n$ is an ordered n-tuple $(x^1, x^2, \ldots, x^n)$ of real numbers x^1, x^2, $\ldots$, x^n. Let us consider n open intervals (a^i, b^i) in the reals, $i = 1, 2, \ldots, n$. An *open coordinate parallelepiped*, or *open coordinate box* in $\mathbb{R}^n$ is the set

$$\mathcal{K}_n = \{\, x(x^1, x^2, \ldots, x^n) \mid a^i < x^i < b^i,\ i = 1, 2, \ldots, n\}.$$

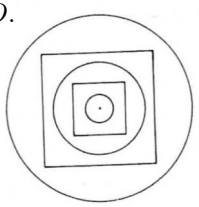
We consider a subset $O \subseteq \mathbb{R}^n$ open in the "product" topology if for any point $x \in O$ there is an open coordinate box $\mathcal{K}_n$ such that $x \in \mathcal{K}_n \subseteq O$. We can easily check that the family of such subsets O satisfies (O1) – (O3), and hence is a topology in $\mathbb{R}^n$.

[1)] And consequently of any finite number.

The same topology appears to be induced by the Euclidean metric of $\mathbb{R}^n$ as a metric space. In fact, for any pair of points $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n)$ in $\mathbb{R}^n$, introduce their Euclidean distance by the formula

$$\varrho(x, y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2}.$$

A pair $(\mathbb{R}^n, \varrho)$ is a metric space denoted by $\mathbb{E}^n$ and called *n-dimensional Euclidean space*. As above, we can consider a subset $O \subseteq \mathbb{R}^n$ *open* if for any point $x \in O$ there exists an open ball $B(x, r)$ such that $B(x, r) \subset O$.



The family of all such open sets satisfies the definition of a topology in $\mathbb{R}^n$, and is called *natural*, or *metric* topology (induced by the Euclidean metric). It is easy to prove that the "product" topology of the n-dimensional real space $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ and the natural topology of the n-dimensional Euclidean space $(\mathbb{R}^n, \varrho)$ coincide (hint: to any ball centred at x, a cube centred in x can be inscribed, and vice versa).

Example 1.9 More generally, any metric space (X, d) endowed with its natural metric topology, is a topological space. Recall how the metric d defines a topology. We consider a subset of X open in the metric topology if and only if it is open with respect to the metric d, i.e. a subset $O \subseteq M$ is *open in the metric (natural) topology* when with each of its points, it includes some open ball $B(x, r)$ centered at the point x. We check directly that (O1) – (O3) hold.

Example 1.10 Normed real vector spaces are topological spaces. Any real vector space with a norm $(V, \|\cdot\|)$ has a natural metric $d(x, y) = \|x - y\|$ for $x, y \in V$. This metric turns V to a topological space if we take on V a metric topology corresponding to d, called *topology of norm on* V.

Example 1.11 Real vector spaces with scalar product are topological spaces. If $(V, (\cdot, \cdot))$ is a real vector space with scalar product we define the corresponding norm by $\|x\| = (x, x)^{1/2}$ for each $x \in V$ and consider the *topology of norm on* V.

Example 1.12 Any nonempty set X together with the family of open sets $\{\varnothing, X\}$ is a topological space, both the topology and the space are called *antidiscrete* or *indiscrete*. Every subset $A \subset X$ is open-and-closed [53, p. 31].

Example 1.13 Any nonempty set X with the family of open sets $\tau = \mathcal{P}(X)$ is a topological space, both the topology and the space are called *discrete*. Note that the discrete topology is generated by the discrete metric.

Example 1.14 Let X be any set. If we take as open sets a family of all subsets having finite complement we obtain on X the so-called *topology of finite complements*.

Example 1.15 Let X be an infinite set. If we take, as open sets, a family of all subsets having countable complement we obtain the so-called *topology of countable complements* on X.

Example 1.16 Assume the set $X = C^0\langle a, b\rangle$ of all real functions continuous on $\langle a, b\rangle$. Open sets are just all subsets of X that, with any of its elements, say f_0, contain, for a suitable $\varepsilon > 0$, all $f \in X$ such that $\sup\{|f_0(x) - f(x)|\colon\ x \in \langle a, b\rangle\} < \varepsilon$. This topology is a metric topology induced by the metric from Example 1.3, or, if we consider X as a vector space, induced by the norm $\|f\| = \sup\{f(x)\colon\ x \in \langle a, b\rangle\} = \max\{f(x)\colon\ x \in \langle a, b\rangle\}$. Similarly for continuous complex functions on a closed interval.

Example 1.17 Let X be a set ordered by a relation of ordering $\leq$. The so-called *interval topology* on X consists of subsets which contain, with each of its points, a set of the form $\{z \in X\colon\ a \leq z \leq b\}$, called *interval*, for some $a, b \in X$.

Example 1.18 Let A be a commutative ring with unit. Recall that an ideal $I \subseteq A$ in A, $I \neq A$ is a prime-ideal if $a \cdot b \in I \Longrightarrow a \in I$ or $b \in I$.

The spectrum $Spec(A)$ of the ring A is a set of all prime-ideals in A. On $Spec(A)$ we define a topology as follows. For any fixed ideal $J \subseteq A$ (not necessarily prime) take the set $\mathcal{O}_J = \{P \in\ Spec(A)\colon\ J \not\subseteq P\}$.

Then the system $\tau = \{\mathcal{O}_J\colon\ J$ is an ideal in A distinct from $A\}$ is a topology on $Spec(A)$. Any ideal contains at least zero element, hence $\mathcal{O}_{(0)} = \varnothing$. Further $\mathcal{O}_A = Spec(A)$, therefore (O1) holds. Let $\mathcal{O}_I, \mathcal{O}_J$ be from τ,

$$\mathcal{O}_I = \{P \in\ Spec(A)\colon\ I \not\subseteq P\}, \quad \mathcal{O}_J = \{P \in\ Spec(A)\colon\ J \not\subseteq P\}.$$

Then the intersection can be written as $\mathcal{O}_I \cap \mathcal{O}_J = \mathcal{O}_K$ where $K = IJ$ is the product of ideals. For a system of ideals $\{\mathcal{O}_{I\alpha}\colon\ \alpha \in A\}$ the union of corresponding sets is $\bigcup\limits_{\alpha} \mathcal{O}_{I\alpha} = \mathcal{O}_L$, where $L = \sum\limits_{\alpha \in A} I_\alpha$ is the sum of ideals.

1.2 Generating of topologies

1.2.1 Closed sets

Complements of open sets play also an important role, and have dual properties.

Definition 1.3 A subset $F \subset X$ is called *closed*, if its complement $X \backslash F$ is open.

Note that due to the de Morgan formulae the following can be proved:

(F1) the empty set $\varnothing$ and the set X are closed;
(F2) the intersection of any system of closed sets is a closed set;
(F3) the union of any finite number of closed sets is a closed set.

There is a topology generated (uniquely) by the system of closed sets. If a system $\mathcal{F}$ of subsets of X satisfies (F1) – (F3) then the set of complements $X \setminus F$ of members F from the system $\mathcal{F}$ has the properties (O1) – (O3) of open sets and is called the *topology generated on X by the family of closed sets $\mathcal{F}$*.

Example 1.19 In the classical algebraic geometry the so-called *Zariski topology* was defined for affine and projective varieties. Let $\mathcal{A}_n$ be an affine space. The Zariski topology is defined by specifying its closed sets: the set $\mathcal{F}$ of all algebraic sets (affine varieties) in $\mathcal{A}_n$ satisfies (F1) – (F3). Similarly, the projective Zariski topology in the projective space is given by Zariski-closed sets which are just all projective varieties (zero sets of homogeneous ideals).

1.2.2 Closure operator. Accumulation points

Let A be a subset of X. The intersection of all closed subsets in X containing A is called the *closure of* A and is denoted as $\overline{A}$; i.e. the closure of A is the smallest close set containing A. A point $x \in X$ is from $\overline{A}$ if and only if each neighbourhood of x intersects A.

The concept of a topological space can be considered as an axiomatization of the notion of the "closeness" of a point to a set: a point is close to a set if it belongs to it closure.

On the other hand, a point $x \in X$ is called an *accumulation point* or *limit point* of A if every neighbourhood of x contains at least one point of $A \setminus \{x\}$, i.e. each neighbourhood of x has a common point with A which is different from x. The family of all accumulation points of A in X is the *derived set* of A and is denoted here as A'.

The following properties hold true:

1. *The closure $\overline{A}$ of a subset $A \subset X$ is the union of A and all its accumulation (limit) points; $\overline{A} = A \cup A'$.*
2. *A set is closed if and only if it contains all its accumulation points.*
3. *A set is closed if and only if it is equal to its closure.*

A set whose closure is the whole space is said to be *dense* in the space. For example, the set of all points in $\mathbb{E}^3$ with rational coordinates is dense in $\mathbb{E}^3$.

A set $A \subset X$ is *co-dense* if $X \setminus A$ is dense in X, and *nowhere dense* if $\overline{A}$ is co-dense, i.e. $X \setminus \overline{A}$ is dense in X.

The closure operator $A \mapsto \overline{A}$ in the given topology has the properties:

Theorem 1.2 (K. Kuratowski) *Let A, B be subsets of the topological space (X, τ). Then the following conditions hold:*

$$(1)\ \ \overline{\varnothing} = \varnothing, \quad (2)\ \ A \subset \overline{A}, \quad (3)\ \ \overline{A \cup B} = \overline{A} \cup \overline{B}, \quad (4)\ \ \overline{\overline{A}} = \overline{A}.$$

It can be verified that the topology is by its closure operator uniquely determined. That is, we have another way how to generate a topology:

Theorem 1.3 *Let (X, τ) be a topological space and let $cl\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a map satisfying for any $A, B \in \mathcal{P}(X)$*

$$A \subset cl\, A, \quad cl\, \varnothing = \varnothing, \quad cl\, X = X, \quad cl(A \cup B) = cl\, A \cup cl\, B, \quad cl(cl\, A) = cl\, A.$$

Then there exists a unique topology on the set X such that $cl\, A = \overline{A}$ holds for all $A \in \mathcal{P}(X)$.

Indeed, if we take all sets $G \in \mathcal{P}(X)$ such that $cl(X \setminus G) = X \setminus G$ we get just the announced topology.

1.2.3 Interior, exterior, boundary

Let $A \subset X$ be a subset in a topological space. Then every point $x \in X$ has exactly one of the following three properties (we speak about *interior*, *exterior* and *boundary* points of A, accordingly; alternative definitions are possible):

(1) there exists a neighbourhood of x which is contained in A;
(2) there exists a neighbourhood of x which is contained in $X \setminus A$;
(3) every neighbourhood of x intersects both the sets A and $X \setminus A$.

The union of all open subsets in the space X that are contained in A is called the *interior of* A and is denoted by $Int\,A$; a point x lies in the interior of A (is an interior point of A) if and only if A is a neighbourhood of x; equivalently, if and only if x has the property (1). The points of $Int\,A$ are *interior points* of A.

Theorem 1.4 *A subset $A \subseteq X$ is open if and only if $A = Int\,A$.*

We can introduce *exterior of* A as $\mathcal{E}xt\,A = Int(X \setminus A)$; a point $x \in \mathcal{E}xt\,A$ if and only if it satisfies (2), and is called an *exterior point of* A. Obviously, $\mathcal{E}xt\,\varnothing = X$. Similarly as above, it is possible to take interior (or exterior) as an axiomatic notion.

Fundamental properties of the interior operator are for every $A, B \in \mathcal{P}(X)$: $Int A \subset A$, $Int\,\varnothing = \varnothing$, $Int\,X = X$, $Int(Int A) = Int A$, $Int A \cap Int B = Int(A \cap B)$. The family of open sets for the topology generated by interior operator consists just from the sets for which $Int\,A = A$.

The set $\delta A = \overline{A} - Int\,A = \overline{A} \cap \overline{(X \setminus A)}$ is called a *boundary* or *frontier* of A. The points from δA are called *boundary (or frontier) points* of A. A point x belongs to δA if and only if it has the property (3).

Among others, the following identities (useful in proofs) can be checked for a subset A of a topological space:

$$Int\,A \subset A \subset \overline{A} = A \cup A' = A \cup \delta A = Int\,A \cup \delta A,$$

$$A \setminus \delta A = Int\,A, \quad X = Int\,A \cup \delta A \cup \mathcal{E}xt\,A \text{ (disjoint union)},$$

$$\delta A = \overline{A} \cap \overline{(X \setminus A)} = \delta(X \setminus A), \qquad \delta\varnothing = \varnothing,$$

$$\varnothing = Int\,A \cap \delta A = \delta(X \setminus A) \cap \mathcal{E}xt\,A = Int\,A \cap \mathcal{E}xt\,A = \delta A \cap \mathcal{E}xt\,A.$$

Theorem 1.5 *A subset A of a topological space satisfies*

$$X \setminus \overline{A} = Int(X \setminus A), \quad X \setminus Int\,A = \overline{(X \setminus A)}.$$

1.2.4 The lattice of topologies. Ordering

On the same set X, more (or even many) topologies can be defined, and they are naturally ordered by inclusion of corresponding systems of open sets.

Let τ and $\tilde{\tau}$ be two topologies, i.e. families of open sets satisfying (O1)–(O3), on the same set X. We say that τ is *finer*, or *bigger* than $\tilde{\tau}$ when the systems satisfy $\tilde{\tau} \subseteq \tau$; in this case, $\tilde{\tau}$ is called *coarser* or *smaller*. Of course there are uncomparable topologies if the underlying set is at least two-element.

From the algebraic point of view, the family of all topologies on the same underlying set, ordered by inclusion, forms a lattice. Indeed, the antidiscrete topology is smaller than any other topology on the same set while discrete topology is the biggest one.

Example 1.20 On a one-element set $X = \{a\}$ there exists a unique topology; the discrete and antidiscrete topology coincide. On a two-element set $X = \{a, b\}$ there are four distinct topologies:

$$\mathcal{O}_0 = \{\varnothing, \{a, b\}\}, \qquad \mathcal{O}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\},$$
$$\mathcal{O}_2 = \{\varnothing, \{a\}, \{a, b\}\}, \qquad \mathcal{O}_3 = \{\varnothing, \{b\}, \{a, b\}\}.$$

Here $\mathcal{O}_0$ is the smallest one, $\mathcal{O}_1$ is the biggest one, $\mathcal{O}_2$ and $\mathcal{O}_3$ are uncomparable. It is interesting to notice how the number of topologies increases with the increasing number of elements of the underlying set X.

1.2.5 Metrization problem

A natural question arises in connection with examples: if we are given a topology is it possible to generate it by some metric, or at least pseudometric? The answer is negative.

A topological space that can be assigned a metric inducing the given topology is called *metrizable*.

The antidiscrete topology on at least two-element set is not metrizable.

The topologies $\mathcal{O}_2$ and $\mathcal{O}_3$ on a two-element set from Example 1.20 are not metrizable (because they are not Hausdorff, which we explain later).

The discrete space is metrizable by the discrete metric.

A great deal of work in general topology was done during examining metrizable spaces, their subspaces and metrizability conditions. Metrization problem was solved in 1951 by a Canadian mathematician R.H. Bing. To be able to formulate necessary and sufficient conditions for a topological space to be metrizable we need to know more about special types of bases and about separation properties which we mention later. Metrization theorems and properties of metrizable spaces are postponed to next sections.

1.2.6 Cover, subcover

Let (X, τ) be a topological space and $A \subseteq X$ a subset. A collection of (open) subsets in X:

$$\mathcal{U} = \{U_\alpha \colon U_\alpha \in \tau,\ \alpha \in \mathcal{J}\}, \qquad \mathcal{J} \text{ is some index set,}$$

is an *(open) cover (or covering)* of A if $A \subset \cup_{\alpha\in\mathcal{J}} U_\alpha$.

Note that the equality holds in the last formula when $X = A$, i.e. $X = \cup_{\alpha\in\mathcal{J}} U_\alpha$.

If $\{U_\alpha \colon U_\alpha \in \tau, \alpha \in \mathcal{J}\}$ is (open) cover of X, under its *(open) subcover* we mean any system $\{U_\alpha \colon U_\alpha \in \tau, \alpha \in \tilde{\mathcal{J}}\}$ where the index set $\tilde{\mathcal{J}} \subseteq \mathcal{J}$.

An (open) cover $\{V_\beta \colon V_\beta \in \tau, \beta \in \mathcal{K}\}$ is a *refinement* of the (open) cover $\{U_\alpha \colon U_\alpha \in \tau, \alpha \in \mathcal{J}\}$ of X if each (open) subset V_β is contained in some U_α.

1.2.7 Bases. Countability Axioms

As we have seen, a topology on a set X can be given by distinguishing a system τ of subsets in X (called "open") satisfying the axioms (O1) – (O3), but it is not the only possibility. We have already mentioned that a topology can be determined, or generated, also in some other way: by the set of closed sets

provided they satisfy (F1) – (F3), by the families of neighbourhoods $\mathcal{U}(x)$ of points x from the topological space satisfying (a) – (d), or by the interior or the closure operator. Here we show yet another possibility, namely to give a suitable part of the family of open sets from which all open sets are generated by set union.

Definition 1.4 Let (X, τ) be topological space. A *neighbourhood base of a point* $x \in X$, or a *local base of* $x \in X$ is a subset $\mathcal{B}(x)$ of the set $\mathcal{U}(x)$ of all neighbourhoods of x such that every neighbourhood of x contains a neighbourhood in $\mathcal{B}(x)$.

Example 1.21 Let $X = \mathbb{R}^n$, or more generally, let X be any metric space. The set of open balls with radius $1/n$, $n = 1, 2, \ldots$ around a fixed point x forms a (countable) neighbourhood base of x.

Definition 1.5 A *base of the topology* τ on X is a system $\mathcal{B}$ of subsets in X such that

1. $\mathcal{B} \subset \tau$.
2. For any point $x \in X$ and any neighbourhood U of x there is a subset $V \in \mathcal{B}$ such that $V \subset U$.

We can give another characterization: a subsystem $\mathcal{B}$ of τ forms a base of the topological space (X, τ) if and only if any non-empty set from τ can be expressed as a union of the sets from the system $\mathcal{B}$.

Obviously, a topological space can have many bases. The following theorem characterizes systems of sets that can serve as a base for some topology.

Theorem 1.6 *A family of sets $\mathcal{B}$ is a base of a topology on a set $X = \cup\{B : B \in \mathcal{B}\}$ if and only if for any $A, B \in \mathcal{B}$ and for any $x \in A \cap B$, there exists $C \in \mathcal{B}$ such that $x \in C \subset A \cap B$ holds. The topology is uniquely determined by its base, and it is in fact the coarsest (smallest with respect to set inclusion) topology on X containing the given base $\mathcal{B}$.*

Example 1.22 Let $X = \mathbb{R}$. The set of intervals $\mathcal{B} = \{\langle a, b) : \ a < b, a, b \in \mathbb{R}\}$ is a base for a topology on $\mathbb{R}$, called *Sorgenfrey topology*. The corresponding topological space is a *Sorgenfrey straightline*. Notice that $\bigcup_{n=2}^{\infty} (a + \frac{b-a}{n}, b) = \langle a, b)$. The Sorgenfrey topology is finer than the usual topology on $\mathbb{R}$.

As mentioned above not every system is a base of a topology. The following weaker concept can be viewed as a compensation, it simplifies considerations concerning possibility of generating a topology by a system of sets.

Definition 1.6 A *subbase of the topology* τ on X is a system $\mathcal{S}$ of subsets in X such that

1. $\mathcal{S} \subset \tau$. 2. Every open set from τ is a union of finite intersections of sets in $\mathcal{S}$.

Every base of topology is also a subbase.

Example 1.23 The system of all intervals (a, ∞) and $(-\infty, a)$, $a \in \mathbb{R}$ is a subbase of the usual topology of $\mathbb{R}$ but not its base.

Theorem 1.7 *Any system $\mathcal{S}$ of sets is a subbase of a uniquely defined topology on the set $X = \bigcup \{S\colon\ S \in \mathcal{S}\}$.*

Indeed, we take all possible finite intersections of sets from $\mathcal{S}$ and use Theorem 1.6 to check that they form a base.

The definition of a topological space is very general. Not many interesting theorems can be proved about all topological spaces. Various classes of topological spaces are studied, ranging from fairly general to more and more special. One type of restrictions is concerned with cardinality of bases.

Recall that a system of sets is *countable* if the system includes at most a countable family of members (i.e. there is a one-one map of elements of the system into natural numbers $\mathbb{N}$).

Definition 1.7 A topological space satisfies the *first countability axiom*, and is called *first countable*, if every point possesses a countable neighbourhood base.

A space satisfies the *second countability axiom*, and is called *second countable*, if it possesses a countable base (of the topology).

A topological space X is called *Lindelöf* if each open cover of X has a countable subcover.

If a space contains a countable dense subset it is called *separable*.

Theorem 1.8 *Any metric (metrizable) space is first countable.*

Theorem 1.9 *If the space is second countable then each its base has a countable subbase.*

The second countability axiom is the strongest one from the list of conditions just mentioned:

Theorem 1.10 *If the topological space is second countable then it is first countable, Lindelöf and separable.*

In the second countable space, all sets containing the fixed point x form obviously a countable neighbourhood base of x. To construct a countable dense subset we choose one element from each member of a fixed countable base (we need the Axiom of Choice); to prove the Lindelöf property is also possible. However in metrizable spaces the following holds:

Theorem 1.11 *If the topological space is metrizable then second countability, Lindelöf property and separability are equivalent.*

Before giving more details it is convenient to introduce new concepts and more terminology.

1.2.8 Sequences in topological spaces, nets

As far as the role and behaviour of convergent sequences is concerned there is a great difference between first coutable spaces and general topological spaces.

Assume a topological space X, a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points from X, i.e. x_n belongs to X for all $n \in \mathbb{N}$, and a fixed point $x \in X$.

We say that the sequence $\{x_n\}$ *converges* to the point x in X, or that x is a *limit point* of the sequence $\{x_n\}$ if for any neighbourhood $U \in \mathcal{U}(x)$ of x there exists a natural number $n \in \mathbb{N}$ such that for all $m \geq n$, $m \in \mathbb{N}$, the point x_m belongs to the neighbourhood U; we use the usual notation $\lim\limits_{n\to\infty} x_n = x$.

We say that $x \in X$ is an *accumulation point* of a sequence $\{x_n\}$ if for any neighbourhood U and for any natural $n \in \mathbb{N}$ there exists $m \geq n$, $m \in \mathbb{N}$ such that $x_m \in U$.

Similarly as in metric spaces (particularly as in real numbers) we can prove:

Theorem 1.12 *If X is a first countable topological space, $A \subset X$, $x \in X$, then the following holds:*

(1) the point x belongs to the closure $\overline{A}$ if and only if there exists a sequence $\{x_n\}$ of points from A, $x_n \in A$, such that $\lim\limits_{n\to\infty} x_n = x$;

(2) the point x is an accumulation point of the sequence $\{x_n\}$ if and only if there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that x is its limit point;

(3) the point x is an accumulation point of the set A if and only if there exists a sequence $\{x_n\}$ of points from the set $A \setminus \{x\}$ such that $x = \lim\limits_{n\to\infty} x_n$.

If we omit the assumption on first countability the theorem is no more true (we can construct examples [85]), because convergence of sequences depends not only on the sequence itself but also on the type of ordering of the local base for the point x. To substitute sequences in general topological spaces, we need to find some more general concept which would "work".

Recall that the relation of *directing* on a set D is an ordering of D which satisfies: if d and d' belong to D then there exists $d'' \in D$ such that $d \leq d''$ and $d' \leq d''$, and the pair $(D, \leq)$ is a *directed set*.

Definition 1.8 A *net* $\{x_d \colon d \in (D, \leq)\}$ is an arbitrary function (map) from a non-empty directed set $(D, \leq)$ to the topological space X.

A point $x \in X$ is said to be a *limit point* of a net $\{x_d \colon d \in D\}$ in X if for every neighbourhood U of x there exists an element $d_0 \in D$ such that for any $d \in D$ satisfying $d_0 \leq d$, x_d belongs to U; we write $x = \lim\limits_{d\in D} x_d$.

To demonstrate applications of the concept let us mention:

Theorem 1.13 *Let A be a subset of a topological space X. A point $x \in X$ belongs to the closure $\overline{A}$ if and only if there exists a net $\{x_d \colon d \in D\}$, $x_d \in A$ for $d \in D$, such that $x = \lim\limits_{d\in D} x_d$.*

Also subnets and accumulation points of a net can be introduced (for more details, [85]).

1.3 Continuous maps

In the family of maps from one topological space to the other, we prefer maps which "preserve topological structure." We start with continuous maps and show that they "pull back" open sets, i.e. preserve topological structure in one direction. Then we pass to one-one onto continuous maps with continuous inverse which peserve topological structure in both directions, and are topological equivalences.

After some experience with generalizing continuity of a real-valued function of real variable(s) to continuity of maps between metric spaces, it might seem quite natural to formulate continuity of maps between topological spaces also in terms of neighbourhoods. Recall that a subset U of (X,τ) is a neighbourhood of a point $x \in X$ if there is an open subset $O \subseteq X$ such that $x \in O \subset U$. If U itself is an open set we speak about an *open neighbourhood*.

1.3.1 Continuous maps of topological spaces

Let (X,τ) and (X',τ') be topological spaces.

Definition 1.9 A map $f\colon X \to X'$ is *continuous* in the point $x \in X$ if for any neighbourhood U' of the point $f(x) \in X'$ there exists a neighbourhood U of $x \in X$ such that $f(U) \subset U'$, i.e. $f(y) \in U'$ for each $y \in U$.

A map f is *continuous* if it is continuous in all points of the set X.

It is convenient to have some criteria for continuity of maps formulated in terms corresponding to various methods of defining or generating topologies. The notion of continuity is particularly easy to formulate in terms of open sets.

Theorem 1.14 *Let (X,τ), (X',τ') be topological spaces and $f\colon X \to X'$ a map of X to X'. The following properties are equivalent:*

(1) $f\colon X \to X'$ is continuous;
(2) inverse images of all open subsets of X' (sets from τ') are open in (X,τ);
(3) inverse images of all closed subsets of X' are closed in (X,τ);
(4) inverse images of all members of a subbase for X' are open in X;
(5) inverse images of all members of a base for X' are open in X;
(6) for every subset $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$;
(7) for every subset $B \subset X'$ we have $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$;
(8) for every point $x \in X$ and every net $\{x_d\colon d \in D\}$ in X with $x = \lim_{d\in D} x_d$, the net of images $\{f(x_d)\colon d \in D\}$ in X' has a limit point equal $f(x)$.

Further equivalences can be found e.g. in [53, p. 47].

Note that a continuous real-valued function of one real variable is continuous according to our new definition, [53, p. 49]. Important point is that continuity is preserved under maps composition; in the proof, the equality $(gf)^{-1}(A) = f^{-1}(g^{-1}(A))$ is used:

Theorem 1.15 *The composition of two continuos maps is continuos.*

Note that any map $f\colon X \to X'$ is continous whenever the topological space X is discrete, or whenever X' is antidiscrete.

Theorem 1.16 *Let X, X' be topological spaces and let $A \subseteq X$ have a subspace topology. Suppose $f: X \to X'$ is continuous. Then the restriction $f|_A: A \to X'$ is continuous.*

Theorem 1.17 (Glueing Lemma) *Let X and Y be topological spaces, $X = A \cup B$, where A and B are closed (open) subsets of X. Let $f_1: A \to Y$ and $f_2: B \to Y$ be continuous maps such that $f_1(x) = f_2(x)$ for all $x \in A \cap B$.*

Then the map $g: X \to Y$ defined by $g(x) = \begin{cases} f_1(x) \text{ for } x \in A, \\ f_2(x) \text{ for } x \in B, \end{cases}$ *is continuous.*

Note that without any assumption on A and B, the theorem is false.

1.3.2 Homeomorphisms

Together with topological spaces, we consider maps ("morphisms") that preserve topological structure:

Definition 1.10 A map $f: X \to X'$ is a *homeomorphism* (or a *topological map*) if the following conditions are satisfied:

1. f is one-one and onto map, i.e. there exists an inverse map f^{-1};
2. both the maps f and f^{-1} are continuous.

In this case the spaces X and X' are called *homeomorphic* which is denoted by

$$X \cong X'.$$

Since the identity map $\mathrm{id}_X: X \to X$ is a homeomorphism, the composition gf of two homeomorphisms f and g as well as the inverse map f^{-1} are again homeomorphisms the following can be checked:

1. $X \cong X$ – *reflexivity*,
2. $X \cong X' \Rightarrow X' \cong X$ – *symmetry*,
3. $X \cong X'$ and $X' \cong X'' \Rightarrow X \cong X''$ – *transitivity*.

Therefore the binary relation " $\cong$ " on the class of all topological spaces is an equivalence relation. Under homeomorphisms, open (closed) sets are mapped again onto open (closed) sets, hence the map f induces one-one onto correspondence between the topologies of X and X'. The topology of two spaces belonging to the same equivalence class is in a sense the same. That is why we identify homeomorphic spaces.

Theorem 1.18 *Let (X, τ), (X', τ') be topological spaces and $f: X \to X'$ a map of X to X'. The following properties are equivalent:*

(1) $f: X \to X'$ is a homeomorphism;
(2) G is open in τ if and only if $f(G)$ is open in τ';
(3) F is closed in τ' if and only if $f^{-1}(F)$ is closed in τ;
(4) O is open in τ' if and only if $f^{-1}(O)$ is open in τ;
(5) A is closed in τ if and only if the image $f(A)$ is closed in τ';
(6) U is a neighbourhood of $x \in X$ if and only if $f(U)$ is a neighbourhood of $f(x) \in X'$;
(7) for every subset $A \subset X$ we have $f(\overline{A}) = \overline{f(A)}$;
(8) for every net $\{x_d : d \in D\}$ in X,

$$x = \lim_{d \in D} x_d \text{ holds if and only if } f(x) = \lim_{d \in D} f(x_d).$$

Further equivalences can be found e.g. in [53, p. 54]. Continuous maps and homeomorphisms of abstract spaces were first considered by M. Fréchet (1910).

If $f: X \to Y$ is a one-one map of topological spaces, and if $f: X \to f(X)$ is a homeomorphism when we give $f(X)$ the induced topology from Y, we call f an *embedding* of X to Y.

Recall that a map between topological spaces is *open (closed)* if the image of each open (closed) set is open (closed).

Theorem 1.19 *A one-one onto map of topological spaces is a homeomorphism if and only if it is continuous and open.*

1.3.3 Topological invariants

If a property of a topological space is preserved under homeomorphisms it is called a topological property or a *topological invariant*.

The object of topology is to study topological properties. Roughly, every property defined in terms of open sets and in terms of set theory is a topological invariant.

We should say at once that there is no hope of classifying all topological spaces. However, there are techniques which anable us to decide whether two spaces are homeomorphic or not. Showing that two spaces are homeomorphic is rather a geometrical problem, involving the construction of a specific homeomorphism between given spaces, and the techniques used vary with the problem.

On the other hand, a problem of an entirely different nature is attempting to prove that two spaces are not homeomorphic to one another: in this case we look for topological invariants trying to find a topological property in which the spaces differ. It might be one of the well-known topological properties (some of them will be discussed in the sequel) such as countability, existence of special bases, connectedness, compactness, separation properties, or an algebraic structure, such as a group or ring constructed from the space (e.g. fundamental group, homotopy groups, homology groups), or number (e.g. Euler number defined for the surface, Betti numbers) etc.

1.4 Constructions of new topological spaces from given spaces

1.4.1 Projectively and inductively generated topologies (initial and final)

Let us describe methods of generating topologies based on the concept of a continuous map. The following four constructions are particularly useful: (topological) product, subspace, sum (= disjoint union) and quotient. Note that first two constructions are particular cases of a more general construction of the so-called projectively generated topologies while sum and quotient are particular cases of the so-called inductively generated topologies.

Theorem 1.20 *Let X be a set, $\{(Y_t, \tau_t) : t \in T\}$ a family of topological spaces and $\{f_t : t \in T\}$ a system of maps where $f_t: X \to Y_t$. In the class of all topologies on X that make all maps f_t continuos there exists a coarsest topology τ. One of its bases consists of all sets of the form $\bigcap_{i=1}^{k} f_{t_i}^{-1}(V_i)$ where V_i is open in Y_{t_i}, $t_1, t_2, \ldots, t_k \in T$ for $i = 1, 2, \ldots, k$.*

The topology τ is called the topology *projectively determined, or projectively generated, by the system of maps* $\{f_t\colon\ t \in T\}$, also *initial topology*.

Notice that all sets of the form $f_t^{-1}(V_t)$ where V_t is open in Y_t form a subbase for the initial topology [53, p. 51].

Theorem 1.21 *A map f of a topological space (X, τ) to a topological space (X', τ') whose topology is generated projectively by a family of maps $\{f_t\colon\ t \in T\}$ where f_t is a map of X' to X'_t, is continuous if and only if every composite map $f_t f$ is continuous for $t \in T$.*

Now let us assume the "dual" situation, when all arrows in the considered maps are reversed.

Theorem 1.22 *Let X be a set, $\{(Y_t, \tau_t)\colon\ t \in T\}$ a system of topological spaces and $\{f_t\colon\ t \in T\}$ a system of maps where $f_t\colon Y_t \to X$. In the class of all topologies on X that make all maps f_t continuos there exists a finest topology τ. Open sets of this topology are exactly all sets $G \subseteq X$ satisfying $f_t^{-1}(G) \in \tau_t$ for all $t \in T$.*

The topology τ is called the topology *inductively generated on X by the system of maps* $\{f_t\colon\ t \in T\}$, also *final topology*.

Theorem 1.23 *Let $f\colon (X, \tau) \to (X', \tau')$ be a map of topological spaces and let τ be a topology inductively generated on X by a family of maps $\{f_t\colon\ t \in T\}$ where $f_t\colon Y_t \to X$. Then the map f is continuous if and only if every composition $f \circ f_t$ is continuous for $t \in T$.*

1.4.2 Subspace and product

If $A \subseteq X$, we introduce a *subspace topology* on A induced by the topology τ on X (or *relative topology*) as follows:

$$\tau_A = \{Y \cap U\colon\ U \in \tau\}.$$

The topological space (A, τ_A) is called a *topological subspace* in (X, τ). It can be checked that a subspace topology on a subset $A \subset X$ of a topological space (X, τ) is just the topology projectively generated by the one-element system $\{j\}$ where $j\colon A \to X$ is the "*canonical identical embedding*", $j(y) = y$ for every $y \in A$.

Theorem 1.24 *Every subspace of a first (second) countable space is itself first (second) countable.*

We shall see later that there exist more properties that are hereditary for subspaces.

Example 1.24 An important example of a topological space with a subspace topology is an *n-dimensional sphere* $\mathbb{S}^n(x, r) \subset \mathbb{R}^{n+1}$ with radius r centered at x

$$\mathbb{S}^n(x, r) = \{y \in \mathbb{R}^{n+1}\colon\ \|x - y\| = r\};$$

its subspace topology is induced by the natural topology of $\mathbb{R}^{n+1}$. Notice that instead of the subspace metric, usually a more suitable metric is defined making use of the length of a geodesic segment passing through the pair of points; the arising topology is the same. If $r = 1$ we speak about the *standard n-sphere* $\mathbb{S}^n$. $\mathbb{S}^1(x, r)$ is called a *circle* ($n=1$); the *standard circle* $\mathbb{S}^1$ corresponds to $n = r = 1$.

Let (X_1, τ_1), (X_2, τ_2), …, (X_m, τ_m) be topological spaces. Then on the (finite) Cartesian product $X_1 \times X_2 \times \cdots \times X_m$ we can define the so-called *product topology* $\prod_i \tau_i$ as the topology generated by the (finite) family of projections $p_i\colon\ i = 1, 2, \ldots, m\}$ where $p_i\colon\ \prod_i X_i \to X_i$, $i = 1, 2, \ldots, m$. Alternatively, this topology is determined by taking

$$\mathcal{B} = \{G_1 \times G_2 \times \cdots \times G_m\colon\ G_i \in \tau_i,\ i = 1, \ldots, m\}$$

as its base. The space $(\prod_i X_i, \prod_i \tau_i)$ is called a (finite) *topological product* of the given spaces.

For an arbitrary (infinite) system of spaces, a definition of the product can be modified (the family of indices might be finite or infinite, even uncountable). The cartesian product $X = \prod_{t \in T} X_t$ [85] consists of functions

$$x\colon\ T \to \bigcup_{t \in T} X_t \ \text{ that satisfy } x_t := x(t) \in X_t \text{ for all } t \in T.$$

The system of projections $\{p_t\colon\ t \in T\}$ where $p_t(x) = x_t$ generates projectively an initial topology on the product X. One of its subbases is formed by the family of all sets $p_t^{-1}(V)$ where V is open in X_t, $t \in T$, and all possible finite intersections of such sets form its base.

Example 1.25 The space $\mathbb{R}^n = \mathbb{R}^1 \times \cdots \times \mathbb{R}^1$ with natural product topology (i.e. product of n copies of natural topologies of factors) is widely used.

As a consequence of Theorem 1.10 we have a statement which is often used in the calculus (continuity of a map of $X = \mathbb{R}^n$ can be examined "componentwise"):

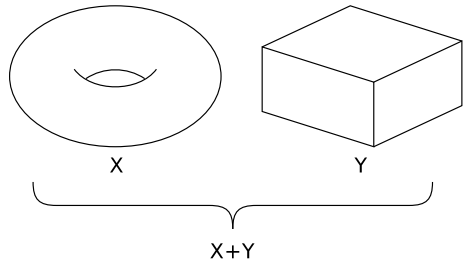
Theorem 1.25 *A map f of a topological space (X, τ) to the product space $(\prod_t X_t, \prod_t \tau_t)$ is continuous if and only if for every projection the composition $p_t \circ f$ is continuous.*

Example 1.26 The *box topology* on $\prod_{n \in \mathbb{N}} X_n$ has as base all sets of the form $V_1 \times V_2 \times \cdots$ where V_n is open in X_n. The box topology contains the product toplogy, and the two coincide if and only if for all but finitely many values of n, X_n is an antidiscrete space [14, p. 56].

1.4.3 Sum and quotient

For simplicity, let us introduced first a *sum* (disjoint union) of two sets X and Y by means of a trick:

$$X + Y = (\{0\} \times X) \cup (\{1\} \times Y).$$



Of course, the sets X and Y do not have to be disjoint, that is why we cannot use the union directly when we want to have the disjoint juxtaposition of a copy of X and one of Y. Obviously, we can introduce the one-one maps $f_1\colon X \to X + Y$, $x \mapsto (0, x)$ for $x \in X$ and $f_2\colon Y \to X + Y$, $y \mapsto (0, y)$ for $y \in Y$ (and we may treat X, Y as subsets of $X + Y$).

If (X,τ) and (Y,τ') are topological spaces, a new topology on $X+Y$ can be given by a family of open sets

$$\{U+V:\ U\in\tau, V\in\tau'\},$$

and denoted $\tau+\tau'$. The pair $(X+Y,\tau+\tau')$ is called the *topological sum* of the topological spaces (X,τ) and (Y,τ'). We can verify that the topology $\tau+\tau'$ is inductively generated on $X+Y$ by a family of maps $\{f_1,f_2\}$.

The same formal trick can be used in general. If $\{(X_t,\tau_t):\ t\in T\}$ is a system of topological spaces then on the set $\sum_{t\in T}X_t=\bigcup_{t\in T}(\{t\}\times X_t)$ we introduce a topology $\sum_{t\in T}\tau_t$ inductively by a system of maps

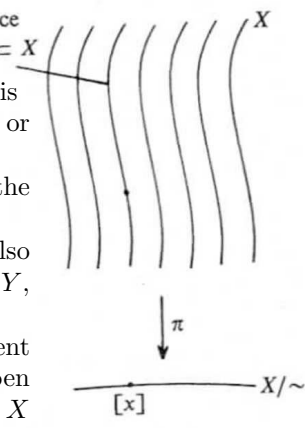
$$\{j_t:\ t\in T\},\quad j_t\colon X_t\to\sum_{t\in T}X_t,\quad j_t(x)=(x,t).$$

The topology $\sum_{t\in T}\tau_t$ is the finest topology that contains the system of sets $\{\{t\}\times G\colon G\in\tau_t,\ t\in T\}$, and every j_t is a (continuous) embedding of X_t to the sum.

1.4.4 The quotient topology

Now we would like to introduce a topology on a "quotient space" in a natural way. Recall that if X is a set and $\sim$ an equivalence relation on X then we denote by $[x]$ the equivalence class of $x\in X$ and $X/\sim$ is the set of equivalence classes. The canonical projection is $\pi\colon X\to X/\sim,\ \pi(x)=[x]$. The equivalence relation determines a partition $\mathcal{P}$ of X into equivalence classes, and vice versa, any partition into disjoint classes defines an equivalence relation.

If $\sim$ is an equivalence relation on a topological space X take the quotient space $Y=X/\sim$ the points of which are just the members of the partition. On Y, we take the finest topology for which π is continuous.



Definition 1.11 The topology inductively determined on $Y=X/\sim$ by the one-element system $\{\pi\}$ (the finest topology on $X/\sim$ such that π is a continuous map) is called the *quotient topology* [75] or the *identification topology* [14].

$X/\sim$ endowed with the topology thus defined is the *quotient* of X by the relation $\sim$.

If $y\in Y=X/\sim$ then the inverse image $\pi^{-1}(y)$ is also called the *fibre* over y and X is the fibred space over Y, the space Y is a *base space* of X.

Note that a subset G of Y is open in the quotient topology if and only if its inverse image $\pi^{-1}(G)$ is open in X. We can think of Y as the space obtained from X by identifying each subset P of the partition $\mathcal{P}$ (defined by the equivalence relation) to a single point.

Theorem 1.26 *Let Z be a topological space and let $Y=X/\sim$ be a quotient space (with a quotient topology). A map $f\colon Y\to Z$ is continuous if and only if the composition $f\pi\colon X\to Z$ is continuous.*

Instead from an equivalence relation we sometimes start from an onto map. Let X be a topological space. If $f\colon X \to Y$ is an onto map the subsets $f^{-1}(y)$, where $y \in Y$, form a partition of X arising from an equivalence relation: $x \sim z$ if and only if $f(x) = f(z)$ for $x, z \in Z$. If we take on Y the finest topology for which f is continuous, $X/\sim$ is obviously homeomorphic with Y, and a map $g\colon Y \to Z$ is continuous if and only if $gf\colon X \to Z$ is continuous. It justifies the following:

Definition 1.12 Let $f\colon X \to Y$ be an onto map and let the topology on Y be the largest one for which f is continuous. Then we call f an *identification map*.

1.5 Connectedness

1.5.1 Path-connected spaces

Definition 1.13 A *path* p in a topological space X is a continuous map $p\colon I = \langle 0, 1\rangle \to X$ such that $p(0) = x$ is a *beginning point* and $p(1) = y$ is an *end point* of the path respectively, and p is said to *join* x to y.

A topological space is called *path-connected* if any of its two points can be joined by a path. A subset is *path-connected* if it is path-connected in subspace topology.

Notice that a path is a parametrized curve, *not* a set of points. We can introduce $|p| = p(I)$ for the trajectory (set of points that are images under p).

If p is a path in X, and if $f\colon X \to Y$ is a continuous map, then the composition fp is a path in Y.

Theorem 1.27 *The following holds:*

1. *Path-connectedness is preserved by surjective continuous maps.*
2. *If $h\colon X \to Y$ is a homeomorphism and X is path-connected then Y is also path-connected.*
3. *Path-connectedness is a topological invariant.*
4. *In a topological space X, let $\mathcal{M}$ be a collection of path-connected sets, with a common point. Then the union of the elements of $\mathcal{M}$ is path-connected.*

On the set of paths, a composition is defined as follows. If p is a path from x to y and q is a path from y to z then $p * q$ is a path from x to z defined by $(p * q)(t) = p(2t)$ for $t \in \langle 0, 1/2\rangle$, $(p * q)(t) = q(2t - 1)$ for $t \in \langle 1/2, 1\rangle$ (Theorem 1.17 is used). The inverse path $-p$ to p, from y to x, is defined by $(-p)(t) = p(1 - t)$, $t \in I$. Any point can be connected with itself, e.g. by a constant path. We can see that we get an equivalence relation on X putting $x \sim y$ if and only if x can be connected with y by a path. A topological space is path-connected if and only if this equivalence relation defines a single equivalence class.

A *component of path-connectedness* of a point x is a maximal path-connected subset of X containing x. The space X is covered by pairwise disjoint components.

The idea of path-connectedness is quite natural and is adequate e.g. in the study of polyhedra. The following idea, however, is more applicable.

1.5.2 Connected topological spaces

We now deal with spaces which consists of "one piece" only. There are several equivalent ways how to formalize this property.

Definition 1.14 A topological space X is *connected* if the whole space and the empty set are the only subsets in X which are at the same time open and closed. In the opposite case, we call the space *disconnected*.

The following equivalent characterizations of connected spaces can be given.

Theorem 1.28 *Let X be a topological spaces. The following properties are equivalent:*

1. *X is connected.*
2. *X is not the union of two disjoint nonempty open sets.*
3. *X is not the union of two nonempty subsets A, B satisfying $A\cap\overline{B}=\overline{A}\cap B=\varnothing$, i.e. neither of the sets contains a point or a limit point of the other.*

Recall that A and B satisfying $A \cap \overline{B} = \overline{A} \cap B = \varnothing$ are called *separated*.

Definition 1.15 A subset of a topological space X is *connected* if it is connected in the subspace topology; equivalently, if it is not the union of two nonempty separated sets.

The following can be proved:

1. *For topological spaces, as well as for subsets, connectedness is preserved by surjective continuous maps.*
2. *The image of a connected topological space (subset) under a continuous map is connected.*
3. *Connectedness is a topological invariant.*

Example 1.27

1. Any at least two-point set in the discrete topology is disconnected.
2. The sum of at least two nonempty topological spaces is disconnected.
3. Any subset of an antidiscrete space is connected.
4. The following metric spaces (with natural topology) are connected: the space of real numbers $\mathbb{R}$, any interval in $\mathbb{R}$, Euclidean n-space $\mathbb{E}^n$, any ball or cube in $\mathbb{E}^n$.
5. Any at least two-element set of the set of rational numbers (with subspace topology) is disconnected.

The following theorem can be used to prove that $\mathbb{E}^n$ is connected:

Theorem 1.29 *Let X be a topological space, and let M_0 and M_w, $w \in W$ be connected subsets of X. Assume $M_0 \cap M_w \neq \varnothing$ for each $w \in W$. Then the union $M_0 \cup (\cup_{w\in W} M_w)$ is connected.*

Theorem 1.30 *Let X be a topological space, let $\mathcal{M}$ be a system of connected subsets of X such that $\bigcup \mathcal{M} = X$ ($\mathcal{M}$ is a cover of X). Assume that for any M, N from $\mathcal{M}$ there exists a finite set $\{M_0 = M, M_1, \ldots, M_n = N\}$ such that $M_i \cap M_{i+1}$ is nonempty for $i = 0, 1, \ldots, n$. Then X is connected.*

By means of the previous theorems we can prove the following:

1. *The product of two (or finite number of) connected spaces is connected.*
2. *The product of a system of topological spaces is connected if and only if each of the factors is connected.*
3. *A closure of a connected subset of a topological space is connected.*

On the other hand, there are disconnected sets with a connected closure (e.g. the disconnected set of all rational numbers $\mathbb{Q}$ has a connected closure $\mathbb{R}$), and there are connected sets with disconnected interior.

Let M be a subset of a topological space X, $x \in M$. The *component* $C(M, x)$ of M that contains x is the union of all connected subsets of M that contain x; the *component of* x is $C(x) = C(\{x\}, x)$. Note that components are connected.

4. *Any topological space can be presented as a disjoint union of components of some of its points. Each component is closed.*
5. *If A and B are separated in X, then every connected subset M of $A \cup B$ lies either in A or in B.*

As a consequence, we can check:

6. *Every path-connected topological space is connected.*

The converse is false in general: Let M be the graph of $f(x) = \sin(1/x)$ in $\mathbb{R}^2$, $0 < x \leq 1/\pi$, together with the points $(0, 1)$, $(0, -1)$. M is connected, but there is no path from $(0, 1)$ (or from $(0, -1)$) to any other point of M.

Theorem 1.31 *For open subsets in $\mathbb{R}^n$ (in $\mathbb{E}^n$) with subspace topology, connectedness and path-connectedness are equivalent.*

1.6 Separation properties

Now we shall discuss axioms of separation which bring another kind of restrictions concerning separation of points and closed sets in topological spaces. We mention properties that become particularly useful when the question of metrizability of a topology should be answered.

"Separation" in the title means two things: either separation of points or closed sets by open neighbourhoods, or by means of special functions. The separation axioms are usually organized in a hierarchy from weaker to stronger properties.

A motivation is as follows. Given two distinct points in a metric space, we can always find disjoint open sets containing them, for if d is a metric and $d(x, y) = \varepsilon > 0$ then the sets

$$U_x = \{z\colon\ d(x, z) < \varepsilon/2\} \qquad \text{and} \qquad U_y = \{z\colon\ d(y, z) < \varepsilon/2\},$$

for instance are disjoint open neighbourhoods.

At the beginning, this property was considered quite "natural" for a topological space to admit. For this reason F. Hausdorff included it in his original definition of "topological space". Recall that the first fundamental book including the field of general topology was published in 1914 by Felix Hausdorff under the title Grudzüge der Mengenlehre; in 1927 Hausdorff published an extensively revised second edition under the title Mengenlehre.

Later it was found that topologies which do not satisfy such a condition can be very useful, too, e.g. the Zariski topology used in algebraic geometry.

1.6.1 The Hausdorff separation axiom

Nowadays we formulate the possibility to separate two points by disjoint neighbourhoods as an additional axiom which a topological space may or need not have:

Definition 1.16 A topological space is called *Hausdorff space*, or T_2*-space*, if for any two different points there exist disjoint neighbourhoods.

As already noted any metric space is Hausdorff, and only Hausdorff topologies can be metrizable.

If we are inspired and influenced by Euclidean pictures too much then the property of "non-Hausdorff" at first glance might seem unreasonable or even strange. But we can find such non-Hausdorff topologies among examples already listed above. Let us mention the following two weaker classes of spaces.

Definition 1.17 A topological space is T_0*-space* if for any two different points there exists a neighbourhood of one of them not containing the other point.

A space is T_1*-space* if for any two distinct points there exist neighbourhoods of each of them not containing the other point.

By definition the implications $T_2 \Rightarrow T_1 \Rightarrow T_0$ hold, but cannot be reversed.

Example 1.28 To show that there are spaces which are not even T_0 it is sufficient to take any at least two-element set X with antidiscrete topology $\{\varnothing, X\}$.

Example 1.29 If we take a three-element set $X = \{a, b, c\}$ together with the collection of open sets $\{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$ we can check that the space is T_0 but not T_1 (e.g. c cannot be separated from b).

Example 1.30 Consider an infinite set X with the topology of finite complements on X. The topology is T_1 but not T_2.

The following properties are concerned the separation of closed sets.

Definition 1.18 We say that a topological space satisfies *regularity condition*, or is *regular*, if for any closed set A of the topological space X and any point x from X not belonging to A there exists a neighbourhood U of x and a subset B of X satisfying $A \subset Int\, B$ such that U and $Int\, B$ are disjoint.

Note that B plays here the role of a *neighbourhood of the set* A, and $Int\, B$ is an open neighbourhood of A; we use this terminology even later. We say that a space satisfies T_3*-separation axiom* if it is T_1 and regular.

Example 1.31 We can give example of a space which is T_2 but cannot be T_3 because it is not regular: suppose $X = \mathbb{R}$, A is the set of all rationals of the form $1/n$ where n runs over reals. We take topology with open sets of the form $H \backslash B$ where H is a subset of $X = \mathbb{R}$ open in the usual topology and B is a subset of A. The subset A cannot be separated from the point 0 in this topology.

Definition 1.19 A topological space satisfies *normality condition*, or is *normal*, if for any pair of disjoint closed subsets of the space there exist a pair of their disjoint neighbourhoods. We say that a space satisfies T_4-separation axiom if it is T_1 and normal.

Example 1.32 On an arbitrary set, the discrete topology is normal.

If the space is T_{i+1}, $i = 0, 1, 2, 3$ then it is also T_i. As examples show the implications cannot be reversed. Intuitively speaking, the above conditions give informations how rich is the topology with respect to the possibility of separation of objects (points, closed subsets) of the space. Because closed sets and neighborhoods are preserved under homeomorphisms we have:

Theorem 1.32 *The property "to be a T_i-space" is a topological invariant.*

1.6.2 Separation by continuous functions

There exists yet another type of separability, namely separation of objects (points, closed subsets) by means of values of continuous functions in case there exist enough "suitable" functions.

Definition 1.20 A topological space X is *completely regular* if for any closed subset A of the topological space X and any point x from X not belonging to A there exists a continuous real-valued function defined on X such that $f(x) = 0$, $f(y) = 1$ for every $y \in A$ and $0 \leq f(z) \leq 1$ for all $z \in X$.

Note that any completely regular space is regular.

A topological product of two Sorgenfrey straightlines is completely regular but is not normal. A T_3-space need not be T_4.

There exist normal T_0-spaces which are neither regular nor T_2. For normal spaces, we can prove that there exists already enough functions for separation:

Theorem 1.33 (Urysohn's Lemma) *A topological space X is normal if and only if for any pair of disjoint closed subsets A, B of the space there exists a continuous real-valued function defined on X, with values in the interval $\langle 0, 1 \rangle$, such that $f(x) = 0$ for $x \in A$ and $f(y) = 1$ for $y \in B$.*

Theorem 1.34 *The following can be proved:*

1. *Any metric (or pseudo-metric) space is normal and completely regular.*
2. *A closed subset of a normal space is normal.*
3. *Any normal T_1-space (i.e. T_4-space) is a completely regular T_2-space.*

Theorem 1.35 *If A, B are closed disjoint subsets of a topological space X and a, b are any real numbers then there exists a continuous real-valued function on X such that $f(A) = a$, $f(B) = b$ and the values of f on X are (strictly) between maximum and minimum of the two-element set $\{a, b\}$.*

The question *When does a function defined on a subset of topological space admit an extension to the whole space?* is one of fundamental questions of topology. Let us mention the following result of classical calculus:

Theorem 1.36 (Tietze extension theorem for metric spaces) *Any real-valued cuntinuous function defined on a closed subset of a metric space can be extended over the whole space.*

More generally for topological spaces, normality is just the condition under which a real-valued function satisfying the above assumptions can be extended. We can formulate the following interesting and useful characterization of normal spaces:

Theorem 1.37 (Tietze extension theorem for topological spaces) *The space X is normal if and only if for any continuous real-valued map f which is defined on a closed subset A of X (with subspace topology) there exists a continuous extension of f defined on the whole space X.*

Theorem 1.38 *Any regular Lindelöf topological space is normal.*

Some of the separation properties are inherited by spaces constructed from the given spaces, e.g. by subspaces or products:

Theorem 1.39 *The following holds:*

1. *A subspace of a T_2-space is itself a T_2-space.*
2. *Topological product of regular spaces is regular.*
3. *A subspace of a regular space is regular.*
4. *Any product of completely regular spaces is completely regular.*
5. *Any subspace of a completely regular space is completely regular.*

1.6.3 Tychonoff spaces

Definition 1.21 A *Tychonoff space*, or *$T_{3\frac{1}{2}}$-space*, is a completely regular T_1-space.

It can be proved that Tychonoff spaces can be characterized just as embeddings to cubes [53, p. 61]:

For a unit interval in reals we use the notation $I = \langle 0, 1\rangle$. If T is an arbitrary index set the product $I^T = \langle 0, 1\rangle^T$ is called the *unit cube*, and we suppose I^T to be endowed with the product of natural topologies on I. Because any two closed intervals in $\mathbb{R}$ are homeomorphic, the unit cube can be considered instead of an arbitrary cube $\langle a, b\rangle^T$.

Moreover, without loss of generality, we can work with unit cubes instead of arbitrary products of intervals (parallelepipeds): the product space $X = \prod_{t\in T}\langle a_t, b_t\rangle$, topologized by product of natural topologies on $\langle a_t, b_t\rangle$, is homeomorphic with I^T.

Theorem 1.40 (Tychonoff Embedding Theorem) *A topological space is Tychonoff if and only if it is homeomorphic to a subspace of a cube I^T.*

First we prove the following.

Theorem 1.41 (Embedding Lemma) *Let $\mathcal{F}$ be a set of continuous maps on a topological space (X,τ) such that for every $f \in \mathcal{F}$, $f\colon (X,\tau) \to (Y_f,\tau_f)$. Let us construct a map F as follows:*

$$F\colon (X,\tau) \to \bigcap_{f\in\mathcal{F}} (Y_f,\tau_f), \quad F(x) = \{f(x)\}_{f\in\mathcal{F}}.$$

Then

* *F is a continuous map.*
* *F is one-one if and only if the system $\mathcal{F}$ distinguishes points, i.e. if for any pair of distinct points x, $y \in X$ there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*
* *If F distinguishes points and closed sets (i.e. for any closed subset A of X and any point x not in A there exists $f \in \mathcal{F}$ such that $f(x)$ is not in $\overline{f(A)}$) then F is an open map of (X,τ) to $F(X)$ endowed with a subspace topology (induced from the product topology).*

The proof of Tychonoff Theorem is based on the following considerations. If the space is homeomorphic to a subspace of some cube then it is Tychonoff (since each cube, its subspaces and all products of such spaces are Tychonoff spaces). On the other hand, if the space is Tychonoff, i.e. completely regular and T_1, then the set $C_r(X,\tau)$ of all real continuous functions on X distinguishes between points and close sets, and also between two distinct points (since one-point subsets are closed). The same is true for any function $f \in C_r(X,\tau)$ satisfying: $f(x) \in \langle 0,1\rangle$ for all $x \in X$. Let $\mathcal{F}$ be a set of such functions. Then the corresponding map F from Theorem 1.41 is the required embedding.

1.7 Compactness

This section is devoted to one of the most beautiful properties which a topological can have, to be as much close to finite sets as possible. The following is known for subsets in Euclidean space:

Theorem 1.42 *A subset X of $\mathbb{E}^n$ is closed and bounded if and only if every open cover of X (with the subspace topology) has a finite subcover.*

Bounded subsets in $\mathbb{E}^n$ are just those contained in some ball which has centre the origin and finite radius.

Example 1.33 Consider the closed unit interval $I = \langle 0,1\rangle$ with subspace topology induced from usual topology of the real line. Consider a system of open sets

$$\langle 0,1/10), \quad (1/3,1\rangle \quad \text{and} \quad (1/(n+2),1/n), \qquad n \in \mathbb{Z},\ n \geq 2,$$

which form an infinite open cover of $\langle 0,1\rangle$. We do not need all of these sets in order to cover I. We can obviously manage with only a finite number of them which form a finite subcover, e.g.

$$\langle 0,1/10), \quad (1/3,1\rangle \quad \text{and} \quad (1/(n+2),1/n), \qquad n \in \mathbb{Z},\ 2 \leq n \leq 9.$$

Example 1.34 Consider an infinite open cover of the plane by open balls of radius 1 and centres with integer coordinates. If we remove a single ball the remaining family of balls fails to cover the plane (because the center of the removed ball is not contained in their union).

1.7.1 Compact topological spaces

Motivated by the above examples we make the following definition.

Definition 1.22 A topological space X is *compact* if every open cover of X has a finite subcover.

In dimension one, we have the celebrated *Heine-Borel Theorem*:

Theorem 1.43 *Any closed interval of the real line is compact.*

The real line itself and the plane are not compact.

Compact subsets of a Euclidean space (with subspace topology) are precisely those subsets which are closed and bounded.

Note in advance that compactness is an important topological property of a space: a homeomorphic image of a compact space is compact, so compactness is a topological invariant. The following argument is used: compactness is preserved by any onto continuous map.

Theorem 1.44 *The continuous image of a compact space is compact.*

In many proofs, an equivalent characterization of compactness is useful. Recall that a space is said to have the *finite intersection property* (abbreviated *f.i.p.*) if every collection $\mathcal{F}$ of closed sets with property (1) has also the property (2):

(1) $F_1 \cap \cdots \cap F_k \neq \varnothing$ for each finite subcollection $\{F_1, \ldots, F_k\} \subset \mathcal{F}$.

(2) $\cap\{F\,;\, F \in \mathcal{F}\} \neq \varnothing$.

The following can be proved:

* *A topological space X is compact if and only if X has the f.i.p.*
* *Every closed subset of a compact space is compact in its subspace topology.*
* *Every infinite subset of a compact space has an accumulation point.*
* *If A is a compact subset of a Hausdorff space X, and if $x \in X$ which is not in A, then there exist disjoint neighbourhoods of x and A.*

As a corollary we get

Theorem 1.45 *A compact subset of a Hausdorff space is closed.*

Theorem 1.46 *Two disjoint compact subsets of a Hausdorff space have disjoint open neighbourhoods.*

As a consequence we have:

Theorem 1.47 *Every compact Hausdorff space is normal.*

A one-one onto continuous map need not have a continuous inverse, and so it need not be a homeomorphism. However, if the map goes from a compact space to a Hausdorff space then its inverse is continuous:

Theorem 1.48 *A one-one, onto, and continuous map from a compact space to a Hausdorff space is a homeomorphism.*

1.7.2 Compactification

As we have seen, compact spaces have many nice and advantageous properties which make them very similar to finite sets. It motivates us to wonder whether it is possible to construct something like a "compact extension" of a topological space, an idea analogous to completion of a metric space. We know from the Calculus that every metric space can be turned into a complete metric space by adding "ideal points" that correspond to classes of co-final non-convergent Cauchy sequences. Is it possible to develop something analogous here?

We would like to find a homeomorphism of the given space to (a subspace of) a compact topological space. We can ask under what conditions the given space can be "embedded", by means of a homeomorphism, into a compact space.

There exist several constructions with different results. The simplest one, but not always advisable, is the so-called Alexandroff one-point compactification which consists in adding a single point and new open sets containing it.

Theorem 1.49 *Let (X, τ) be a T_1-space (here τ is the family of open sets for X). Let p (sometimes ∞ is used) denote a point not in X. The* 1-point compactification *of X is the topological space X^* obtained as follows. As a point set, $X^* = X \cup \{p\}$. A base for the topology τ^* on X^* is*

$$\mathcal{B}_{X^*} = \tau \cup \mathcal{G},$$

where

$$\mathcal{G} = \{X^* \setminus F\colon\ F \text{ is compact and closed in } X\}.$$

Explicitly, open sets of the topology are

$$\tau^* = \{G \subset X^*\colon\ G \in \tau \text{ or } X^* \setminus G \text{ is compact and closed in } (X, \tau)\}.$$

Then (X^*, τ^*) is a compact topological space and (X, τ) is its subspace.

Even for a compact space X, we can create a one-point compactification X^*, but in this case $\{p\}$ is an open neighbourhood of p, X is compact in (X^*, τ^*), and p is an isolated point of X^*. Hence the topological space X need not be dense in its one-point compactification. Another disadvantage of this construction is an unpleasant fact that not every continous function on X has a continuous extension to X^*.

It motivates us to find a better definition.

Definition 1.23 A pair (Y, c) is called a *compactification* if Y is a compact topological space and $c\colon X \to Y$ is a homeomorphic embedding of X in Y such that $\overline{c(X)} = Y$, i.e. the image is dense in Y.

If the space Y is Hausdorff we speak about a *Hausdorff compactification*.

Note that the Alexandroff one-point compactification need not be a compactification in the sense of this definition.

Every space which is embeddable in a compact space, i.e. there exists a homeomorphism $f: X \to M$ onto a subspace $M = f(X)$ of Y, has a compactification, namely, the pair $(\overline{f(X)}, \iota f)$ where ι is embedding of M to $\overline{M}$.

On the class of compactifications of a topological space X, an ordering relation can be introduced. If X is T_2 and we consider Hausdorff compactifications only the ordering admits nice additional properties. It can be checked that if a T_2-space X has at least one Hausdorff compactification then X is (up to a homeomorphism) a dense subset of a compact T_2-space, therefore X is Tychonoff. That is, the only candidates to spaces admitting Hausdorff compactification are Tychonoff spaces. Moreover, if there exists a maximal Hausdorff compactification of a T_2-space X then it is unique up to a homeomorphism.

Theorem 1.50 *A space X has a compactification if and only if X is a Tychonoff space.*

There exist a construction of the so-called Čech-Stone compactification βX of a Tychonoff space X, [53, p. 221, 225], and it can be verified that βX is the largest element in the family of all compactifications of X.

1.7.3 Local compactness

Definition 1.24 A topological space X is *locally compact* if, for each $x \in X$, there exists an open neighbourhood U_x of x such that $\overline{U}_x$ is compact (i.e. each of its points has a compact neighbourhood) [53, p. 196].

Each compact space is also locally compact.

The space $\mathbb{E}^n$ with usual topology is not compact but is locally compact.

Theorem 1.51 *The following holds:*

* *Every locally compact topological space is a Tychonoff space.*
* *A topological space is locally compact if and only if it is homeomorphic to an open subspace of a compact space.*
* *A finite product of locally compact spaces is locally compact (note that for infinite product the statement is no more true).*
* *The sum is locally compact if and only if all the summands are locally compact.*

1.7.4 Partition of unity

We now show a tool often used e.g. in the theory of vector bundles and their sections, roughly speaking, to take "local solutions" of a problem for construction of a global "section" by means of "weights".

Recall that a system of subsets $\{H_\alpha : \alpha \in \mathcal{J}\}$ in a topological space (X, τ) is called *locally finite* if for each point $x \in X$ there is a neighbourhood intersecting only a finite number of sets from the system.

If $f: X \to \mathbb{R}$ is a real function on a topological space (X, τ) its *support* is a closure of all points $x \in X$ such that $f(x) \neq 0$; we use the notation $\operatorname{supp} f$.

Definition 1.25 A *partition of unity* is a family $\{f_\alpha : \alpha \in \mathbb{N}\}$ of continuous maps of X to $\langle 0, 1\rangle$ which satisfy

1. The system $\{\operatorname{supp} f_\alpha : \alpha \in \mathbb{N}\}$ is locally finite, that is, every point $x \in X$ has a neighbourhood in which the f_α vanish for all but a finite number of α.
2. For every point $x \in X$, we have $\sum_{\alpha\in\mathbb{N}} f_\alpha(x) = 1$.

The partition of unity is said to be *subordinate* to a given open cover $\mathcal{U}$ of X if for every α the support of $\operatorname{supp} f_\alpha$ is entirely contained in one of the sets of the cover $\mathcal{U}$.

It can be verified that for any locally finite cover of a normal (Hausdorff) topological space there exists a partition of unity.

1.7.5 Paracompactness

Metrizable spaces, in particular the Euclidean space, have the nice property that every open cover admits a partition of unity subordinate to it. Let us characterize now the class of topological spaces with the same property:

Definition 1.26 A Hausdorff topological space X is called *paracompact* if every open cover of X has a locally finite open refinement, that is, if for every open cover $\mathcal{V}$ of the space X there exists an open cover $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of X such that the following holds:

1. $\mathcal{W}$ is locally finite, i.e. every $x \in X$ has a neighbourhood that intersects W_λ for only finitely many λ;
2. $\mathcal{W}$ is a refinement of $\mathcal{V}$, i.e. every W_λ is containd in a set of $\mathcal{V}$.

Note that in contrast to the definition of compactness, in the definition of paracompactness the term "refinement" cannot be replaced by "subcover" (a counterexample is given in [53, p. 373]).

The Euclidean space $\mathbb{E}^n$ is paracompact. Also every discrete space, Hausdorff space and Lindelöf T_2-space are paracompact.

Theorem 1.52 (Stone) *Every metrizable space is paracompact.*

As a consequence, all subspaces of metrizable spaces are paracompact.

Theorem 1.53 *Every paracompact space is normal.*

Notice that paracompactness is an invariant of closed maps (The Michael Theorem, [53, p. 385]).

Theorem 1.54 *A closed subspace of a paracompact space is paracompact.*

The following equivalent characterization of paracompact T_2-spaces is of great importance [75, p. 124] and [53, p. 375]:

Theorem 1.55 *A Hausdorff space is paracompact if and only if it has the property: every open cover admits a partition of unity subordinate to it.*

Theorem 1.56 *For any T_1-space X the folowing conditions are equivalent:*
The space X is paracompact.
Every open cover of X has a locally finite partition of unity subordinate to it.
Every open cover of X has a partition of unity subordinate to it.

1.8 Metrization of a topological space

We can check that a topology τ on a set X can be induced by a metric on X if and only if the topological space (X, τ) can be embedded into a metrizable space. Indeed, if a fixed embedding is known then as a distance of two points from X, we can take distance of their images with respect to a fixed metric in the image space.

Further, note that if (X_1, d_1) and (X_2, d_2) are metric spaces then the cartesian product $X_1 \times X_2$ endowed with the product metric defined by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2} \quad \text{for} \quad (x, y) \in X_1 \times X_2$$

is a *product metric* of metric spaces (X_1, d_1) and (X_2, d_2). Note that the metric topology of a product of metric spaces coincides with the product of metric topologies of X_1 and X_2. More generally, if $X_1, X_2, \ldots, X_i, \ldots$ are metric spaces, the topology on X_i being induced by a metric d_i, then

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

defines a metric on $\prod X_i$ which induces the product topology [14, p. 56].

Example 1.35 The "countable" cube $I^{\mathbb{N}}$ is metrizable ($\mathbb{N}$ are natural numbers). Besides the above metric d, we can take, for example, the metric $\varrho(x, y) = \sum\limits_{i=1}^{\infty} \frac{1}{2^i} \cdot |x_i - y_i|$. $I^{\mathbb{N}}$ is also separable, completely regular, therefore regular, and second countable. Also any subspace of $I^{\mathbb{N}}$ is a separable metrizable topological space (with restricted metric).

So we can easily see: if T is a countable index set then the unit cube I^T is a metrizable topological space. It is interesting to find out which topological spaces can be metrizable this way.

1.8.1 Metrization Theorems

The answer comes in the following form:

Theorem 1.57 (Urysohn Metrization Theorem) *Let (X, τ) be a T_1-space. Then the following conditions are equivalent:*
(1) (X, τ) is metrizable separable topological space.
(2) (X, τ) is regular and second countable.
(3) (X, τ) is homeomorphic to a topological subspace of a cube I^T where T is a countable set.

Theorem 1.58 (On Metrization; R.H. Bing, 1951) *The following conditions are equivalent in any topological space* (X, τ)*:*

(1) (X, τ) *is metrizable.*

(2) (X, τ) *is* T_1*-space and has a* σ*-locally finite base.*

(3) (X, τ) *is regular* T_1*-space and has a* σ*-discrete base.*

To give explanation for the above terminology recall that a system $\mathcal{A}$ of subsets of a topological space X is *locally finite* if every point x of X has a neighbourhood U that intersects at most finite number of sets from $\mathcal{A}$. A system $\mathcal{B}$ is *σ-locally finite* if it is a union of a countable set of such locally finite systems. A system $\mathcal{A}$ of subsets is *discrete* if every point x of X has a neighbourhood U that intersects at most one set from $\mathcal{A}$. A system $\mathcal{B}$ is *σ-discrete* if it is a union of a countable set of such discrete systems.

1.8.2 Some properties of metric and metrizable spaces

As we have seen, any metrizable (metric) space is Hausdorff and first countable. A metrizable space is second countable if and only if it is separable.

Two metrics are *equivalent* if they define the same open sets (the same topology). Obviously, two metrics on the same set X are equivalent if and only if for any point $x \in X$ as a center, for any ball in first of the metrics there is a ball in the second metric contained in it, and vice versa.

Recall that a sequence $\{x_n\}$ of points of a metric space (X, d) has a *limit point* $x \in X$ if the sequence of real numbers $\{d(x_n, x)\}$ has zero as a limit, that is

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x_n, x) = 0.$$

If d and d' are equivalent metrics on X then:

$$\lim_{n\to\infty} x_n = x \text{ holds in } (X, d) \text{ if and only if } \lim_{n\to\infty} x_n = x \text{ holds in } (X, d').$$

A map of metric spaces $f\colon (X, d) \to (Y, \varrho)$ is continuous (is continuous in $x \in X$) if and only if $f\colon (X, d') \to (Y, \varrho)$ is continuous (is continuous in $x \in X$).

1.8.3 Complete metric spaces

A metric space (X, d) is *complete* if each Cauchy sequence of points from (X, d) has a limit point in X.

Theorem 1.59 (Cantor Theorem) *A metric space is complete if and only if for any sequence* $\{F_n\}_{n\in\mathbb{N}}$ *of closed sets such that* $F_{n+1} \subset F_n$ *for each* $n \in \mathbb{N}$ *and* $\operatorname{diam}(F_n) \to 0$ *the intersection* $\bigcap_{n\in\mathbb{N}} F_n$ *is non-empty.*

By means of the following result, S. Banach succeeded to prove that the set of continous real functions defined on the closed interval $\langle a, b\rangle$ which have a derivative in at least one point from the interval is nowhere dense (in the topology of uniform convergence). Roughly, there are "much more" continuous functions which have not derivative in any point.

Theorem 1.60 *If* $\{G_n\}_{n\in\mathbb{N}}$ *is a sequence of dense open subsets in a complete metric space then* $\bigcap_{n\in\mathbb{N}} G_n$ *is dense in* X*.*

Theorem 1.61 *Every compact metric space is complete.*

On the other hand, there are complete metric spaces which are not compact.

Example 1.36 The real line $\mathbb{R}$ together with the usual metric is complete (cf. *Cauchy-Bolzano Theorem*) because real sequences that are not bounded has no limit point in $\mathbb{R}$.

Example 1.37 Any open interval (a, b) in $\mathbb{R}$ with subspace metric and subspace topology is not complete (e.g. $\lim_{n\to\infty}(a + \frac{b-a}{n}) = a$ is not a point from (a, b)).

Accounting the fact that (a, b) and $\mathbb{R}$ are homeomorphic, we conclude that completeness is not a topological invariant. On the other hand, notice that completeness is an invariant under isometries.

A complete subspace of a metric space is closed.

A subspace A of a complete metric space is a complete if and only if A is closed.

A product of two (finite number of) complete metric spaces is complete.

Example 1.38 $\mathbb{E}^n$ is a complete metric space.

Every metric space is isometric to a subsace of a complete metric space.

Every metrizable space is embeddable in a completely metrizable space.

1.9 Topological algebraic structures

In geometry we often come across objects that carry an algebraic structure (of a groupoid, group, ring, vector space etc.), at the same time a topology and both structures fit together nicely, are "compatible" in the sense that all algebraic operations involved are continuous with respect to the given topology and suitable product topologies.

1.9.1 Topological groups

Let us start with a typical example, the circle.

Example 1.39 The set of complex numbers of unit modulus can be thought of as the unit circle $\mathbb{S}^1$ in the Cartesian plane $\mathbb{E}^2$, and the topology of $\mathbb{S}^1$ is the subspace topology inherited from the plane. Multiplication of complex numbers defines the group structure. It can be checked that the group multiplication

$$\mu\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1, \qquad (e^{i\varphi}, e^{i\psi}) \mapsto e^{i(\varphi+\psi)}$$

as well as the inverse map in the group $\iota\colon \mathbb{S}^1 \to \mathbb{S}^1$, $e^{i\varphi} \mapsto e^{-i\varphi}$, is a continuous map.

Definition 1.27 A *topological group* G is both a Hausdorff topological space and a group such that the multiplication $\mu\colon G \times G \to G$, $(g, h) \mapsto g \cdot h$ as well as the inverse map $\iota\colon G \to G$, $g \mapsto g^{-1}$ are continuos.

These two conditions can be merged into one, the axiom for topological groups: The map $G \times G \to G$, $(x, y) \mapsto x \cdot y^{-1}$ is continuos. Then the map $x \mapsto x^{-1}$, $G \to G$ can be thought of as a composition of two continuous maps

$y \mapsto (e, y)$, $G \to G \times G$, and $(x, y) \mapsto x \cdot y^{-1}$, $G \times G \to G$, therefore it is continuous, even a homeomorphism. Also $(x, y) \mapsto x \cdot y$ is continuous, as a composition of two continuous maps $(x, y) \to (x, y^{-1})$ and $(x, y) \mapsto x \cdot y^{-1}$.

If the group is written additively the axiom takes the form: the subtraction $(g, h) \mapsto g - h$ is continuous.

Example 1.40 The real line with addition of reals as the group operation and $r \mapsto -r$ as the inverse map is a topological group.

Any abstract group with the discrete topology is a topological group.

A topological group has the same topological structure locally near each point. To check it we use the following. The *left translation* by the element x in the topological group G is a map $L_x : G \to G$ defined by $L_x(g) = xg$. Left translations are homeomorphisms, $L_{x^{-1}}$ is the inverse of L_x, [14, p. 75]. Similarly, right translations $R_x : G \to G$, $R_x(g) = gx$ are homeomorphisms. Now if x and y are two points of the topological group G then the left translation $L_{yx^{-1}}$ is a homeomorphism which sends x to y. The structure of a topological group is completely determined by a neighbourhood base at the identity element of the group.

Theorem 1.62 *In a connected topological group, any neighbourhood of the identity element e is a set of generators for the whole group.*

A subset H of a topological group G is a *topological subgroup* of G if it is algebraically a subgroup and in addition has a subspace topology. H itself is then a topological group because $H \times H \to H$, $(g, h) \mapsto g \cdot h^{-1}$ for $g, h \in H$, is continuos.

Theorem 1.63 *If G is a topological group with the identity element e denote by K the connected component of e. Then K is a closed normal subgroup of G.*

Most topological groups important in geometry arise as matrix groups. Denote by $\mathbb{M}$ the set of all $n \times n$ matrices with real entries together with matrix multiplication $\mu : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$. Now we topologize the groupoid $(\mathbb{M}, \mu)$ as follows. We identify each element (a_{ij}) of the set $\mathbb{M}$ with a point $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn})$ of the Euclidean space $\mathbb{E}^{n^2}$, the identification gives us a topology (of the product space) on $\mathbb{M}$.

Denote by $\pi_{ij} : \mathbb{M} \to \mathbb{E}^1$ the projection of a matrix (a_{ij}) to its ij-th entry a_{ij}. The matrix multiplication is continuous because all composite functions $\pi_{ij}\,\mu$ are (the ij-th entry of the product matrix is a polynomial in the entries of both matrices), [14, p. 76]. We make use of this considerations in examples.

Example 1.41 The general linear group $GL(n, \mathbb{R})$, in short $GL(n)$, consists of invertible matrices with real entries, matrix multiplication defines the group operation. Because $GL(n, \mathbb{R}) \subset \mathbb{M}$, we give $GL(n)$ the subspace topology from $\mathbb{M}$. Matrix multiplication $GL(n) \times GL(n) \to GL(n)$ as well as the inverse function are continuous by similar arguments as above, [14, p. 76]. $GL(n)$ is a topological group with the identity element $E = (\delta^i_j)$. $GL(n)$ is not compact, it is an open subset of $\mathbb{M}$. $GL(n)$ is not connected, it has exactly two components. Matrices with positive determinant form a component of the identity matrix E, and matrices with negative determinant form the second component, together they define partition into two disjoint open subsets.

Example 1.42 The orthogonal group $O(n)$ consists of orthogonal $n \times n$ matrices with real entries, characterized by $A.A^T = E$, $\sum a_{ij}a_{kj} = \delta^i_j$, or equivalently $A^T = A^{-1}$. $O(n)$ is a compact topological subgroup of $GL(n)$ [14, p. 75].

Example 1.43 The special orthogonal group $SO(n)$ is a compact topological subgroup of $O(n)$ consisting of orthogonal matrices which have determinant equal 1, i.e. $SO(n)$ is a component of the identity matrix E.

1.9.2 Topological vector spaces

Let $\mathbb{K}$ be equal $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.28 A $\mathbb{K}$-vector space V with a topological structure is a *topological vector space* if its linear and topological structure are compatible in the following sense:

(1) The subtraction $V \times V \to V$, $(u, v) \mapsto u - v$ is continuous.
(2) Multiplication by scalars $\mathbb{K} \times V \to V$, $(k, v) \mapsto kv$ is continuous.

The first axiom says that the additive group $(V, +)$ is a topological group.

Examples of topological spaces occur within the range of applications of topology, particularly in functional analysis, and they have played an important role in the formation of the notion of topological space.

First describe the situation for finite-dimensional vector spaces. The space $\mathbb{K}^n$, with usual (product) topology, is a topological vector space, and every isomorphism $\mathbb{K}^n \to \mathbb{K}^n$ is a homeomorphism. Therefore every n-dimensional vector space V has exactly one "usual" topology for which some, and consequently any, isomorphism onto $\mathbb{K}^n$ is a homeomorphism. This topology makes V a topological vector space, and satisfies the Hausdorff Axiom:

Theorem 1.64 *The usual topology on a finite-dimensional vector space is the only one that makes it into a Hausdorff topological space.*

Infinite-dimensional case is more interesting, Hilbert spaces (with inner product which makes the corresponding metric complete) can be even classified, Banach spaces (normed vector spaces which are complete) and Fréchet spaces (endowed with a family of seminorms) are useful [75, p.26-28], but we need here only finite-dimensional spaces.

Also topological rings are studied, we do not need the concept here.

1.10 Fundamental group

We would like to present construction of topological invariants called homotopy groups. Homotopy groups are used in algebraic topology to classify topological spaces. Roughly, homotopy groups record information about the basic shape, or holes, of a topological space. The first and simplest is the fundamental group (also first homotopy group) which records information about loops in a space. To any path-connected topological space we assign an abstract group, and to any continuous map of topological spaces we assign a homomorphism of the corresponding groups; the construction has a functorial character. Elements of the group are equivalence classes of closed curves around a fixed point. Instead

of homeomorphisms, we use a weaker equivalence relation between maps called homotopy.

The fundamental group is an invariant of the topological space in the following sense: homeomorphic spaces have isomorphic fundamental groups (fundamental group is an invariant of the so-called homotopy type of the space). It anables us to distinguish between two path-connected topological spaces: we calculate and compare their fundamental groups. If the groups are not isomorphic the spaces cannot be homeomorphic. If the groups are isomorphic we must look for other invariants in which the spaces might differ (e.g. higher homotopy groups or homology groups).

1.10.1 Homotopic maps

The object of this part is to explain exactly the concept of a continuous deformation.

Definition 1.29 Let $f,\ g\colon X \to Y$ be continuous maps of topological spaces. We say that f is *homotopic* to g if there exists a continuos map $F\colon X \times I \to Y$ such that for all points $x \in X$

$$F(x,0) = f(x), \qquad F(x,1) = g(x).$$

The map F is called a *homotopy from f to g* and we write $f \simeq_F g$.

Note that we can take the parameter $t \in I$ as time. We start in time $t = 0$ and have the map f. In time $t = 1$ we have the map g. As time runs from 0 to 1 the map f is continuously deformed to g.

If a map $F\colon X \times I \to Y$ is given we write $F_t(x) = F(x,t)$ for all $x \in X$ and for all $t \in I$.

To give a homotopy F means equivalently to give a parametrized system of maps $F_t(x) = F(x,t)$ with a parameter space I in the following sense.

Definition 1.30 Let $X,\ Y,\ C$ be topological spaces and let $\{f_c\colon\ X \to Y,\ c \in C\}$ be a system of maps such that the map $f\colon X \times C \to Y$ given by $f(x,c) = f_c(x)$ is continuous. Then the system of maps $\{f_c\}$ is called a *parametrized system* of maps with the *parameter space* C.

On the other hand if a continuous map $f\colon X \times C \to Y$ is given the above formula defines a parametrized system $\{f_c\}$.

Example 1.44 A sequence of continuous maps $f_n\colon X \to Y$ is a parametrized system with a parameter space $(\mathbb{N}, \delta)$ where δ is a discrete topology on the set of natural numbers.

Example 1.45 If $f,\ g\colon X \to M$ are continuous maps of a topological space to a convex subset $M \subset \mathbb{R}^n$ of a euclidean space we can assume a straight-line (linear) homotopy $F(x,t) = (1-t)f(x) + tg(x)$ which for each point x sends $f(x)$ to $g(x)$ along a line segment connecting them; due to convexity the line segment is contained in M.

Homotopies can be composed. Let f, g, $h\colon X \to Y$, let $f \simeq_F g$ and $g \simeq_G h$. Define H by

$$H(x,t) = \begin{cases} F(x,2t) & \text{for } 0 \le t \le 1/2, \\ G(x,2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

The map H is a *composition* or *product* of homotopies F and G, we write $H = F * G$. Continuity of H follows by the Glueing Lemma. Further, $H(x,0) = f(x)$, $H(x,1) = h(x)$. Therefore $f \simeq_H h$. It is obvious that $f \simeq f$, it is sufficient to take $F(x,t) = f(x)$ corresponding to a constant parametrized system $\{f_t = f, t \in I\}$. If $f \simeq_F g$ then $g \simeq_G f$ where $G(x,t) = F(x,1-t)$, $t \in I$. G is continuous because $t \mapsto (1-t)$ is continuous. Another speaking, if a homotopy from f to g is given by the system $\{F_t\colon\ t \in I\}$, then the homotopy from g to f is given by the system $\{F_{1-t}\colon\ t \in I\}$. It proves the following:

Theorem 1.65 *"To be homotopic" is an equivalence relation on the set of continuous maps of topological spaces.*

An equivalence class with respect to this relation is called a *homotopy class* If X and Y are topological spaces denote $[X;Y]$ the set of all homotopy classes (i.e. partition classes corresponding to the equivalence relation $\simeq$) of continuous maps from X to Y.

On the space $Y^X = \{f\colon X \to Y; f \text{ is continuous}\}$ of all continuous maps of X to Y a compact – open topology is defined, denoted c. – o. One of the bases for compact – open topology is the system of sets of the form

$$\bigcap_{i=1}^{k} M(K_i, O_i), \quad K_i \subset X \text{ is compact}, \quad O_i \subset Y \text{ is open},$$

where $M(K,O) = \{f \in Y^X\colon\ f(K) \subset O\}$ is the set of all continuous maps which send the subset $K \subset X$ onto $O \subset Y$.

If f, $g \in Y^X$ and if H is a homotopy from f to g we define a path connecting a point f with a point g as $c(s) = H_s$, $c\colon I \to Y^X$. On the other hand, if we are given a path $c\colon I \to Y^X$, $c(0) = f$, $c(1) = g$, then $H(x,s) = c(s)(x)$ defines a homotopy from f to g.

Theorem 1.66 *Homotopy classes of maps $X \to Y$ are just all path connected components of the topological space* $(Y^X, \text{c. – o.})$.

The equivalence classes, the homotopy classes, are denoted

$$[f] = \{g\colon\ f \simeq g\}, \qquad f, g\colon X \to Y.$$

Sometimes, additional conditions are asked for a homotopy. If, in addition, f and g agree on some subset $A \subset X$ and we wish to deform f to g keeping the values of f on A we ask for a homotopy F with the additional

$$F(a,t) = f(a) \quad \text{for all } a \in A, \text{ for all } t \in I.$$

When such a homotopy exists we say that f is *homotopic to g relative to A* and we write $f \simeq_F g \operatorname{rel} A$.

The relation $\simeq \operatorname{rel} A$ is again an equivalence relation. Homotopies as well as homotopies relative to a fixed set behave "nicely" with respect to composition.

Theorem 1.67 *If f, $\overline{f}\colon X \to Y$ and g, $\overline{g}\colon Y \to Z$, $f \simeq \overline{f}$, $g \simeq \overline{g}$, then $g \circ f \simeq \overline{g} \circ \overline{f}$.*

A similar theorem can be proved for homotopy relative to a fixed set.

Definition 1.31 We say that a map $f\colon X \to Y$ is *homotopic zero* if it is homotopic to a constant map $h\colon X \to \{y_0\}$ for some $y_0 \in Y$.

Example 1.46 Any map $f\colon X \to \mathbb{S}^n$ of a topological space to the standard unit n-sphere, which is not surjective, is always homotopic zero.

1.10.2 Loops

Let I denote the closed unit interval. Let X be a topological space, choose a based point $x \in X$.

Definition 1.32 A *loop* p based at x is a continuous path $p\colon I \to X$ from x to x.

That is, a loop p is a path satisfying $x = y$, $p(0) = p(1) = x$.

Example 1.47 We consider homotopy classes of loops. If p, $\hat{p}\colon I \to X$ are two loops based at the same point x and we wish to deform p continuously to $\hat{p}$ we have to find a homotopy from p to $\hat{p}$ relative to $\{0, 1\}$.

If $e_x\colon I \to \{x\}$ is a constant path from x to x we say that e_x is a *constant loop* based at x.

If p and q are two loops based at the same point of X, we define the *product* $p * q$ to be the loop given by

$$p * q(t) := \begin{cases} p(2t), & 0 \le t \le 1/2, \\ q(2t-1), & 1/2 \le t \le 1 \end{cases}$$

(we multiply from left to the rigt) and the *inverse loop* $p^{-1}(t) = p(1-t)$, $t \in I$.

Unfortunately, this multiplication does not give a group structure on the set of all loops based at a fixed point. We easily check that it is not associative. To resolve the situation we pass to equivalence classes (homotopy classes) relative to $\{0, 1\}$.

1.10.3 Homotopy of paths and loops

Two paths from x to y in X are called *homotopy equivalent* if they are homotopic relative two-element set $\{0, 1\} \subset I$.

Similarly two loops p and q based at x are *homotopy equivalent* if they are homotopic relative two-element set $\{0, 1\} \subset I$. Instead of $p \simeq q \operatorname{rel} \{0, 1\}$ we write $p \simeq_x q$ (p and q are homotopic relative x). Explicitely, it means that there exists a map $H\colon I \times I \to X$ such that

$$\begin{aligned} H(t, 0) &= p(t), \quad H(t, 1) = q(t), && t \in I, \\ H(0, s) &= H(1, s) = x, && s \in I. \end{aligned}$$

The homotopy of paths as well as the homotopy of loops relative x are equivalence relations. We shall refer to the equivalence classes as *homotopy classes*,

denote $\langle p\rangle = \{q\colon\ p \simeq q\}$ for paths from x to y, and denote the homotopy class of a loop p by $\langle p\rangle$.

Two paths from x to y or two loops based at x satisfy: If $p \simeq_{H_s} p'$, then $p^{-1} \simeq_{H_{1-s}} p'^{-1}$. If $q \simeq q'$ such that the product $q * p$ is defined, then also the product $q' * p'$ is defined and $q * p \simeq q' * p'$.

Constructions of homotopies can be described by pictures. As usual, a diagram is much more effective than the formulae.
For homotopy classes of paths we define inverse class and product of classes

$$\langle p\rangle^{-1} = \langle p^{-1}\rangle, \qquad \langle p\rangle \circ \langle q\rangle = \langle p * q\rangle$$

whenever the starting point of q is the endpoint of p.

Let X be a topological space. If $\langle p\rangle$ is a path from x to y and e_x, e_y are constant loops based at x and y then

$$\langle e_x\rangle \circ \langle p\rangle = \langle p\rangle, \quad \langle p\rangle \circ \langle e_y\rangle = \langle p\rangle, \quad \langle p\rangle \circ \langle p^{-1}\rangle = \langle e_x\rangle, \quad \langle p^{-1}\rangle \circ \langle p\rangle = \langle e_y\rangle.$$

If $(p * q) * r$ is defined then $\quad (\langle p\rangle \circ \langle q\rangle) \circ \langle r\rangle = \langle p\rangle \circ (\langle q\rangle \circ \langle r\rangle)$.

1.10.4 Construction of the fundamental group

A topological space with a distinguished point will be called a *pointed space* and denoted $(X; x_0)$.

Let $(X; x_0)$ be a pointed topological space. On the set of all loops based at x_0 consider the equivalence relation homotopy relative $\{0, 1\}$, which we denote $\simeq_{x_0}$, and the partition into equivalence classes determined by this relation. Given a loop p based at x_0 the corresponding equivalence class is denoted $\langle p\rangle$. Multiplication of loops induces a multiplication of homotopy classes via

$$\langle p\rangle \circ \langle q\rangle = \langle p * q\rangle \quad \text{and} \quad \langle p\rangle \circ \langle p^{-1}\rangle = \langle e_{x_0}\rangle.$$

Multiplication $\circ$ of classes is a group operation:

Theorem 1.68 *Let $(X; x_0)$ be a pointed topological space. The set of homotopy classes relative $\{0, 1\}$ of loops based at x_0 is a group with multiplication $\circ$.*

Definition 1.33 The group of homotopy classes relative $\{0, 1\}$ of loops based at x_0 is called the *fundamental group* or the *(first) homotopy group* of the topological space and is denoted here by $\pi_1(X; x_0)$.

Without loss of generality, we can restrict ourselves to path-connected topological spaces because each loop based at x_0 has to be contained in a path-connected component of the point x_0. It can be proved that the fundamental group of a path-connected topological space is in a sense independent of the particular point x_0, so that to such a space, we can assign an abstract group determined, up to an isomorphism, uniquely.

Theorem 1.69 *Let X be a path-connected topological space. Let x, $y \in X$ be two points of X. Then there exists a group isomorphism of the group $\pi_1(X; x)$ with the group $\pi_1(X; y)$.*

If γ is a path connecting x and y (it have to exist, due to path-connectedness, and we fix one) and p is a loop based at the point x, then $(\gamma^{-1} * p) * \gamma$ is a loop based at y and we can define a map

$$\gamma_{\#}\colon \pi_1(X;x) \to \pi_1(X;y), \quad \gamma_{\#}(\langle p\rangle) = \langle(\gamma^{-1} * p) * \gamma\rangle = \langle\gamma^{-1}\rangle \circ \langle p\rangle \circ \langle\gamma\rangle.$$

We can check that the definition is correct and that $\gamma_{\#}$ is an isomorphism with inverse map ${\gamma_{\#}}^{-1} = {\gamma^{-1}}_{\#}$. If p and q are two paths from x to y then the corresponding isomorphisms $\gamma_{\#}$ and $\sigma_{\#}$ differ up to an inner automorphism of the group $\pi_1(X;x)$, $\sigma_{\#} = \gamma_{\#} \circ \mathrm{ad}(\gamma * \sigma^{-1})$; recall that an inner automorphism of a group $(G, \circ)$ given by an element $a \in G$ is a map

$$\mathrm{ad}(a)\colon G \to G, \qquad \mathrm{ad}(a)(g) = a \circ g \circ a^{-1}, \quad g \in G.$$

So to any path-connected topological space we can assign an abstract group, called *first homotopy group* which is, up to an isomorphism, given as the fundamental group of an arbitrary fixed point of the topological space. Note that if the group is not commutative we have no natural way how to identify fundamental groups in two different points, the isomorphism is not canonical, it depends on a homotopy class of the particular path connecting both points. In special case when the group $\pi_1(X;x_0)$ is abelian, the isomorphism is canonical.

As continuous maps $f\colon X \to Y$ between two pointed spaces with distinguished points $x_0 \in X$ and $y_0 \in Y$ we take maps satisfying also $f(x_0) = y_0$, and a homotopy of such maps will be homotopy relative $\{x_0\}$.

Theorem 1.70 *Let $f\colon (X;x_0) \to (Y;y_0)$ be a continuous map of path-connected topological spaces with distinguished points, $f(x_0) = y_0$. Then the map*

$$\pi_1 f\colon \pi_1(X;x_0) \to \pi_1(Y;f(x_0)), \quad \pi_1 f\,\langle p\rangle = \langle fp\rangle, \quad p \in \pi_1(X;x_0)$$

is a group homomorphism. Here $(fp)(t) = f(p(t))$.

Theorem 1.71 *Let $(X;x_0)$ and $(Y;y_0)$ be path-connected topological spaces with distinguished points. Two homotopy equivalent continuous maps, $f \simeq_{x_0} g$, of $(X;x_0)$ to $(Y;y_0)$ determine the same homomorphism of fundamental groups, $\pi_1 f = \pi_1 g$.*

For simple, the map π_1 is often denoted π_*.

Theorem 1.72 *Let $f\colon (X;x_0) \to (Y;y_0)$ and $g\colon (Y;y_0) \to (Z;z_0)$ be continuous maps of pointed spaces, $f(x_0) = y_0$, $g(y_0) = z_0$. Then*

$$\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$$

where $\pi_1(g)\colon \pi_1(Y;y_0) \to \pi_1(Z;z_0)$ and $\pi_1(f)\colon \pi_1(X;x_0) \to \pi_1(Y;y_0)$.

The identity map $\mathrm{id}_X\colon (X;x_0) \to (X;x_0)$ induces the identity homomorphism of groups, $\pi_1(\mathrm{id}_X) = \mathrm{id}_{\pi_1(X;x_0)}$. If $h\colon (X;x_0) \to (Y;y_0)$ is a homeomorphism then Theorem 1.72 with $f = h$, $g = h^{-1}$ gives $\pi_1(h^{-1}) \circ \pi_1 h = \pi_1(\mathrm{id}_X)$. Similarly for $f = h^{-1}$, $g = h$ we get $\pi_1 h^{-1} \circ \pi_1(h^{-1}) = \pi_1(\mathrm{id}_Y)$. Therefore the following holds:

Theorem 1.73 *Homeomorphic path-connected topological spaces have isomorphic fundamental groups.*

The fundamental group is an algebraic invariant of a topological space. If the fundamental groups of spaces are not isomorphic then the spaces are not homeomorphic.

Definition 1.34 A path-connected space with the trivial fundamental group is called *simply connected*

Example 1.48 The fundamental group of a one-element space is trivial.

Example 1.49 The fundamental group of any convex set in $\mathbb{R}^n$ is trivial. Each loop can be deformed to a constant loop based at the same point by means of a linear homotopy $H(t,s) = st\colon \mathbb{R}^n \times I \to \mathbb{R}^n$. Notice that the so-called contractible spaces have the same property.

Example 1.50 The fundamental group of the unit circle (1-sphere) $\mathbb{S}^1$ is an infinite cyclic group isomorphic to the additive group of integers. Similarly the fundamental group of the set of nonzero complex numbers,

$$\pi_1(\mathbb{S}^1, 1) \simeq (\mathbb{Z}, +) \simeq \pi_1(\mathbb{C} - \{0\}, 1).$$

For the exact proof, [14, 26, 112, 115, 136].

Example 1.51 The n-sphere $\mathbb{S}^n$ for $n \geq 2$ has the trivial fundamental group, and is simply connected.

Notice that there are topological spaces that have isomorphic fundamental groups although they are not homeomorphic.

Example 1.52 The fundamental group of the projective space $\mathbb{P}^n$ is equal $\mathbb{Z}_2$ provided $n \geq 2$.

The following theorem is useful if we already know fundamental groups of topological spaces and wish to calculate the fundamental group of the product space.

Theorem 1.74 *If X with a distinguished point x_0 and Y with a distinguished point y_0 are two path-connected topological spaces then the fundamental group $\pi_1(X \times Y; (x_0, y_0))$ is isomorphic to the direct sum $\pi_1(X; x_0) \oplus \pi_1(Y; y_0)$ of fundamental groups of factors.*

Example 1.53 If m, $n \geq 2$ then the product $\mathbb{S}^m \times \mathbb{S}^n$ has a trivial fundamental group.

Example 1.54 The fundamental group of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is, up to an isomorphism, $(\mathbb{Z}, +) \oplus (\mathbb{Z}, +)$.

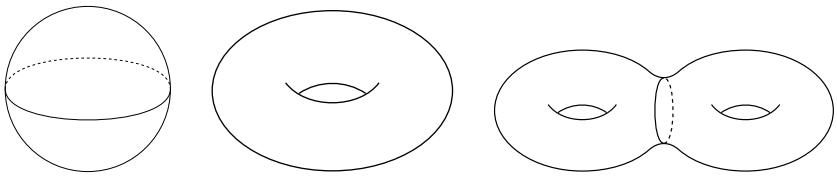
1.11 Topological manifolds

A topological space that locally resembles Euclidean space is a useful generalization called a manifold. We say that a topological space is a locally Euclidean space of dimension n if for each point there exists a homeomorphism mapping some open neighbourhood of the point onto an open subset in $\mathbb{E}^n$. First let us pay attention to special case $n = 2$.

1.11.1 Surfaces

Reducing mathematical balast as much as possible we can say that a *surface*, or a 2-manifold, is a topological space in which each point has a neighbourhood homeomorphic to the plane $\mathbb{E}^2$ (or equivalently, homeomorphic to the open unit disc in $\mathbb{E}^2$), and for which any two distinct points possess disjoint neighbourhoods (for technical reasons, to avoid pathological cases; i.e. the space is T_2). Some authors require also existence of a countable base. Notice that a surface *need not* be embedded into any Euclidean space, we are free of having to work inside some Euclidean space (although we are able to manage, up to a homeomorphism, so that to have a vizualization).

Well-known examples of surfaces that "live" in $\mathbb{E}^3$, such as the two-sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \colon \|x\| = 1\}$, the standard torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the surface of a cube or the surface of an infinite cylinder $\mathbb{S}^1 \times \mathbb{E}^1$ fit nicely this definition when they are given the subspace topology from Euclidean space. What properties can help us to distinguish between two surface? We can use already mentioned topological invariants such as compactness. The important role plays the number of components of boundary, also the number of holes (intuitively speaking), and the possibility to orient the surface.



The figures vizualizes the *sphere*, the *torus* and the *double torus* (*pretzel*).

We now turn briefly to the idea of orientation. For surfaces embedded into the Euclidean 3-space, the idea of existence of a continuous vector field of unit normal vectors (orthogonal to tangent planes in respective points) is often used. In case of a surface that need not be embedded we have to find another way. Note that a circle is oriented if it is given a sense of rotation. There are just two senses called "anticlockwise" or positive, and "clockwise" or negative. Here we explain rather roughly how to make use of oriented circles. Imagine that we are able to translate a (very small) circle in the surface so that the center moves along a curve (we are rather intuitive, not too precise now: what is a translation along a curve on a surface? and what is a circle? we work up to a homeomorphism ...).

We call a surface *orientable* if the operation of translating a small, oriented

circle round a simple (without self-intersections) closed curve always preserves the sence of the circle.

If there is a closed curve on a surface round which the effect of translation on the circle is to reverse the sense the surface is *non-orientable*.

The two-sphere is an example of an orientable, compact surface without any hole.

A better, more precise, but longer way how to explain orientation is to develop first the theory of triangulable spaces, [14], then we are able to give exact definitions.

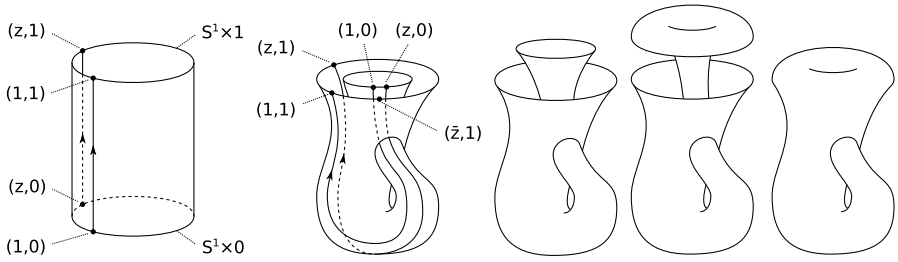
Some important examples of 2-manifolds can be generated as quotient spaces. The following construction shows one possible way how to present some of the well-known surfaces.

Let X be a topological space and $\varphi\colon X \to X$ be a homeomorphism. Consider an equivalence relation on $X \times \langle 0,1\rangle$ given as $(x,0) \sim (\varphi(x),1)$, and every other point (x,t) for $t \in (0,1)$ is equivalent only to itself. Denote by $X \times \langle 0,1\rangle/\varphi$ the quotient space of $X \times \langle 0,1\rangle$ by this relation, together with the quotient topology.

Example 1.55 (*Torus*) If $X = \mathbb{S}^1$ and $\varphi\colon\ \mathbb{S}^1 \to \mathbb{S}^1$ is the identity map then $X\colon\ \langle 0,1\rangle/\varphi$ is, up to a homeomorphism, the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, an orientable compact surface with one hole.

Example 1.56 (*Klein Bottle*) If $X = \mathbb{S}^1 = \{z \in \mathbb{C}\colon\ \|z\| = 1\}$ and $\varphi\colon\ \mathbb{S}^1 \to \mathbb{S}^1$ is the complex conjugation, $\varphi(z) = \overline{z}$, then $X \times \langle 0,1\rangle/\varphi$ is homeomorphic to the Klein bottle, a non-orientable compact surface with no holes. There is no surface in $\mathbb{E}^3$ homeomorphic to the Klein bottle, therefore this surface cannot be vizualized correctly and faithfully in $\mathbb{E}^3$; a projection of this surface to $\mathbb{E}^3$ has always self-intersection [75, pp. 46, 47]. On the other hand, a parametrization in higher dimension (in $\mathbb{E}^5$, in $\mathbb{E}^4$) can be given. Namely, the map $f\colon\ \langle 0,2\pi\rangle \times \langle 0,2\pi\rangle \to \mathbb{E}^5,\ f(x,y) = (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle to $\mathbb{E}^5$. Klein bottle can be embedded even to $\mathbb{E}^4$, e.g. if we use the map $g\colon\ \langle 0,2\pi\rangle \times \langle 0,\pi\rangle \to \mathbb{E}^4$,

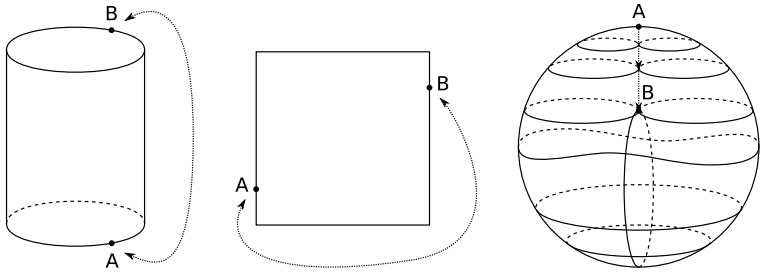
$$g(x,y) = ((2+\cos x)\cos 2y, (2+\sin x)\sin 2y, \sin x \cos y, \sin x \sin y).$$



The figure shows how the Klein bottle is constructed step by step.

Example 1.57 (*Projective Plane*) To create a surface homeomorphic with a projective plane we either start with a unit square and identify two and two

opposite sides in the opposite directions, or factorize the two-sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \colon \|x\| = 1\}$ by reflexion $\varphi \colon \mathbb{S}^2 \to \mathbb{S}^2$ in the origin, $\varphi(x) = -x$. Even this surface can be embedded in $\mathbb{E}^4$ but not in $\mathbb{E}^3$, it cannot be oriented, is compact and has no holes. The map $h \colon \mathbb{S}^2 \to \mathbb{E}^4$ of the unit sphere (embedded in $\mathbb{E}^3$) to $\mathbb{E}^4$ given by $h(x, y) = (x^2 - y^2, xy, xz, yz)$ induces an embedding of the real projective plane to $\mathbb{E}^4$.



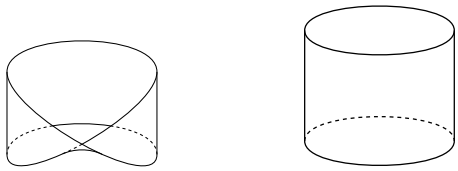
The figure shows projections of the *Klein bottle* and the *projective plane*.

Example 1.58 (*Surface of Infinite Cylinder*) Take an equivalence reation $\sim$ on $\langle -1, 1\rangle \times \mathbb{R}$ which identifies $(-1, y)$ with $(1, y)$ for $y \in \mathbb{R}$ (other points form one-element equivalence classes). Then $\langle -1, 1\rangle \times \mathbb{R}/\sim$ is homeomorphic to the cylinder surface $\mathbb{S}^1 \times \mathbb{E}^1$, or to a one-sheeted hyperboloid (given e.g. by the equation $x^2 + y^2 - z^2 - 1 = 0$). The surface is orientable but is not compact.

If we wish to allow a surface to have an edge or boundary, as in the case of the so-called Möbius strip or the cylinder, then we have to modify the definition: we allow in addition points which have neighbourhoods homeomorphic to the upper half-plane $\mathbb{E}^2_+$ (consisting of points of the plane with y-coordinate greater or equal zero).

Example 1.59 (*Möbius strip (band)*) If $X = \langle -1, 1\rangle$ and $\varphi(x) = -x$ then $X \times \langle 0, 1\rangle/\varphi$ is, up to a homeomorphism, a Möbius strip. The surface has a single component of boundary and cannot be oriented. Indeed, if we translate a small oriented circle once round the "central circle" of the Möbius strip the sense of the circle is reversed. Two Möbius strips glued together along the boundaries become a Klein Bottle, [75, p. 49].

Example 1.60 (*Cylinder of finite height*) If $X = \langle -1, 1\rangle$ and $\varphi(x) = x$ for $x \in X$ then the factor space $X \times \langle 0, 1\rangle/\varphi$ is, up to a homeomorphism, the cylinder $\mathbb{S}^1 \times \langle 0, 1\rangle$.

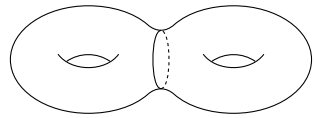


The figure shows the *Möbius strip* and the *cylinder*.

To construct an orientable compact surface with two holes we make use of another type of factorization. Combining the sum and the quotient we can glue topological spaces together. If X and Y are topological spaces, X_0 is a subspace of X and $\varphi\colon\ X_0 \to Y$ is a continuous map we can generate an equivalence relation on the sum $X + Y$ by setting $\varphi(x) \sim x$ for all $x \in X_0$ (we identify the points x of X_0 with their images $\varphi(x)$ in Y). Then we take the quotient space $X + Y/\sim$ and say that it is obtained by means of the attaching map φ.

Note that the equivalence classes are either one-element, arising from points that are neither in X_0 nor in $\varphi(X_0)$, or of the form $\varphi^{-1}(y) + y \subset X + Y$ where $y \in \varphi(X_0)$. Particularly, if X_1 and X_2 are surfaces we construct the so-called "connected sum" $X_1 \# X_2$: we remove a point p_i from the surface X_i, $i = 1, 2$, then glue the punctured surfaces together by means of an appropriate homeomorphism φ, and obtain $X_1 \# X_2 := (X_1 \setminus \{p_1\} + X_2 \setminus \{p_2\})/\sim$. Notice that up to a homeomorphism, the resulting space is uniquely determined.

Example 1.61 (*Double Torus*) Let us take two tori, puncture each of them, choose an appropriate homeomorphism and glue the punctured tori together as pictured below.



1.11.2 Hypersurface

A subset of H of a topological space X is a *hypersurface* if each point $x \in H$ has a neighbourhood in X and a homeomorphism h of U onto an open set in $\mathbb{E}^n$ such that $h(U \cap H) = h(U) \cap \Pi$ where Π is a hyperplane in $\mathbb{E}^n$.

1.11.3 Topological manifold

Definition 1.35 A *topological n-dimensional manifold*, in short *n-manifold* is a second countable Hausdorff topological space X such that each point x of X has a neighbourhood homeomorphic to an open subset of $\mathbb{E}^n$.

We may, if we wish, choose each homeomorphism φ_x so that $\varphi_x(x) = 0$ and so that the image of φ_x is a ball $B(0, \varepsilon)$ in $\mathbb{E}^n$. Certain translation in $\mathbb{E}^n$ is used, and properties of open sets are applied, [169, p. 109].

Example 1.62 $\mathbb{E}^n$ is an n-manifold: $\mathbb{E}^n$ is a metric space, therefore T_2-space; $\mathbb{E}^n$ is separable because the set of all points in $\mathbb{E}^n$ with rational coordinates is dense in $\mathbb{E}^n$.

Note that the dimension of the Euclidean space $\mathbb{E}^n$ is a topological invariant:

Theorem 1.75 *An open subset of the topological space $\mathbb{E}^n$ cannot be homeomorphic with an open subset of $\mathbb{E}^m$ provided $m \neq n$.*

Theorem 1.76 *Every topological n-manifold* (according to our Definition 1.35) *is locally compact.*

Theorem 1.77 *Every topological n-manifold is paracompact.*

Example 1.63 Surfaces are 2-manifolds. A hypersurface is a topological manifold of dimension $(n-1)$. The unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ is an $(n-1)$-manifold embeddable to $\mathbb{E}^n$.

1.11.4 Charts, atlas

A *chart* (or a *coordinate system*) in a topological manifold is a pair (U, φ) where $U \subset M$ is an open subset and $\varphi\colon U \to B$ is a homeomorphism onto an open subset $\varphi(U) = B$ of $\mathbb{R}^n$; $\varphi^i = x^i \circ \varphi$ are *coordinate functions* of the chart. For short, we often denote them by the same symbol x^i, and we identify $(x^i) \equiv (U, \varphi)$. The inverse map φ^{-1} is called a *(local) parametrization* of a manifold, [28].

Two charts in a topological manifold M

$$(x^i) = (U, \varphi),\ \varphi\colon U \to \mathbb{R}^n \quad \text{and} \quad (x'^i) = (U', \varphi'),\ \varphi'\colon U' \to \mathbb{R}^n,$$

respectively, induce a map $\varphi \circ \varphi'^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ with domain $\varphi'(U \cap U')$ and $\varphi' \circ \varphi^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ with domain $\varphi(U \cap U')$, respectively (the domains are open, maybe empty).

A family of charts in X form a C^0*-atlas* on X if the domains of charts cover X (X is the union of chart domains, i.e. each point of X is contained in the domain of at least one chart).

Example 1.64 On the 2-sphere $\mathbb{S}^2$, an atlas with two charts can be constructed by means of the strereographic projection. On the torus, at least three charts are necessary.

Up to now, we were interested in continuous maps, called also of the class C^0. In geometry we often need differentiability.

The charts are C^r*-related* if each of $\varphi \circ \varphi'^{-1}$ and $\varphi' \circ \varphi^{-1}$ is of the class C^r on its respective domain; $r = 0, 1, \ldots, n, \ldots, \infty, \omega$ (this condition is considered to hold trivially if U and U' do not meet).

The maps $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$, called *transfomations of the coordinate system* (x^i) and (x'^j), respectively, will be written in short as

$$x'^i = x'^i(x^1, \ldots, x^n) \quad \text{and} \quad x^i = x^i(x'^1, \ldots, x'^n). \tag{1.1}$$

1.11.5 The classification of compact connected 2-manifolds

To show the power of topology we would like at least to sketch how surfaces can be classified, although we cannot develop here all theory necessary for correct proofs.

It can be proved that any compact surface (i.e. "closed", without "boundary") is homeomorphic either to the sphere, or to the sphere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips. No two of the surfaces are homeomorphic, [14, 87].

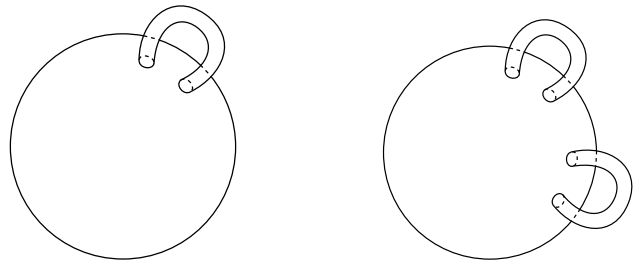
Recall how a handle can be attached to the sphere. Let us start with interpreting the torus and the double torus in a new way. Create a connected sum of the sphere and the torus: $\mathbb{S}^2\#\mathbb{T}^2$ is obviously homeomorphic to $\mathbb{T}^2$ again, but it is often vizualized as a sphere with a handle attached. By iteration, we construct a sphere with two handles, homeomorphic to the doble torus $\mathbb{T}^2\#\mathbb{T}^2$, in general a sphere with r handles, which is homeomorphic to the r-torus (surface of genus r). The sphere is just the 2-sphere with 0 handles.

If we remove a disc from the sphere and then, to the hole, we glue the Möbius strip along its whole boundary we say that we attached a "cross-cap" to the sphere. Denote by $\mathcal{C}(q,r)$ the homeomorphism class of surfaces containing a 2-sphere with q cross-caps and r handles attached; if $q \geq 1$ the surface cannot be embedded into $\mathbb{E}^3$. The sphere is just the 2-sphere with 0 handles, an element from $\mathcal{C}(0,0)$. The 2-sphere with a single cross-cap, an element of $\mathcal{C}(1,0)$, is a projective plane. The 2-sphere with two cross-caps, i.e. element of $\mathcal{C}(2,0)$, is just the Klein bottle. The number of cross-caps can be reduced:

Theorem 1.78 *Let X be a compact connected 2-manifold. Then X is in $\mathcal{C}(q,r)$ where $0 \leq q \leq 2$ and $r \geq 0$.*

Theorem 1.79 *Let X_1, $X_2 \in \mathcal{C}(q,r)$. Then X_1 and X_2 are homeomorphic if and only if $q_1 = q_2$ and $r_1 = r_2$* [128, p. 163].

Theorem 1.80 *Every compact connected surface in $\mathbb{E}^3$ is in $\mathcal{C}(0,r)$, i.e. it is the 2-sphere with r handles, $r \geq 0$.*



2 MANIFOLDS WITH AFFINE CONNECTION

2.1 Differentiable manifolds

Recall that a real-valued function $f: U \to \mathbb{R}$ defined on an open subset $U \subset \mathbb{R}^n$ is *differentiable of the class* C^r provided all partial derivatives of orders $\leq r$ exist and are continuous at every point of U, *smooth* (of the class C^∞) provided all partial derivatives of all orders exist, and *analytic* (of the class C^ω) if it has all derivatives of all orders and coincides with its Taylor series near each point.

A map φ from $U \subset \mathbb{R}^m$ to $\mathbb{R}^n$ is *differentiable of the class* C^r if each real function $x^i \circ \varphi$, $1 \leq i \leq n$, is differentiable of the class C^r (is smooth if $r = \infty$, is real analytic if $r = \omega$, respectively).

2.1.1 Differentiable structure (complete atlas)

Definition 2.1 A *C^r-atlas* $\mathcal{S}$ on M (of the class C^r where r is a non-negative integer, or ∞, or ω) is a collection of charts in M such that

- the domains cover M (M is the union of chart domains, i.e. each point of M is contained in the domain of some chart).
- any two charts are C^r-related.

We call a C^r-atlas *complete* if it contains each map that is C^r-related with all maps of the atlas (i.e. is maximal relative to the above two conditions). Each atlas on a manifold is contained in a unique complete atlas, [133]. A complete atlas is sometimes called a *differentiable structure* (of the class C^r).

Of course, any topological manifold is C^0 (has an atlas of the class C^0), but there are topological manifolds on which no C^1-structure can be defined[2)]. If a C^r-structure is given on M then M carries a C^s-structure for any s lower than r (it is sufficient to add all maps that are C^s-related with the given atlas, and take the complete C^s-atlas which contains them).

On the other hand, from a complete C^1-atlas, we can always select a complete C^ω-atlas (another speaking, from a C^1-differentiable manifold we can create an analytic one; the proof comes from Whitney).

A C^∞-atlas is called *smooth*, and defines a smooth structure. A C^ω-atlas is called *(real) analytic*, and determines a (real) analytic structure.

[2)] The proof was given by M. Kervaire, [531].

Definition 2.2 A *C^r-differentiable manifold* of dimension n is a topological manifold furnished with a complete C^r-atlas $\mathcal{S}$. It will be (usually) denoted by M_n or $(M_n, \mathcal{S})$ here.

If not said otherwise C^r-differentiable manifolds are called here *C^r-manifolds* or *manifolds* in short.

Note that if (U, φ) is a chart in M and $V \subset U$ an open subset then $(V, \varphi_{|V})$ is also a chart in M.

If M is given by an atlas $\mathcal{S}$, and U is an open subset in M (i.e. a topological subspace, with induced topology), then the set of all coordinate systems from $\mathcal{S}$ such that the domain is contained in U, form an atlas $\mathcal{S}'$ on U (since the domains cover U); $(U, \mathcal{S}')$ is a *submanifold* of $(M, \mathcal{S})$. For more details, e.g. [133].

Another simple construction, how to get new manifolds from old ones, is the product. Given e.g. the C^r-manifolds $(M_n, \mathcal{S})$ and $(N_k, \mathcal{T})$, then for any $(U, \varphi(x^i)) \in \mathcal{S}$ and $(V, \psi(y^j)) \in \mathcal{T}$, the products

$$\varphi \times \psi \colon\ U \times V \to \mathbb{R}^{n+k}, \quad (p, q) \mapsto (x^1(p), \dots, x^n(p), y^1(q), \dots, y^k(q))$$

are coordinate systems (charts) in the Hausdorff space $M_n \times N_k$, any two such product charts overlap in the class C^r, and $M_n \times N_k$ furnished with the set of all product charts is the *product manifold* of M_n and N_k; obviously, $\dim M_n \times N_k = n + k$. By iteration, we get products of any finite number of manifolds.

2.1.2 Smooth map, diffeomorphism

Let $(M_n, \mathcal{S})$ and $(N_k, \mathcal{T})$ be C^r-manifolds (given by complete atlases). A map $\Phi \colon M_n \to N_k$ is called *of the class C^r* (*C^r-mapping*) if for any pair of charts $(U, \varphi) \in \mathcal{S}$ and $(V, \psi) \in \mathcal{T}$, the map $\psi \circ \Phi \circ \varphi^{-1}$, called the *coordinate expression of Φ with respect to the pair (φ, ψ) of charts*, is of the class C^r. It can be checked that a mapping is differentiable iff its coordinate expressions with respect to some atlases on M_n and N_k are differentiable. In short, we write $\Phi \in C^r$.

In local coordinates $(x^i) = (U, \varphi)$ around a point $p \in M_n$ and $(y^I) = (V, \psi)$ around the image $\Phi(p) \in N_k$ the map $\Phi \colon M_n \to N_k$ reads

$$y^I = y^I(x^1, \dots, x^n), \quad I = 1, \dots, k. \tag{2.1}$$

A differentiable map $f \colon M_n \to \mathbb{R}$ is called a (differentiable real) *function*. The set[3)] of all smooth functions on M_n is usually denoted by $\mathcal{F}(M)$.

A bijective mapping Φ between n-dimensional manifolds M_n and N_n will be called here a *C^r-diffeomorphism* if Φ and $\Phi^{-1} \in C^r$. If such a C^r-diffeomorphism exists, the manifolds M_n and N_n are called *C^r-diffeomorphic*. Identity mappings, inverses of C^r-diffeomorphisms and compositions of C^r-diffeomorphisms are C^r-diffeomorphisms again.[4)]

The important role of diffeomorphisms lays in the fact that they preserve the differentiable structure. *Manifold theory* can be defined as the study of

[3)] Endowed with functional addition and multiplication, $\mathcal{F}(M)$ is a module.

[4)] In short, we write "diffeomorphism" instead of C^r-diffeomorphism in what follows; if not said otherwise, r is supposed to be high enough, or $r = \infty, \omega$, respectively.

those objects preserved by diffeomorphisms; diffeomorphic manifolds are the same from the view-point of manifold theory.

An *immersion* $\Phi\colon M_n \to N_k$ $(n \le k)$ is a C^r-mapping such that the differential $T\Phi_p$ is one-one (injective) for all $p \in M$.

The following conditions are equivalent:

1. $T\Phi_p$ is one-one.
2. The Jacobi matrix of $T\Phi_p$ has rank n relative to one (and hence to every) choice of coordinate systems, i.e. $\operatorname{rank}\|\partial y^I/\partial x^i\| = n$.
3. If (y^j), $j = 1, \dots, k$ are coordinates at $\Phi(p)$ in N_k then there are integers $1 \le j_1 \le \cdots \le j_n$ such that $(y^{j_1} \circ \Phi, \dots, y^{j_n} \circ \Phi)$ are coordinate functions around p in M_n.

An *embedding* $\Phi\colon M_n \to N_k$ of a manifold M into N is a one-one immersion such that the induced mapping $M \to \Phi(M)$ is a homeomorphism onto the topological subspace $\Phi(M)$ of N.

Theorem 2.1 (Whitney's Theorem on embedding) *Any C^1-manifold of dimension n is diffeomorphic with some analytic submanifold of the space $\mathbb{R}^{2n+1}$.*

As a consequence, we get: On any C^1-manifold there is an analytic atlas which is a subset of the given C^1-atlas.

2.1.3 Tangent vector, tangent space, tangent bundle

The crucial step in generalizing calculus from $\mathbb{E}_n$ (or $\mathbb{R}^n$, respectively) to an arbitrary manifold is the following elegant definition, which, let us say, axiomatizes the directional derivative in Euclidean vector spaces, known from calculus.

Definition 2.3 Let $p \in M_n$ be a point. A *tangent vector* at p is a real-valued mapping $\mathbf{v}\colon \mathcal{F}(M_n) \to \mathbb{R}$ that is

- $\mathbb{R}$-linear, i.e. $\mathbf{v}(af + bg) = a\mathbf{v}(f) + b\mathbf{v}(g)$, and
- "Leibnizian", i.e. $\mathbf{v}(fg) = \mathbf{v}(f)g(p) + f(p)\mathbf{v}(g)$

for all $a, b \in \mathbb{R}$ and any $f, g \in \mathcal{F}(M_n)$.

On the set T_pM of all tangent vectors to M_n at $p \in M$, a linear structure[5)] is introduced which makes T_pM a real n-dimensional vector space. Let us set

$$TM = \{(p, \mathbf{v})\,;\ p \in M,\ \mathbf{v} \in T_pM\}.$$

We make TM a differentiable manifold of dimension $2n$ in the following natural way, [28, 133]. If a (smooth) atlas on M is given we construct the so-called "adapted" atlas[6)] on TM: if (x^i) are local coordinates on $U \subset M$ we take $(x^i, \frac{\partial}{\partial x^i})$ as local coordinates on $TU \subset TM$. If a point p has coordinates

[5)] The linear structure is given by the usual functional addition $(\mathbf{v} + \mathbf{w})(f) = \mathbf{v}(f) + \mathbf{w}(f)$ and scalar multiplication by reals, $(a\mathbf{v})(f) = a\mathbf{w}(f)$, $f \in \mathcal{F}(M)$, $a \in \mathbb{R}$.

[6)] Adapted to the structure of vector bundle TM.

$(x^1, \dots, x^n)$ and $\mathbf{v} = v^i \partial_i|_p$ is a tangent vector at p then $(p, \mathbf{v}) \in TM$ has coordinates (x^i, v^i). Two such charts overlap smoothly.

It can be checked that TM is Hausdorff and second countable. TM together with this differentiable structure is the so-called *tangent bundle* of M, the vector bundle $p_M \colon TM \to M$. Its (linear) fibre $p_M^{-1}(p)$ over the base point $p \in M$ is isomorphic to T_pM, the isomorphism being $(p, \mathbf{v}) \to \mathbf{v}$.

To define partial differentiation on a manifold, we "move" the function[7)] to Euclidean space using a coordinate system, and then take the usual partial derivatives: if $f \in \mathcal{F}(M)$, $\varphi = (x^1, \dots, x^n)$ is a coordinate system in M around p and $u^1, \dots, u^n$ are natural coordinate functions of $\mathbb{R}^n$, let

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)), \qquad 1 \le i \le n.$$

It can be verified that a tangent vector is a local object, a straightforward computation shows that the function

$$\partial_i|_p \equiv \frac{\partial}{\partial x^i}\Big|_p \colon \mathcal{F}(M) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x^i}(p),$$

is a tangent vector to M at p, $\langle \partial_1|_p, \dots, \partial_n|_p \rangle$ form a basis of the tangent space T_pM, and

$$Z = \sum_{i=1}^{n} \mathbf{v}(x^i) \frac{\partial}{\partial x^i}\Big|_p \quad \text{for all} \quad Z \in T_pM.$$

An important idea of differential calculus is to approximate smooth objects by linear objects. Near each point $p \in M$, the manifold M is approximated by the tangent (linear) space T_pM.

2.1.4 Differential map

Now we approximate a C^r-differentiable (or smooth) mapping $\Phi \colon M \to N$ around any point $p \in M$ by a linear transformation of tangent spaces. We use the fact that allows us to "push forward" tangent vectors: if $\mathbf{v} \in T_pM$ then the correspondence $f \mapsto \mathbf{v}(f \circ \Phi)$ defines a map $T\Phi_p(\mathbf{v}) \colon \mathcal{F}(N) \to \mathbb{R}$ which is a tangent vector at $\Phi(p)$ as can be verified.

Definition 2.4 Let $\Phi \colon M_n \to N_k$ be a C^{r+1}-mapping. For each $p \in M$,

$$T\Phi_p \colon T_pM \to T_{\Phi(p)}N, \qquad \mathbf{v}(f \circ \Phi) \mapsto T\Phi_p(\mathbf{v})(f)$$

(denoted sometimes by Φ_* or $d\Phi$) is a *differential map* or *tangent map*[8)] of Φ at the point p (obviously, $T\Phi_p \in C^r$).

[7)] We pull it back, form a "pull-back" of f.

[8)] A differential map gives rise to a tangent map $T\Phi \colon TM \to TN$ of tangent bundles. The construction has a functorial character.

In local coordinates $\varphi = (x^i)$ about p and $\psi = (y^i)$ about $\Phi(p)$[9)]

$$T\Phi_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{i=1}^{k} \frac{\partial(y^i \circ \Phi)}{\partial x^j} \frac{\partial}{\partial y^i}\Big|_{\Phi(p)} = \frac{\partial(y^i \circ \Phi)}{\partial x^j} \frac{\partial}{\partial y^i}\Big|_{\Phi(p)}, \qquad 1 \le j \le n.$$

Hence the matrix of the (linear) mapping $T\Phi_p$ with respect to these coordinate bases, called the *Jacobian matrix* of Φ at p relative φ and ψ, is

$$\left(\frac{\partial y^i \circ \Phi}{\partial x^j}\Big|_p\right)_{j=1,\dots,n}^{i=1,\dots,k}. \tag{2.2}$$

If $\Phi\colon M_n \to N_k$ and $\Psi\colon N_k \to P_m$ are differentiable (smooth) maps then $T(\Psi \circ \Phi)_p = T\Psi_{\Phi(p)} \circ T\Phi_p$; the subscript $p \in M$ is often omitted. The classical chain-rule formula for the Jacobian matrix of a composite map follows.

For a smooth map $\Phi\colon M_n \to N_k$, the differential $T\Phi_p$ at $p \in M$ is a linear isomorphism if and only if there is a neighbourhood $U \ni p$ in M such that the restriction $\Phi_{|U}$ is a diffeomorphism of U onto $\Phi(U) \subset N$ (the proof can be given in local coordinates about p; the statement is a version of the classical Inverse Function Theorem).

If the differential $T\Phi_p$ of a smooth map Φ is a linear isomorphism near each point $p \in M$ then Φ is called a *local diffeomorphism*. If a local diffeomorphism is one-to-one and onto then it is a diffeomorphism.

2.1.5 Curve, tangent vector of a curve

Let M be a differentiable manifold and $I \subset \mathbb{R}$ an arbitrary interval.

A *(parametrized) curve of class C^r in M* is a map $\ell\colon I \to M$ which can be extended into a C^r-map of some open interval $J \supset I$ into M. Particularly, we speak about a *parametrized curve about $p \in M$* if $I = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ and $\ell(0) = p$.

If we are going to integrate a closed interval $I = [a, b]$ is preferable; sometimes, $\ell\colon [a, b] \to M$ is called a *curve segment*.

As a submanifold of $\mathbb{R}$, I has a coordinate system given by the identity map u of I; for any $t \in \mathbb{R}$, we get the unit vector $\frac{d}{du}(t) \in T_t(\mathbb{R})$ at t (in the "positive" direction determined by u).

In what follows, we usually suppose an open interval I.

To introduce a tangent (velocity) vector of a curve $\ell\colon I \to M$ in a natural way, the differential map $T\ell\colon T\mathbb{R} \to TM$ may be used.

Definition 2.5 Let $\ell\colon I \to M$ be a curve, I open. The *velocity vector*, or *tangent vector* of ℓ at $t \in I$ is given by

$$\lambda(t) = \ell'(t) = T\ell\left(\frac{d}{du}\Big|_t\right) \in T_{\ell(t)}(M).$$

[9)] Einstein's summation convention is used.

The tangent vector λ of a curve ℓ has the following properties:

- Directional derivative: $\lambda(f) = \ell'(t)(f) = \dfrac{df(\ell)}{du}(t)$.
- Coordinate expression: If (x^i) are local coordinates around $\ell(t) \in M$ then

$$\lambda(t) = \ell'(t) = \frac{d(x^i \circ \ell)}{du}(t)\, \partial_i\big|_{\ell(t)},$$

i.e. $\lambda(t) = \lambda^i(t)\partial_i\big|_{\ell(t)}$, where $\lambda^i(t)$ are *components* of the tangent vector $\lambda(t)$.

- Reparametrization (consequence of the chain rule): If $\ell\colon I \to M$ is a curve and $\sigma\colon J \to I$ is a smooth function on an (open) interval J then the curve $\overline{\ell} = \ell \circ \sigma\colon J \to M$, called a *reparametrization*, has the velocity vector

$$\overline{\ell}'(\overline{t}) = (d\sigma/du)(\overline{t})\ell'(\sigma(\overline{t})) \quad \text{for all } \overline{t} \in J.$$

- Effect of a map (consequence of the chain rule): A map $\Phi\colon M \to N$ carries a curve $\ell\colon I \to M$ to the curve $\Phi \circ \ell\colon I \to N$, and the tangent map of Φ preserves velocities,

$$T\Phi(\ell'(t)) = (\Phi \circ \ell)'(t).$$

Note that reparametrizations of the form $\overline{t} = t+\text{const}$, are not of importance, hence if we wish, we may suppose that $0 \in I$.

A curve is *regular* provided $\ell'(t) \neq 0$ for all $t \in I$.

2.1.6 Vector field, flow

Definition 2.6 A *vector field* on a manifold M is a map X that assignes to each point $p \in M$ a vector $X_p = X(p) \in T_pM$ from the tangent vector space[10].

A vector field X is called *differentiable of the class* C^r ($r \geq 1$[11]), in short *C^r-field*, if real-valued functions Xf on M, $p \mapsto (Xf)(p) = X_p(f)$, $p \in M$, are of the class C^r for all $f \in \mathcal{F}(M)$. If we introduce pointwise addition of vector fields and multiplication of vector field by smooth functions the set $\mathcal{X}(M)$ of all smooth vector fields on M is a module[12] over the ring $\mathcal{F}(M)$.

In local coordinates (x^i) on $U \subset M$, the so-called *coordinate vector fields*

$$\partial_i = \frac{\partial}{\partial x^i}, \qquad i = 1, \dots, n,$$

on U are smooth (since $\partial_i f = \frac{\partial f}{\partial x^i}$), independent, hence form the so-called coordinate basis at any point, and any smooth vector field X on M_n can be written on U as $X = X^i\partial_i$, where $X^i = Xx^i$ are smooth functions, the so-called *components* of X.

A (differentiable) vector field on a manifold represents a differential equation. We say that a curve $\ell\colon I \to M$ is an *integral curve* of a vector field $X \in \mathcal{X}(M)$

[10] Another speaking, $X\colon M \to TM$ is a vector field if $p_M \circ X = id_M$ where id_M is the identity map on M and $p_M\colon TM \to M$ is the canonical projection, see later.

[11] In the case $r = 0$, the vector field is called *continuous*.

[12] Formally, the definition of a module over a commutative ring with unit is the same as the definition of a vector space over a field.

provided $\ell' = X$ (that is, at each point, the curve has velocity prescribed by the field, $\ell'(t) = X_{\ell(t)}$ for all $t \in I$). In terms of local coordinates, when X^i denote components of X the above condition yields[13] a system of first-order ordinary differential equations $dx^i/dt = X^i(x^j)$, more precisely,

$$\frac{d(x^i \circ \ell)}{dt} = X^i(x^1 \circ \ell, \dots, x^n \circ \ell), \qquad 1 \le i \le n. \tag{2.3}$$

For a C^r-differentiable vector field and a fixed point p, there is always an integral curve ℓ_p defined near $0 \in \mathbb{R}$ and starting at $\ell(0) = p$[14]; moreover, we can always consider a maximal[15] integral curve starting at p, which is unique (but it is not obliged to be defined on the entire real line $\mathbb{R}$), [133]. A vector field is *complete* if any of its maximal integral curves is defined on $\mathbb{R}$.

If we collect (maximal) integral curves of $X \in \mathcal{X}(M)$ starting at various points we get a map $\varphi(p,t) = \ell_p(t)$; for a complete vector field, $\varphi\colon M \times \mathbb{R} \to M$ is the so-called flow. If p is fixed then $t \mapsto \varphi(p,t)$ is an integral curve $\varphi_p(t)$ while if t is fixed then $p \mapsto \varphi(p,t)$ gives a function $\varphi_t\colon M \to M$ that lets every point "flow" for exactly the time t; sometimes, $\{\varphi_t\colon t \in \mathbb{R}\}$ is refered to as the flow of the complete X.

If X is not complete then for each point p, there exists its open neighbourhood U and an interval $J = (-\varepsilon, \varepsilon) \ni 0$ such that a *local flow* $\varphi\colon U \times J \to M$ is defined; φ is smooth for sufficiently small U, J (which follows by the theory of differential equations).

Local flows have the following properties, stating that a local flow is a *local one-parameter group*: φ_0 is the identity map on U, $\varphi_s \circ \varphi_t = \varphi_{s+t}$ whenever $s, t, s+t$ are in J, and for $t \in J$, each (t-th) stage $\varphi_t\colon U \to \varphi_t(U)$ is a diffeomorphism.

Lemma 2.1 *If X is a C^r-differentiable vector field on M_n ($r \ge 0$) such that $X_p \neq 0$ at a point $p \in M$ then there is a coordinate system near p such that $X = \partial/\partial x^1$.*

Remark 2.1 The proof for $X \in C^1$ can be given e.g. by means of local flows [133], and for a continuous field X, i.e. $X \in C^0$, see [664].

In terms of maps and bundles, a vector field $X \in \mathcal{X}(M)$ is a differentiable section of the tangent bundle TM, i.e. a differentiable map $X\colon M \to TM$ is a vector field iff $p_M \circ X = id_M$ (where id_M is the identity map on M and $p_M\colon TM \to M$ is the canonical projection).

[13] Since the right-hand side is independent of the parameter t the field X can be interpreted as the velocity of a steady state flow of a fluid through the manifold.

[14] In fact, the existence of an integral curve starting at a point is guaranteed for a continuous vector field due to theorems on differential equations; if at least C^1-differentiability is satisfied then uniqueness can be proved.

[15] Every maximal integral curve is either constant, or one-to-one, or simply periodic, that is, there is a smallest $c > 0$ such that $\ell(t+c) = \ell(t)$, called a period, and ℓ is one-to-one on some interval $[a, a+c)$.

2.1.7 Distributions

A *k-dimensional distribution* on a set $A \subseteq M$ is a map $D\colon A \to TM$ that assigns to each point $x \in A$ a k-dimensional subspace $D_x \subseteq T_xM$ of the tangent space. We say that D is *smooth* (*differentiable*) on A if A is open, and for each point $x \in A$ there are k smooth (differentiable) independent vector fields $X_1, \ldots, X_k$ which span D_y for all y in some neighbourhood of x.

Under a *tangent distribution along a curve* we mean here a map that associates to any point $\ell(t)$ of a regular parametrized curve ℓ in M a one-dimensional subspace in the tangent space $T_{\ell(t)}M$ which is the span of the tangent vector of the curve at the point.

2.1.8 One-forms

At any point $p \in M_n$, we can construct the dual vector space T_p^*M of the tangent space T_pM, called a *cotangent space* of M at p, the elements of which are linear forms, i.e. linear maps of T_pM to $\mathbb{R}$, called *covectors* sometimes.

One-forms on M are objects dual to vector fields:

Definition 2.7 A one-form[16] ω on M is a map that assigns to each point p an element ω_p of the cotangent space T_p^*M[17]. If $X \in \mathcal{X}(M)$ then $\omega(X) = \omega_p(X_p)$; ω is smooth provided $\omega(X)$ is a smooth function for any $X \in \mathcal{X}(M)$.

Let $\mathcal{X}^*(M)$ denote the set of all 1-forms on M.

Definition 2.8 The *differential* of $f \in \mathcal{F}(M)$ is a one-form df such that $(df)(\mathbf{v}) = \mathbf{v}(f)$ for every tangent vector $\mathbf{v}$ to M.

Hence the differential converts functions to one-forms. In fact, the map $(df)_p\colon T_pM \to \mathbb{R}$ is linear for any $p \in M$; moreover, $(df)(X) = Xf$ is smooth if $X \in \mathcal{X}(M)$. In local coordinates (x^i) on U, $\omega = \omega(\partial_i)dx^i$, and $df = \frac{\partial f}{\partial x^i}dx^i$ on U.

Note that similarly as in the case of tangent spaces, we can collect cotangent spaces, and create the so-called cotangent bundle[18] with the underlying set

$$T^*M = \{(p, \omega); p \in M, \omega \in T_p^*M\},$$

and present a 1-form as a smooth section of the projection $T^*M \to M$.

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[16] In the classical terminology, they are called *covariant vectors*. The explanation follows.

[17] That is, ω assigns a real number to every tangent vector, and is linear on the tangent space at each point.

[18] That is, introduce the topology as well as the differentiable structure.

2.2 Tensor fields and geometric objects

2.2.1 Tensors on a vector space

Let $V_1, \ldots, V_s, W$ be real vector spaces. A map $F\colon V_1 \times \ldots \times V_s \to W$ is called ($\mathbb{R}$-)*multilinear* provided F is ($\mathbb{R}$-) linear in each "slot".

Definition 2.9 If V is a real vector space, V^* its dual, $r \geq 0$ and $s \geq 0$ are integers, $(r,s) \neq (0,0)$, then a multilinear function $F\colon (V^*)^r \times V^s \to \mathbb{R}$ is called a *tensor of type* (r,s) *over* V, or of contravariant degree r and covariant degree s.

Remark that more often, vector slots are written first, and covector slots follow.

Note that the established definition used in geometry for tensors is inconsistent with "covariant" and "contravariant" as used in the category theory.

The set of all tensors of type (r,s) on V is denoted $T^r_s(V)$. Tensors of type $(0,0)$ are just real constants, $T^0_0(V) = \mathbb{R}$. Tensors from $T^r_0(V)$ are called *contravariant tensors*. Particularly elements of $T^1_0(V) = V$ are sometimes called[19] *contravariant vectors*. Elements of $T^0_s(V)$ are *covariant tensors*; accordingly, elements of $T^0_1(V) = V^*$ are sometimes called *covariant vectors* or *linear forms*.

$T^r_s(V)$ endowed with functional addition (or subtraction)

$$\begin{aligned}(A \pm B)(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s)\\ = A(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s) \pm B(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s)\end{aligned}$$

and multiplication by reals

$$(rA)(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s) = rA(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s)$$

is a vector space.

As well known, if V is finite-dimensional we can fix a base $\langle e_i \rangle$, take the dual base $\langle f^i \rangle$ in V^*, and calculate the so-called *components* of a tensor F with respect to base linear forms and base vectors, $F^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = F(f^1, \ldots, f^r, e_1, \ldots, e_s)$. Due to multilinearity, every tensor can be expressed by means of its components. In components, linear operations read

$$(A \pm B)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = A^{i_1 \cdots i_r}_{j_1 \cdots j_s} \pm B^{i_1 \cdots i_r}_{j_1 \cdots j_s}, \qquad (rA)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = rA^{i_1 \cdots i_r}_{j_1 \cdots j_s}.$$

Superscripts are sometimes called *contravariant indices*, while subscripts are *covariant indices*.

A tensor F of a type (r,s) (with r,s sufficiently high) is called *symmetric* with respect to a pair of indices of the same kind (i.e. both covariant or both contravariant) if vectors (or 1-forms, respectively) on these positions can be interchanged unless the value of the tensor is changed, and *antisymmetric* if after interchanging, only the sign is changed. For illustration, if $F(\omega, \mathbf{v}_1, \mathbf{v}_2) = F(\omega, \mathbf{v}_2, \mathbf{v}_1)$ holds, or equivalently, in components, $F^i_{jk} = F^i_{kj}$, we say that F is symmetric in the two lower indices (or in j,k). If $F(\omega, \mathbf{v}_1, \mathbf{v}_2) = -F(\omega, \mathbf{v}_2, \mathbf{v}_1)$,

[19] Any $\mathbf{v} \in V$ behaves as a linear form on V^*, $\mathbf{v}\colon v^* \to \mathbb{R}$, if we put $\mathbf{v}(\omega) := \omega(\mathbf{v})$.

or equivalently $F^i_{jk} = -F^i_{kj}$, we say that F is antisymmetric in the two lower indices (or in j, k). Similarly in upper indices.

For any tensor of type (r, s) with $s \geq 2$ or with $r \geq 2$, we can define a tensor symmetric with respect to a distinguished pair of "lower" or "upper" indices, respectively, by means of an operation called *symmetrization* (without division) and denoted by $(\ ,\ )$. For a tensor of a type (r, s) with $s \geq 2$ or with $r \geq 2$, we define a tensor antisymmetric in a pair of indices of the same type by means of *antisymmetrization* (called also *alternation*) without division, denoted by $[\ ,\ ]$[20]. E.g. we introduce

$$A^{i_1\cdots i_r}_{(j_1 j_2)\cdots j_s} = A^{i_1\cdots i_r}_{j_1 j_2 j_3\cdots j_s} + A^{i_1\cdots i_r}_{j_2 j_1 j_3\cdots j_s},$$

$$A^{i_1\cdots i_r}_{[j_1 j_2] j_3\cdots j_s} = A^{i_1,\cdots i_r}_{j_1 j_2 j_3\cdots j_s} - A^{i_1\cdots i_r}_{j_2 j_1 j_3\cdots j_s}.$$

Sometimes, an operation of *cycling* (without division) is useful, introduced for three indices of the same type and denoted by $(\ ,\ ,\ )$[21]. E.g. we introduce

$$A^{i_1\cdots i_r}_{(j_1 j_2 j_3) j_4\cdots j_s} = A^{i_1\cdots i_r}_{j_1 j_2 j_3 j_4\cdots j_s} + A^{i_1\cdots i_r}_{j_2 j_3 j_1 j_4\cdots j_s} + A^{i_1\cdots i_r}_{j_3 j_1 j_2 j_4\cdots j_s}.$$

Besides linear operations that are defined for tensors of the same type, any two tensors can be multiplied. If $A \in T^r_s(V)$ and $B \in T^p_q$ then $A \otimes B \in T^{r+p}_{s+q}(V)$, a *tensor product* of A and B, is introduced by

$$\begin{aligned}(A \otimes B)&(\omega^1, \ldots, \omega^{r+p}, \mathbf{v}_1, \ldots, \mathbf{v}_{s+q}) \\ &= A(\omega^1, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_s)\, B(\omega^{r+1}, \ldots, \omega^{r+p}, \mathbf{v}_{s+1}, \ldots, \mathbf{v}_{s+q}).\end{aligned} \tag{2.4}$$

In components, $(A \otimes B)^{i_1\cdots i_r i_{r+1}\cdots i_{r+p}}_{j_1\ldots j_s j_{s+1}\cdots j_{s+q}} = A^{i_1\cdots i_r}_{j_1\ldots j_s} B^{i_{r+1}\ldots i_{r+p}}_{j_{s+1}\cdots j_{s+q}}$.

If $A \in T^r_s(V)$ and $r, s \geq 1$, a *contraction* $A \mapsto c^p_q A$ of A over p, q ($1 \leq p \leq r$, $1 \leq q \leq s$) associates to A a tensor $c^p_q A \in T^{r-1}_{s-1}(V)$ (in fact, a trace of a linear map)

$$\begin{aligned}&c^p_q A(\omega^1, \ldots, \omega^{r-1}, \mathbf{v}_1, \ldots, \mathbf{v}_{s-1}) \\ &= \sum_m A(\omega^1, \ldots, \omega^{p-1}, f^m, \omega^{p+1}, \ldots, \omega^r, \mathbf{v}_1, \ldots, \mathbf{v}_{q-1}, e_m, \ldots, \mathbf{v}_{q+1}, \ldots, \mathbf{v}_s).\end{aligned}$$

Remark 2.2 Formally the same definition and results can be given for a K-multilinear map of a direct product of modules over a ring K if we replace "vector space" by "module" and $\mathbb{R}$ by K, [133]. A dual module can be also well defined.

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[20] The operation of symmetrization (antisymmetrization, respectively) generalizes the way how we obtain a symmetric (antisymmetric) part of a bilinear form, only halving is omitted. It can be easily checked that the result is a tensor of type (r, s) again.

[21] Cycling sends a tensor of type (r, s), $s \geq 3$ (or $r \geq 3$, respectively) to a tensor of the same type.

2.2.2 Tensors on manifolds

For tensor fields, there are several equivalent definitions.

A *tensor field* of type (r,s) on a manifold can be introduced either as a tensor of type (r,s) over the $\mathcal{F}(M)$-module $\mathcal{X}(M)$, that is, as an $\mathcal{F}(M)$-linear function $F\colon\ \mathcal{X}(M)^s \times \mathcal{X}^*(M)^r \to \mathcal{F}(M)$, or as a map F which maps any point $p \in M$ into a tensor F_p of the prescribed type (r,s) on the tangent space T_pM at p, depending (differentiably, or smoothly) on p.

The set of all tensor fields of type (r,s) on M is denoted $T^r_s(M)$. Multiplication of tensors by functions and addition of tensors of the same type can be introduced in an obvious way (pointwise). Tensors of type $(0,0)$ are constant functions. Elements of $T^r_0(M)$ are *contravariant tensor fields*. Elements of $T^0_s(M)$ are *covariant tensor fields*. Particularly, one-forms are $(0,1)$, since $X \mapsto \omega(X)$ is the $\mathcal{F}(M)$-linear map from $\mathcal{X}(M) \to \mathcal{F}(M)$ for a 1-form ω, and every $(0,1)$ field arises in this way; we have the identification $T^0_1(M) = \mathcal{X}^*(M)$. A vector field is $(1,0)$ since for $X \in \mathcal{X}(M)$, $\omega \to \omega(X)$ is $\mathcal{F}(M)$-linear, and any $(1,0)$ tensor field arises in this way; $T^1_0(M) = \mathcal{X}(M)$.

If $A \in T^r_s(M)$ and $B \in T^p_q(M)$ then the tensor product $A \otimes B \in T^{r+p}_{s+q}(M)$ of tensor fields is given by

$$\begin{aligned}&(A\otimes B)(\omega^1,\dots,\omega^{r+p},X_1,\dots,X_{s+q})\\ &\quad = A(\omega^1,\dots,\omega^r,X_1,\dots,X_s)\,B(\omega^{r+1},\dots,\omega^{r+p},X_{s+1},\dots,X_{s+q}).\end{aligned} \tag{2.5}$$

In local coordinates (x^i) on $U \subset M$, the tensor field F of type (r,s) has components

$$F^{i_1\cdots i_r}_{j_1\cdots j_s}(x) = F(dx^{i_1},\dots,dx^{i_r},\partial_{j_1},\dots,\partial_{j_s}).$$

Consider another local chart (V,ψ), $\psi = (x'^1,\dots,x'^n)$, and the coordinate change

$$x'^i = x'^i(x^1,\dots,x^n), \quad \text{respectively} \quad x^i = x^i(x'^1,\dots,x'^n). \tag{2.6}$$

Accounting

$$dx^{j_m} = \frac{\partial x^{j_m}}{\partial x'^{i_\alpha}} dx'^{i_\alpha} \quad \text{and} \quad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^\beta}\frac{\partial x'^\beta}{\partial x^i}$$

we easily find that on overlappings $U \cap U'$ of neighbourhoods, the tensor components transform according to the well-known formula

$$F^{i_1\cdots i_r}_{j_1\cdots j_s}(x) = \frac{\partial x^{i_1}}{\partial x'^{\alpha_1}} \cdots \frac{\partial x^{i_r}}{\partial x'^{\alpha_r}} \frac{\partial x'^{\beta_1}}{\partial x^{j_1}} \cdots \frac{\partial x'^{\beta_s}}{\partial x^{j_s}} F'^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}(x'). \tag{2.7}$$

Any tensor is determined by its components. Locally, we can write

$$F = F^{i_1\cdots i_r}_{j_1\cdots j_s}(x)(dx^{j_1})_x \otimes \cdots \otimes (dx^{j_s})_x \otimes (\partial_{i_1})_x \otimes \cdots \otimes (\partial_{i_r})_x.$$

If we give functions $F^{i_1\cdots i_r}_{j_1\cdots j_s}(x)$ in coordinate neighbourhoods they determine a tensor provided the transformation laws (2.7) hold on overlappings.

A tensor F is *of the class* C^r on a manifold $M \in C^{r'}$ $(r' > r)$, in short $F \in C^r$, if $F^{i_1\cdots i_r}_{j_1\cdots j_s}(x) \in C^r$.

A *contraction* over h, k is an operation which sends a type (r, s) tensor field F $(r, s \geq 1)$ onto a type $(r-1, s-1)$ tensor field, in components,

$$F^{i_1 \cdots i_{r-1}}_{j_1 \cdots j_{s-1}}(x) = F^{i_1 \cdots i_h m\, i_{h+1} \cdots i_{r-1}}_{j_1 \cdots j_k m\, j_{k+1} \cdots j_{s-1}}(x).$$

The construction is based on the fact that there is a unique $\mathcal{F}(M)$-linear function $C\colon T^1_1(M) \to \mathcal{F}(M)$, called $(1,1)$ *contraction*, such that $C(X \otimes \omega) = \omega(X)$ for all $X \in \mathcal{X}(M)$ and $X \in \mathcal{X}^*(M)$.

Symmetrization, antisymmetrization (= alternation) and cycling is introduced for tensor fields in a obvious way.

2.2.3 Geometric objects on manifolds

According to the above, vector fields, one-forms, and tensor fields on manifolds are in local coordinates characterized uniquely by means of a system of functions. This was a motivation for introducing a more general concept, the so-called geometric object. Very often, mathematical models of many properties of objects or processes we meet in physics, mechanics and geometry, are described by (ordered) systems of functions, depending on (coordinates of) a point p moving in some (open subset of) a manifold M_n.

In the classical point of view, geometric properties are those that behave "nicely", are "preserved" under coordinate transformations, in the sense specified below. We find it useful to mention this here, since it is the background even of modern approaches.

Let $(M_n, \{(U_\alpha, \varphi_\alpha)\})$ be an n-dimensional manifold given by an atlas. Let $\varphi\colon U \to \mathbb{R}^n$ and $\varphi'\colon U' \to \mathbb{R}^n$ be two maps with corresponding local coordinates $x = (x^1, \ldots, x^n)$ and $x' = (x'^1, \ldots, x'^n)$, respectively, and let

$$\varphi' \circ \varphi^{-1}\colon\ x'^k = x'^k(x^1, \ldots, x^n) \quad \text{and} \quad \varphi \circ \varphi'^{-1}\colon\ x^k = x^k(x'^1, \ldots, x'^n) \tag{2.8}$$

be the corresponding coordinate transformation on $U \cap U'$ and its inverse.

In the classical approach and terminology, an "object" $\mathcal{A}$ living on a manifold is a *geometric object* if in any local map (U, φ), $\mathcal{A}$ is given by a family of N functions $\mathcal{A}^A(x)$, $A = 1, \ldots, N$,

$$(\mathcal{A}^1(x), \ldots, \mathcal{A}^N(x))$$

which transform[22)] according to the formulas[23)]

$$\mathcal{A}'^A(x') = \mathcal{F}^A(\mathcal{A}^1(x(x')), \ldots, \mathcal{A}^N(x(x')),\ x^1(x'), \ldots, x^n(x'), x') \tag{2.9}$$

(where the functions $\mathcal{F}^A$ depend on the arguments written above, i.e. on the components $\mathcal{A}^1(x)$ of the geometric object and on the transformation $x(x')$) provided the following natural conditions are satisfied, called the "equivalence laws":

[22)] $\mathcal{A}^A(x^1, \ldots, x^n)$ is called the A-th component of the geometric object in local coordinates $(x^1, \ldots, x^n)$; $\mathcal{A}'^A(x'^1, \ldots, x'^n)$ is the A-th component of the geometric object in local coordinates $(x'^1, \ldots, x'^n)$ (at the same point), respectively.

[23)] The equations (2.9) are called the *transformation law of a geometric object* under the given coordinate transformation.

1. *Reflexivity*: if $x'^h = x^h$ then $\mathcal{A}^A = \mathcal{A}'^A$.
2. *Symmetry*: if a coordinate transformation $x'^h = x'^h(x)$ transforms $\mathcal{A}^A$ into $\mathcal{A}'^A$ then the inverse transformation $x^h = x^h(x')$ maps $\mathcal{A}'^A$ into $\mathcal{A}^A$.
3. *Transitivity*: if $\mathcal{A}^A$ is mapped into $\mathcal{A}'^A$ under the coordinate transformation $x'^h = x'^h(x)$ and $\mathcal{A}'^A$ is mapped into $\mathcal{A}''^A$ under the coordinate transformation $x''^h = x''^h(x'(x))$ then $\mathcal{A}^A$ is mapped into $\mathcal{A}''^A$ under the transformation $x''^h = x''^h(x'(x))$.

That is, components of the geometric object do not change under the identity transformation (identity guarantees reflexivity), the inverse transformation guarantees symmetry, and the composed transformation guarantees transitivity of the arising relation.

Well-known examples are functions, tensor fields, and connections.

Remark 2.3 Particulary, in this approach, a *function*[24] (scalar function, sometimes called also an *invariant*) is a geometric object which is given, in any local map, by a (unique real) function $f(x^1, \dots, x^n)$ of coordinates, and under the coordinate transformation $x'^h = x'^h(x)$, the transformation law is $f'(x') = f(x(x'))$, that is,

$$f'(x'^1, x'^2, \dots, x'^n) = f(x^1(x'), x^2(x'), \dots, x^n(x')).$$

Remark 2.4 In the classical terminology of geometric objects, a contravariant vector[25] λ^h is a geometric object which is given, in any coordinate system (x^i) in M_n, by an ordered n-tuple of functions $\lambda^h(x)$, $h = 1, \dots, n$, that transform (under the coordinate transformation (2.8))

$$\lambda'^h(x') = \lambda^\alpha(x)\frac{\partial x'^h(x)}{\partial x^\alpha}. \tag{2.10}$$

By Lemma 2.1 and Remark 2.1, for any non-vanishing contravariant vector field λ^h there are coordinates such that $\lambda^1 = 1$, $\lambda^j = 0$ for $j = 2, \dots, n$. Moreover, if $\lambda^i(x) \in C^r$ $(r \geq 0)$ it exists a transformation $x' = x'(x) \in C^{r+1}$.

The differential dx^i, i.e. $(dx^1, \dots, dx^n)$, of the coordinates (x) can serve as an example, because,

$$dx'^h = dx^\alpha \frac{\partial x'^h}{\partial x^\alpha}$$

holds.

Let a curve ℓ be given in M_n by $x^h = x^h(t)$, $h = 1, \dots, n$, $t \in I$. Then its tangent vector at $\ell(t_0) \in M_n$ is given by

$$\lambda^h = \frac{dx^h(t)}{dt}\Big|_{t=t_0}.$$

Under the coordinate transformation (2.8), ℓ is given in new coordinates by $x'^h = x'^h(x(t)) = x'^h(t)$, and we can easily check that λ'^h are related to λ^h just by (2.10).

[24] A function is a (0,0) tensor field on a manifold.

[25] A contravariant vector is a $(1, 0)$ tensor field on a manifold, in fact a vector field.

Remark 2.5 A dual geometric object, a covariant vector[26)] is locally given by functions $\mu_i(x)$ which transform

$$\mu'_i(x') = \mu_\alpha(x)\frac{\partial x^\alpha(x')}{\partial x'^i}.$$

The differential df of a real-valued function f on M_n is a covariant vector:

$$df' = \frac{\partial f'(x'^i)}{\partial x'^i}\,\partial x'^i = \frac{\partial f(x(x'))}{\partial x^\alpha}\cdot\frac{\partial x^\alpha}{\partial x'^i}\,dx'^i\,.$$

Remark 2.6 Within the theory of geometric objects, a tensor field of type (r,s) can be introduced as a geometric object defined by a system of functions $F^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x)$ which transform according to (2.7).

Now it is evident that a type $(0,0)$ tensor is a function, $(1,0)$ tensor is a contravariant vector (= vector), and $(0,1)$ tensor is a covector (= 1-form).

Remark 2.7 A geometric object $\mathcal{A} \in C^r$ if its components $\mathcal{A}^A(x) \in C^r$ in all coordinate charts of the defining atlas of the manifold M_n.

2.3 Manifolds with affine connection

2.3.1 Affine connections, manifolds with affine connection

Recall that an *affine connection* (called also *linear connection* sometimes[27)]) on an n-dimensional manifold M is a map ∇ which maps any pair of vector fields to a vector field, $(X,Y) \mapsto \nabla_X Y$, such that

$$\begin{aligned}\nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z,\\ \nabla_X(fY) &= f\nabla_X Y + (Xf)Y,\\ \nabla_{fX+gY}Z &= f\nabla_X Z + g\nabla_Y Z\end{aligned} \tag{2.11}$$

for any vector fields X, Y, Z on M and any differentiable functions f, g on M.

Definition 2.10 We call $\mathbb{A}_n = (M, \nabla)$ a *manifold with affine connection*, or a *manifold with linear connection*.

In local coordinates with respect to a chart (U,φ), $\varphi = (x^1,\dots,x^n)$,

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^h_{ij}\,\frac{\partial}{\partial x^h}\ , \tag{2.12}$$

where the functions $\Gamma^h_{ij}(x)$ characterizing the affine connection ∇ are called *components of affine connection*[28)] ∇ relative to the chart under consideration.

[26)] A covariant vector is a $(0,1)$ tensor field on a manifold M_n, in fact a 1-form.

[27)] Note that in [90], the term affine connection is used in a different meaning.

[28)] or *Christoffel symbols*

The components of a connection satisfy the well-known transformation law on overlaps of neighbourhoods: if (U', φ'), $\varphi' = (x'^1, \dots, x'^n)$ is another local chart then under the coordinate change (2.6) the following holds on the intersection $U \cap U'$:

$$\Gamma'^{h}_{ij}(x') = \left(\Gamma^{\gamma}_{\alpha\beta}(x(x')) \frac{\partial x^{\alpha}}{\partial x'^{i}} \frac{\partial x^{\beta}}{\partial x'^{j}} + \frac{\partial^2 x^{\gamma}}{\partial x'^{i} \partial x'^{j}} \right) \frac{\partial x'^{h}}{\partial x^{\gamma}} . \tag{2.13}$$

Vice versa, ∇ can be fully determined by its components: the set of (differentiable, smooth, real analytic) functions $\Gamma^{k}_{ij}(x)$, $x \in U$, where U runs through the atlas of an n-dimensional manifold M, determines a connection on M provided they satisfy the former transformation law (2.13) on overlaps of neighbourhoods.

Remark 2.8 Note that an object of affine (= linear) connection is a geometric object, which is defined on a manifold M_n by a set of n^3 functions $\Gamma^{h}_{ij}(x)$ on U for each chart (U, φ) provided the transformation law (2.13) holds on overlaps of coordinate domains.

Remark 2.9 A manifold with affine connection $\mathbb{A}_n \in C^r$ if $\Gamma^{h}_{ij}(x) \in C^r$ in all charts $x = (U, \varphi)$ of $\mathbb{A}_n$.[29)]

The *torsion* of a connection ∇ on M is a type $(1,2)$ tensor field S given by

$$S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad X, Y \in \mathcal{X}(M),$$

where $[\ ,\ ]$ is the *Lie bracket*: $[X,Y]f = X(Yf) - Y(Xf)$ for $f \in \mathcal{F}(M)$.

Locally we can write $S = S^{h}_{ij}\, \partial_h \otimes dx^i \otimes dx^j$ where components of the torsion tensor are

$$S^{h}_{ij} = \Gamma^{h}_{ij} - \Gamma^{h}_{ji} \equiv \Gamma^{h}_{[ij]}.$$

The torsion S is a skew-symmetric tensor: $S(Y,X) = -S(X,Y)$. A connection ∇ is called *torsion-free*, or *symmetric* if $S \equiv 0$, that is, iff in local coordinates, $\Gamma^{h}_{ij} = \Gamma^{h}_{ji}$ holds.

2.3.2 Covariant differentiation

An affine connection ∇ induces a covariant derivative of tensor fields relative to a vector field: if T is a tensor of type (r,s) then its *covariant derivative* ∇T is a tensor of type $(r, s+1)$. In components,

$$T^{i_1 \cdots i_p}_{j_1 \cdots j_q, k} \equiv \nabla_k T^{i_1 \cdots i_p}_{j_1 \cdots j_q}(x)$$

$$= \partial_k T^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \sum_{s=1}^{p} \Gamma^{h_s}_{\alpha k} \cdot T^{i_1 \cdots i_{s-1} \alpha i_{s+1} \cdots i_p}_{j_1 j_2 \cdots j_q} - \sum_{s=1}^{q} \Gamma^{\alpha}_{j_s k} \cdot T^{i_1 \cdots i_p}_{j_1 \cdots j_{s-1} \alpha j_{s+1} \cdots i_q}.$$

Here and in what follows, the comma "," denotes the *covariant derivative on* ∇.

[29)] That is why according to the transformation formula (2.13), the manifold must satisfy $M_n \in C^{r+1}$.

Particularly, for a tensor field T of type $(0,s)$,

$$(\nabla_X T)(X_1,\dots,X_s) = X\,T(X_1,\dots,X_s) - \sum_j T(X_1,\dots,\nabla_X X_j,\dots,X_s);$$

especially if g is of type $(0,2)$ then

$$(\nabla_Z g)(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X,\nabla_Z Y). \tag{2.14}$$

For a tensor field T of type $(1,s)$,

$$(\nabla_X T)(X_1,\dots,X_s) = \nabla_X T(X_1,\dots,X_s) - \sum_j T(X_1,\dots,\nabla_X X_j,\dots,X_s).$$

A particular tensor is the *identity tensor* Id[30)], the type $(1,1)$ tensor field on M the components of which are given in any local coordinate system by the *Kronecker delta*

$$\delta_i^h = \begin{cases} 1, & h = i, \\ 0, & h \neq i. \end{cases}$$

For an arbitrary connection ∇ we have $(\nabla_X Id)(Y) = 0$ for any $X, Y \in \mathcal{X}(M)$, which in any local coordinate system reads $\delta^h_{i,j} = 0$.

Tensors with vanishing covariant derivatives are called *covariantly constant*.

2.3.3 Curvature and Ricci tensor

The *curvature* R of a manifold with affine connection $\mathbb{A}_n = (M,\nabla)$ is a tensor field of type $(1,3)$ defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z, \tag{2.15}$$

called sometimes also the *Riemann tensor* of a connection.

Locally $R = R^h_{ijk}\,\partial_h \otimes dx^i \otimes dx^j \otimes dx^k$, where components are

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}\,. \tag{2.16}$$

In spaces with symmetric affine connection, the Riemann tensor has the following properties:

$$\begin{aligned} &\text{(a)} \qquad R^h_{ijk} + R^h_{ikj} = 0, \\ &\text{(b)} \qquad R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \\ &\text{(c)} \qquad R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0. \end{aligned} \tag{2.17}$$

The formula (c) is called *Bianchi identity*.

[30)] $Id\colon\ X \mapsto X$ for any $X \in \mathcal{X}(M)$

We can introduce the *Ricci tensor Ric* of type $(0,2)$ as a trace of a linear map, namely

$$Ric(Z,Y) = \operatorname{trace}\{X \mapsto R(X,Y)Z\} \tag{2.18}$$

(the other possibility, $\operatorname{trace}\{Y \mapsto R(X,Y)Z\}$, differs only by sign).[31] Locally, $Ric = R_{ij}\, dx^i \otimes dx^j$ where

$$R_{ij} = R^\alpha_{i\alpha j} = \partial_\alpha \Gamma^\alpha_{ij} - \partial_j \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{ij}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{i\beta}\Gamma^\beta_{\alpha j}\,. \tag{2.19}$$

After contraction of (2.17c) over h and l we obtain

$$R^\alpha_{ijk,\alpha} = R_{ik,j} - R_{ij,k}. \tag{2.20}$$

Let us recall the *Ricci identity* for tensors $T \in C^2$ of type (p,q), which expresses independence of partial derivatives of components of the tensor $\partial_l\partial_m T^{h_1\cdots h_p}_{i_1\cdots i_q} = \partial_m\partial_l T^{h_1\cdots h_p}_{i_1\cdots i_q}$:

$$\begin{aligned} T^{h_1\cdots h_p}_{i_1\cdots i_q,[lm]} &= -T^{\alpha h_2\cdots h_p}_{i_1\cdots i_q} R^{h_1}_{\alpha lm} - T^{h_1\alpha\cdots h_p}_{i_1\cdots i_q} R^{h_2}_{\alpha lm} - \cdots - T^{h_1\cdots h_{p-1}\alpha}_{i_1\cdots i_q} R^{h_p}_{\alpha lm} \\ &\quad + T^{h_1\cdots h_p}_{\alpha i_2\cdots i_q} R^{\alpha}_{i_1 lm} + T^{h_1\cdots h_p}_{i_1\alpha\cdots i_q} R^{\alpha}_{i_2 lm} + \cdots + T^{h_1\cdots h_p}_{i_1\cdots i_{q-1}\alpha} R^{\alpha}_{i_q lm} - T^{h_1\cdots h_p}_{i_1\cdots i_q,\alpha} S^\alpha_{lm}. \end{aligned}$$

2.3.4 Flat, Ricci flat and equiaffine manifolds

Definition 2.11 A manifold with affine connection $\mathbb{A}_n$ (and also a connection ∇) is called *flat* if $S \equiv 0$ and $R \equiv 0$.

A manifold $\mathbb{A}_n$ is flat if and only if around any point, there are local coordinates such that $\Gamma^i_{jk} = 0$ holds. In a flat manifold the Ricci tensor also vanishes, p. 92.

Definition 2.12 A manifold with affine connection $\mathbb{A}_n$ is called *Ricci flat* if the equality $Ric \equiv 0$ holds.

The Ricci tensor is not necessarily symmetric for a general affine connection (even for a symmetric one).

Definition 2.13 A manifold $\mathbb{A}_n$ with a symmetric affine connection is called an *equiaffine manifold* if the Ricci tensor is symmetric.

Since in the case of a symmetric connection, $Ric(X,Y) - Ric(Y,X) = \operatorname{Tr} R(X,Y) = 0$ holds, the Ricci tensor is symmetric if and only if the curvature R is trace-less, i.e. in local coordinates [170, p. 14]: $R^\alpha_{\alpha jk} = 0$ holds.

A manifold $\mathbb{A}_n$ with a symmetric affine connection is an equiaffine manifold if and only if in any coordinate system (x^i) there exists a function $f(x)$ satisfying [135, 170]

$$\Gamma^\alpha_{i\alpha}(x) = \partial_i f(x). \tag{2.21}$$

We remark that this function f is not a "scalar" function on M, i.e. it is not a tensor of type (0,0). In two different coordinate systems, this function may have different values in the given point $x \in M$.

[31] So it are used books: [121, 170, 197, 200].

2.3.5 Parallel transport of vectors and tensors

By means of the concept of covariant derivative on a manifold $\mathbb{A}_n = (M, \nabla)$ with affine connection ∇ we can define parallel transport of vectors and tensors along curves which enables us to compare vectors and tensors in different points of the manifold $\mathbb{A}_n$.

Let $\ell\colon I \to M$, $t \mapsto \ell(t) = x(t) = (x^1(t), \dots, x^n(t))$ ($I \subset \mathbb{R}$ is an open interval and $\ell \subset U \subset M$, (U, φ) with $\varphi = (x^i)$, is local chart) be a differentiable curve in an n-dimensional manifold with affine connection $\mathbb{A}_n = (M_n, \nabla)$, and let $\lambda\,(= \dot{x})$ denote the corresponding tangent vector field ("velocity field") along ℓ.

Under a differentiable *vector field along* ℓ we mean a differentiable mapping $X\colon\ I \to TM$ such that $p_M \circ X = \ell$, that is, $X(t) \in T_{\ell(t)}M$ for any $t \in I$.

Definition 2.14 A *covariant derivative of a vector field X along the curve $\ell(t)$* is given by $\nabla_t X \overset{\text{def}}{=} \nabla_\lambda X$.

In components, this derivative reads

$$\nabla_t X^h = \frac{dX^h(t)}{dt} \;+\; \Gamma^h_{ij}(x(t))\; X^i(t)\; \lambda^j(t), \quad \text{where} \quad \lambda^j(t) = \frac{dx^j(t)}{dt}. \tag{2.22}$$

If X is a smooth vector field defined along a given (local) curve ℓ we can locally extend X to a smooth vector field $\tilde{X}$ defined on an open set containing the image $\ell(I)$ of the curve ℓ, and we can then write (2.22) in the following form: $\nabla_t X^h = \nabla_\alpha \tilde{X}^h\; \lambda^\alpha(t)$. It can be checked that the definition is independent of extension. Analogously we can introduce a *covariant derivative of a tensor field T along the curve ℓ*,

$$\nabla_t T = \nabla_\alpha T\; \lambda^\alpha(t). \tag{2.23}$$

The operator ∇_t and tensors T_1, T_2 satisfy the following formulae:

$$\nabla_t(T_1 \pm T_2) = \nabla_t T_1 \pm \nabla_t T_2; \quad \nabla_t(T_1 \cdot T_2) = \nabla_t T_1 \cdot T_2 + T_1 \cdot \nabla_t T_2. \tag{2.24}$$

Definition 2.15 A vector field X along ℓ is said to be

(a) *parallel along* ℓ if X satisfies the condition

$$\nabla_t X = 0 \quad \text{for any } t;$$

(b) *recurrent along* ℓ if there exists a real function $\varrho\colon I \to \mathbb{R}$ such that

$$\nabla_t X = \varrho\; X \quad \text{for any } t.$$

Analogously, a tensor field T is said to be *parallel along* $\ell(t)$ if T satisfies the condition $\nabla_t T = 0$, for any t, and *recurrent along* $\ell(t)$ if T satisfies the condition $\nabla_t T = \varrho(t)\, T$ for any t where ϱ is a function along ℓ.

The concept of parallelism and recurrence of vectors and tensors along curves is independent of regular reparametrations of the particular curve (as well as of the choice of an extension).

Let $x_0 = \ell(t_0)$ and $x_1 = \ell(t_1)$ be points on the given curve $\ell(t)$. We say that a vector X_1 from the tangent space $T_{x_1}M$ in x_1 is a result of the *parallel transport along* ℓ from the point x_0 to the point x_1 if along $\ell(t)$, there exists a parallel vector field $X(t)$ for which $X(t_0) = X_0$ and $X(t_1) = X_1$. This concept is introduced for tensors in a similar way. According to the properties of parallel transport, it follows that the vector X_1 (tensor T_1, respectively) in x_1 is by the above prescription uniquely determined by the vector X_0 (respectively tensor T_0) in x_0 and by the curve ℓ.

Besides, it follows that if X and Y are parallel vector fields along ℓ then also any combination $\alpha X + \beta Y$, $\alpha, \beta \in \mathbb{R}$, is parallel along ℓ. Therefore linear independence (dependence) of vectors is preserved under parallel transport along all curves. Similar statement holds also for recurrent vector fields, even the function of recurrency is preserved.

A tensor field T is called (*absolutely*) *parallel*, if the parallel transport is independent of the choice of the curve. A necessary and sufficient condition on the tensor T to be absolutely parallel is $\nabla T = 0$.

A tensor field T is called (*absolutely*) *recurrent* if the recurrent transport is independent of the choice of a curve. A necessary and sufficient condition on the tensor T to be absolutely recurrent is $\nabla T = \varrho \otimes T$ where ϱ is a one-form; more precisely, $\nabla_X T = \varrho(X)\, T$ for all X.

In the theory of manifolds with affine connection, manifolds endowed with a particular parallel or recurrent tensor field play an important role. Let us mention e.g. well-known manifolds with the curvature or Ricci tensor of special type[32)]:

$$\begin{aligned}
\nabla R &= 0 && \text{– } \textit{symmetric,}\\
\nabla R &= \varrho \cdot R && \text{– } \textit{recurrent,}\\
\nabla Ric &= 0 && \text{– } \textit{Ricci symmetric,}\\
\nabla Ric &= \varrho \cdot Ric && \text{– } \textit{Ricci recurrent.}
\end{aligned}$$

[32)] Symmetric manifolds were introduced by P.A. Shirokov, [167] and É. Cartan [324], recurrent manifolds were introduced by H.S. Ruse [775, 937].

Petr Alexeevich Shirokov, 1895-1944, was a Russian mathematician, Kazan State University, who did fundamental work in the theory of differential geometry [167, 806, 807].

Élie Joseph Cartan, 1869-1951, was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He also made significant contributions to mathematical physics, differential geometry, and group theory, [30, 323–325].

2.3.6 Geodesics

In the introduction we briefly mentioned geodesics. Now we intend to present an exact definition of geodesics in manifolds with affine connection $\mathbb{A}_n$. A geodesic, that is an object analogous to straight lines in Euclidean spaces, can be characterized in different ways: either from the view-point of the variational theory, as the shortest line connecting two (close) points, or as a curve whose tangent vectors in all of its points are parallel, etc. Here we generalize the second definition for manifolds with affine connection, which we can do without the notion of a length, making use of parallel transport instead.

Definition 2.16 A curve ℓ in $\mathbb{A}_n$ is a *geodesic* when its tangent vector field remains in the tangent distribution of ℓ during parallel transport along the curve.

Equivalently, a curve $\ell(t) \subset \mathbb{A}_n$ is a geodesic if and only if the covariant derivative of its tangent vector $\lambda(t) = \dot{\ell}(t)$ is proportional to the tangent vector itself,

$$\nabla_\lambda \lambda = \varrho(t)\, \lambda\,, \tag{2.25}$$

where ϱ is some function of the parameter t of the curve ℓ.

Remark that if the parameter t on the geodesic is chosen so that $\varrho(t) \equiv 0$ then this parameter is called *natural* or *affine*. A natural parameter is usually denoted by s. We can check that $s = \int \varrho(t)\, dt$.

In components, these equations read

$$\frac{d^2x^h(s)}{ds^2} + \Gamma^h_{ij}(x(s))\frac{dx^i(s)}{ds}\frac{dx^j(s)}{ds} = 0. \tag{2.26}$$

If s and s^* are natural parameters of the same geodesic then $s^* = c_1\, s + c_2$ where c_1 $(\neq 0)$, c_2 are constants.

Approximately since the 1960' geodesics are in the literature frequently defined in this restrictive sense, that is by the condition $\nabla_\lambda \lambda = 0$, see [50–52, 91, 96, 118, 121, 122, 135, 149, 151, 152, 170, 197, 200]. We do not adopt this definition here, but prefer the traditional notion based on (2.25), which is independent of parametrization.

From the equations $\nabla_{\lambda(s)}\lambda(s) = 0$ for geodesics, one of the most important properties of geodesics follows, namely, through a given point x_0 in a given direction (determined by a fixed tangent vector λ_0) there passes exactly one unique geodesic. This property is a consequence of the unicity of the Cauchy problem for these equations that can be presented in the following form:

$$\begin{aligned} &\frac{dx^h(s)}{ds} = \lambda^h(s),\ \frac{\lambda^h(s)}{dt} = -\Gamma^h_{ij}(x(s))\,\lambda^i(s)\lambda^j(s);\\ &x^h(s_0) = x^h_0, \qquad \lambda^h(s_0) = \lambda^h_0. \end{aligned}$$

If $\Gamma^h_{ij} \in C^0$ the existence of a solution is *GUARANTEED*.

If $\Gamma^h_{ij} \in C^1$ there exists a *UNIQUE* solution.

Remark 2.10 Note that a geodesic flow is a flow determined by geodesic curves. A geodesic flow defines a family of curves in the tangent bundle. The derivatives of these curves define a vector field on the total space of the tangent bundle, known as the geodesic spray.

We mention the following definition which is connected with geodesics.

Definition 2.17 A manifold $\mathbb{A}_n = (M, \nabla)$ with affine connection is said to be *geodesic complete* (shortly *complete*) if any geodesic $\gamma(s)$ is defined on M for all values of natural parameters $s \in \mathbb{R}$.

Example 2.1 In an affine coordinates x of a flat $\mathbb{A}_n$ it holds that $\Gamma^h_{ij}(x) = 0$ for $x \in \mathbb{R}^n$. All geodesics ℓ with canonical parameter s in $\mathbb{A}_n$ satisfy the linear equations ℓ: $x^h = x^h_0 + \lambda^h_0 \cdot s$, where x_0 is a point and λ_0 is a direction which define ℓ (*straight line* in an affine space). These straight lines are complete.

In a half flat space $\mathbb{A}_n$ where $\Gamma^h_{ij}(x) = 0$ for $x \in \{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n$: $x^1 > 0$, $x^i \in \mathbb{R}$, $i = 2, \dots, n)\}$ all complete straight lines have $\lambda^1 = 0$. Other straight lines (with $\lambda^1 \neq 0$) are incomplete.

Example 2.2 We show an example of $\mathbb{A}_n \in C^0$ where one geodesic pass through the given point x_0 and the given direction λ_0.

Let $\mathbb{A}_n$ be a space with affine connection ∇ whose components $\Gamma^h_{hh} = f^h(x^h) \in C^0(\mathbb{R})$ for $h = 1, 2, \dots, n$ and all other components are vanishing. As $\Gamma^h_{ij} \in C^0(\mathbb{R}^n)$ we have $\mathbb{A}_n \in C^0$.

We find the geodesics in $\mathbb{A}_n$ that pass through the given point x_0 and the given direction λ_0:

$$x_0 = (x^1_0, x^2_0, \dots, x^n_0) \quad \text{and} \quad \lambda_0 = (\lambda^1_0, \lambda^2_0, \dots, \lambda^n_0). \tag{2.27}$$

The equations (2.26) of a geodesic ℓ: $x^h = x^h(s)$ with canonical parameter s are written in the following form

$$dx^h(s)/ds = \lambda^h, \quad d\lambda^h(s)/ds + f^h(x^h) \cdot (\lambda^h)^2 = 0. \tag{2.28}$$

Each of these equations ($h = 1, 2, \dots, n$), for initial conditions (2.27), has only one solutions

$$\int_{x^h_0}^{x^h(s)} \exp\left(\int_{x^h_0}^{t} f^h(\tau)\, d\tau\right) dt = \lambda^h_0 \cdot s + x^h_0. \tag{2.29}$$

This solution valides locally. In this $\mathbb{A}_n$: $S = 0$ and $R = 0$. In the points x on coordinate hyperplains ($x^i = x^i_0$, i is fixed index) there is no locally affine system of affine flat space.

2.3.7 Some remarks on definitions for geodesics

Johann I. Bernoulli[33], in his letter to Leibniz[34] from 1697, posed a problem: *find a curve on a given surface which passes through a pair of given points and*

[33] Johann I. Bernoulli, 1667–1748, was a great Swiss mathematician, physicists and physician who studied reflection and refraction of light, orthogonal trajectories of families of curves, quadrature of areas by series, and the brachistochrone.

[34] Gottfried Wilhelm Leibniz (also Leibnitz), 1646–1716, a German polymath, mathematician and natural philosopher, who wrote mostly in Latin and French. Among others, he invented calculus independently on Newton, and his notation is the one in general use since.

has the property that its segment connecting the given two points is shorter (or the same) as any other curve segment connecting them.

In the next year 1698 he communicated, again to Leibniz, solution of the problem in general features, explanation of its meaning, and the proof of a geodesic variational problem (see p. 122). So the year 1698 may be considered as the year when the concept of geodesics was born, although the first published work on this topic appeared in the year 1728, and was written by Euler[35)].

In 1696, Johann I. Bernoulli challenged other mathematicians to find the curve which solves the brachistochrone problem (brachistochrone, or brachystochrone, is the curve of fastest descent under gravity), knowing the solution to be a segment of a cycloid. Leibniz, Newton[36)], Jacob Bernoulli[37)] and L'Hospital[38)] solved Bernoulli's challenge.

We can say that the Bernoulli's problem on geodesics together with the problem of Galilei[39)] on the curve of fastest descent the first correct solution of which is also due to Johann Bernoulli[40)], were, so to say, first seats from that the calculus of variations was grown up, planted, and cultivated.

Let us now explain the solution of the problem of Bernoulli, but in the more general setting, and more adapted to the language of modern differential geometry, [50].

The definition of geodesics can be based exclusively on purely geometric considerations. The basic idea of a geometric view-point goes back to Lagrange[41)], and it was explained already in one of his works published in 1806.

Remark 2.11 As already mentioned, two non-equivalent definitions of geodesics can be found in textbooks and other bibliographies:
(D1) A curve $\ell(t)$ in a manifold with affine connection (M, ∇) is a *geodesic* if there exists a vector field along ℓ which is tangent and parallel along ℓ. Another speaking, for the tangent vector $\lambda(t) = \ell'(t)$ the following holds:

$$\nabla_{\lambda(t)}\lambda(t) = \varrho(t)\,\lambda(t). \tag{2.30}$$

[35)] Leonhard Euler, 1707–1783, the Swiss mathematician who was tutored by Johann I. Bernoulli, and worked at the Petersburg Academy and at Berlin Academy of Science.

[36)] Isaac Newton, 1643–1727, an English physicist, mathematician, astronomer, natural philosopher, and alchemist.

[37)] Jacob Bernoulli, 1654–1705, a Swiss mathematician, an elder brother of Johann Bernoulli.

[38)] Guillaume de L'Hospital, 1661–1704, a French mathematician who published the first book ever on differential calculus, and solved a problem about cycloids.

[39)] Galileo Galilei, 1564–1642, an Italian physicist, mathematician, astronomer and philosopher. His achievements include the first systematic studies of uniformly accelerated motion.

[40)] The problem of brachistochrone was posed as follows: Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time. It was solved by Johann I. Bernoulli, Jacob Bernoulli, I. Newton and G. l'Hospital; the solution is the curve called *cycloid* (it was studied and named by Galileo in 1599). Cycloid is the curve defined by the path of a point on the edge of circular wheel as the wheel rolls along a straight line.

[41)] Joseph Louis Lagrange, 1736–1813, the greatest French mathematician of the eighteenth century. The author of *Mécanique analytique*.

Note that under a suitable reparametrization, we can always menage that $\varrho(t) \equiv 0$. In such a case, the parameter is canonical. In Riemannian spaces, the arc length s is always canonical.

(D2) A curve $\ell(t)$ in an affine manifold is a *geodesic* if its tangent vector field $\lambda(t)$ is parallel along ℓ. In other words, the following formula is satisfied:

$$\nabla_{\lambda(t)}\lambda(t) = 0. \tag{2.31}$$

In the "classical sources" (Beltrami, Lagrange, Eisenhart), the first definition was used. If we accept the thesis that the aim of differential geometry is to study geometric objects then (D1) is more close to this approach (geodesics according to (D1) are geometric objects while objects defined by (D2) are not).

During the time, as the methods of differential geometry has been developed, the view-point has been changed significantly. In the not very far history (approximately in the last fifty years, e.g. Helgason [69], and his followers), even the approach to geodesics has been changed. Nowadays, when introducing geodesics, many authors use the second definition (D2), usually for some technical reasons or others. In other words, they admit canonically parametrized geodesics only which arise as solutions of the equations (2.31). According to our opinion, sometimes such a restrictive definition is inadequate; particularly, when we investigate projective transformations of Riemannian spaces, geodesic or almost geodesic mappings of affine manifolds etc., we prefer to consider all possible solutions of (2.30), i.e. geodesics with recurrent tangent fields.

In some papers and books, geodesic lines defined by (D1) (i.e. solutions for (2.31)) are called *unparametrized geodesics* (Eastwood [388], Eastwood, Matveev [389]) or *pregeodesics*, to emphasize that the particular parametrization is not important, while under geodesics the authors understand only those that have canonical parameter s (that is only those curves, that are characterized by the equations (2.31)). Under a non-linear reparametrization, a "geodesic" introduced by (D2) transforms to a curve which formally is no more a geodesic, and becomes an "unparametrized geodesic".

I. Kolář in his publication [91] solves this paradox in an elegant (and quite correct) way: he uses the notion "*geodesic line*" in our classical point of view, and reserves the term "*geodesic path*" for a geodesic line parametrized by a canonical parameter.

In many papers and textbooks on differential geometry and (pseudo-) Riemannian spaces, geodesic lines are introduced similarly as in our text, see T. Levi-Civita [107], L.P. Eisenhart [50–52], H. Cartan [31], I. Kolář, P. Michor, J. Slovák [94], W. Kühnel [101], M. Nakahara [132] R. Penrose [137], A.Z. Petrov [139], A.G. Popov [143], E. Poznak [147], Ž. Radulović, J. Mikeš, M.L. Gavrilchenko [149], P.K. Rashevskij [151, 152], H. Rund [156], J.A. Schouten, D.J. Struik [163], N.S. Sinyukov [170], G. Vranceanu [933], K. Yano, S. Bochner [200], K. Yano [197] etc. Note that also in a monograph by P.K. Rashevskij [152] and E.G. Poznak, E.V. Shikin [143] (published at Moscow University) geodesic lines are understood in our invariant view-point. Besides, the invariant definition is used also in a recent publication [101] by Kühnl.

2.4 Special coordinate systems and reconstructions

2.4.1 Affine coordinates and flat manifolds

Let $\mathbb{A}_n$ be a manifold with symmetric affine connection ∇ and $\Gamma^h_{ij}(x)$ be components of ∇ in a coordinate chart $(U_\alpha, \varphi_\alpha)$. As already mentioned, flat manifolds are characterized as follows: there are local coordinates (about any point) $(U_\alpha, \varphi_\alpha)$ in which $\Gamma^h_{ij}(x) = 0$; such coordinates are called *affine*.

A natural question arises:

how to recognize whether a given $\mathbb{A}_n$ is flat or not?

Suppose that in some coordinate system (x^i) the components $\Gamma^h_{ij}(x)$ of a connection are known. We wish to find a coordinate transformation $x' = x'(x)$ such that $\Gamma^{h'}_{i'j'}(x) = 0$. From (2.3) we get

$$\frac{\partial^2 x^{h'}(x)}{\partial x^i \partial x^j} = \Gamma^h_{ij}(x) \frac{\partial x'^{h'}(x)}{\partial x^h}. \tag{2.32}$$

This system of PDE's can be examined by means of methods explained in Section 2.5, p. 100.

Examining the integrability conditions for (2.32) we find that the system is completely integrable if and only if $R = 0$ and $S = 0$ which is a criterion for (locally) flat spaces in tensor form. Therefore equations (2.32) have a local solution for arbitrary initial conditions $\partial x'^{h'}/\partial x^h(x_0) = x^{h'}_h$ and $\det x^{h'}_h \neq 0$. The above claim is valid in the case $\Gamma^h_{ij}(x) \in C^1$.

These equations (2.32) have unique solution for spaces of the Example 2.2 for $\Gamma^h_{ij}(x) \in C^0$. For $R \neq 0$ or $S \neq 0$ the equations (2.32) are not solvable.

2.4.2 Geodesic coordinates in a point, Fermi and Riemann coordinates

Geodesic coordinates in a point

On the other hand, there are always solutions to (2.32) in a single fixed point, i.e. $\Gamma^{h'}_{i'j'}|_{x_0} = 0$. Such a coordinate system will be called *geodesic in the point* x_0.

A coordinate system (x') with

$$x'^h = a^h_\alpha (x^\alpha - x^\alpha_0) + \frac{1}{2} a^h_k \Gamma^k_{ij}(x_0)(x^i - x^i_0)(x^j - x^j_0),$$

where a^h_i are constants, $\det(a^h_i) \neq 0$, x^i_0 are coordinates of the point x_0 in a coordinate chart $(U_\alpha, \varphi_\alpha)$, is geodesic in the point x_0 which may be verifed easily. These transformations preserve the differentiability order of connections and tensors.

Fermi coordinates

This result can be strengthened in the following sense: we can find a coordinate system (x') in which $\Gamma'^h_{ij}|\,\ell = 0$ along a prescribed curve ℓ (considered as a point set). Such a coordinate system will be called *geodesic along a curve* ℓ, and the coordinates are called *Fermi coordinates.*[42)]

In the Fermi coordinates the connection can lose some orders of differentiability.

[42)] Enrico Fermi, 1901-1954, was an Italian physicist, and was awarded the 1938 Nobel Prize in Physics. He was widely regarded as one of the very few physicists to excel both theoretically and experimentally.

Riemann coordinates

Now we introduce Riemann coordinates on a manifold with torsion-less affine connection as follows. Let (U, φ) be a fixed chart in $\mathbb{A}_n$. Consider the equations of canonically parametrized geodesics (with canonical parameter s) passing through $x_0 \in U$ such that x_0 corresponds just to the value $s = 0$ (that is, $x^h(0) = x_0^h$): (2.26).

In the case $\Gamma_{ij}^h \in C^1$, given the initial data

$$x^h(0) = x_0^h \quad \text{and} \quad \frac{dx^h(0)}{ds} = \xi_0^h$$

the equations (2.31) have the only solution

$$x^h = x^h(s, \xi_0^h).$$

Let us put

$$x^h = x^h(y^1, \dots, y^n) \tag{2.33}$$

where

$$y^h = \xi_0^h \cdot s \tag{2.34}$$

and $x^h(y) = x^h(s, \xi_0^h)$. Now (2.33) can be viewed as a coordinate transformation $x^h = x^h(y)$ since the Jacobian $\mathcal{J}_{x_0} = \det\left(\partial x^h / \partial y^i|_{x_0}\right) = \det(\delta_i^h) = 1$.

The coordinates $(y) = (y^1, \dots, y^h)$ will be called *Riemann coordinates* with the reference point x_0.

We can see that (2.34) are equations for geodesics passing through x_0. Equations of these geodesics in Riemann coordinates take the same form as usual equations for straight lines in Euclidean space. This is just the characteristic property of Riemann coordinates.

On the other hand, the equations of geodesics in Riemann coordinates are

$$\frac{d^2 y^h(s)}{ds^2} + \Gamma_{ij}^h(y(s)) \frac{dy^i(s)}{ds} \frac{dy^j(s)}{ds} = 0;$$

since geodesics passing through x_0 have equations (2.34) it follows

$$\Gamma_{ij}^h(y)\, y^i y^j = 0. \tag{2.35}$$

But it follows from (2.35) that the equations for geodesics read (2.34). Hence (2.35) is a local criterion for a Riemann coordinate system.

If we transform Riemann coordinates y^h by a linear homogeneous transformation $y^{h'} = a_h^{h'}\, y^h$ where $a_h^{h'} = \text{const}$, $|a_h^{h'}| \neq 0$, then the equations for geodesics (2.34) transform to $y^h = \xi_0^{h'} \cdot s$ where $\xi^{h'} = a_h^{h'} \xi^h$. Consequently $y^{h'}$ are also Riemann coordinates with the reference point x_0.

These transformations preserve the differentiability order of connections and tensors.

2.4.3 Pre-semigeodesic coordinates

This coordinates are a generalization of semigeodesic coordinates in Riemannian manifolds, discussed later, see p. 118.

Coordinates in $\mathbb{A}_n$ are called *pre-semigeodesic coordinates* if one system of coordinate lines are geodesics and their natural parameter is just the first coordinate.

Let x^1-curves be geodesics $\ell = (s, x_0^2, x_0^3, \ldots, x_0^n)$ where s is a natural parameter. Substituting this parametrization into the equations for geodesics (2.26) we obtain

$$\Gamma^h_{11}(x) = 0. \tag{2.36}$$

This condition is necessary and sufficient for a coordinate system to be pre-semigeodesic:

Lemma 2.2 *The conditions $\Gamma^h_{11} = 0$, $h = 1, \ldots, n$ are satisfied in U if and only if the parametrized curves*

$$\ell\colon\ I \to U, \qquad \ell(s) = (s, a_2, \ldots, a_n), \quad s \in I,\ a_i \in \mathbb{R}, \quad i = 2, \ldots, n, \tag{2.37}$$

are canonically parametrized geodesics of $\nabla|_U$ (I is some interval, a_k are suitable constants chosen so that $\ell(I) \subset U$).

Proof. Let $\Gamma^h_{11} = 0$ hold for $h = 1, \ldots, n$. Then the local curves with parametrizations (2.37) satisfy

$$d\ell(s)/ds = (\partial_1)_{\ell(s)}, \qquad d^2\ell(s)/ds^2 = 0, \tag{2.38}$$

therefore are solutions to the system (2.26). Conversely, if the curves (2.37) are among solutions to (2.26) then due to (2.38), we get $\Gamma^h_{11} = 0$ from (2.26). □

Hence the pre-semigeodesic chart is fully characterized by the condition that the curves $x^1 = s$, $x^i = \text{const}$, $i = 2, \ldots n$ belong to the geodesics of the given connection in the coordinate neighbourhood. The definition domain U of such a chart is "tubular", a tube along geodesics.

On existence pre-geodesic charts.

We thought that the exists of this chart is a trivial, because the existence of geodesics congruence is possible to constructed analogically as for semigeodesic charts for (pseudo-) Riemannian manifolds, see p. 118. In this case we construct a geodesic congruence so that it passed through at points of a certain hypersurface and at the points have tangent vector which is not tangent of this hypersurface. We remark that this condition in semigeodesic chart is replaced by orthogonality.

This problem is obviously more difficult, than we suppose.

This observed by Z. Dušek and O. Kowalski [386, 387], who precisely proved the existence of pre-geodesic charts in the case when components of affine connection are real analytic functions.

I. Hinterleitner and J. Mikeš [664] proved that pre-semigeodesic chart exists in case if components of connections are differentiable. The existence of this chart is not excluded in case if components are only continuous.

We work locally with the coordinates $x = (x^1, \ldots, x^n)$ and $x' = (x'^1, \ldots, x'^n)$, respectively. For a transformation of coordinates x and x', we write $x'^h = f^h(x^i)$, or $x' = x'(x)$ for short. We start with the formula (2.13) for the transformation of the connection.

Theorem 2.2 *For any affine connection determined by $\Gamma^h_{ij}(x) \in C^r$, $r \geq 1$, there exists a local transformation of coordinates determined by $x' = f(x) \in C^{r+1}$ such that the connection in new coordinates satisfies $\Gamma'^h_{11}(x') = 0$ for $h = 1, \ldots, n$.*

Proof. Let (U, x) be a coordinate system at a point $p = (0, 0, \ldots, 0) \in U \subset M$, $\Gamma^h_{ij}(x) \in C^r$, $r \geq 1$, be components of ∇ on (U, x).

In a neighbourhood of p we construct a set of geodesics, which go through the point x_0 of a hypersurface $\sigma \ni p$ in the direction $\lambda_0(x_0) \neq 0$, which is not tangent to σ.

Let σ and λ_0 be defined in the following way:

$$\sigma\colon\ x^1 = \varphi(x_0^2, \ldots, x_0^n),\ x^i = x_0^i,\ i > 1, \quad \text{and} \quad \lambda_0^h = \Lambda^h(x_0^2, \ldots, x_0^n). \tag{2.39}$$

Then the above considered geodesics are the solutions of the following ODE's

$$\begin{aligned} \frac{dx^h(\tau)}{d\tau} &= \lambda^h(\tau), \\ \frac{d\lambda^h(\tau)}{d\tau} &= -\Gamma^h_{\alpha\beta}(x(\tau))\lambda^\alpha(\tau)\lambda^\beta(\tau) \end{aligned} \tag{2.40}$$

for initial conditions

$$\begin{aligned} x^h(0) &= (\varphi(x_0^2, \ldots, x_0^n), x_0^2, \ldots, x_0^n), \\ \lambda^h(0) &= \Lambda^h(x_0^2, \ldots, x_0^n). \end{aligned} \tag{2.41}$$

for any $(x_0^2, \ldots, x_0^n)$ in the neighbourhood of p.

Remark. From (2.40), (2.41) and from the theory of ODE's [83] follows:

1. If $\Gamma^h_{\alpha\beta}(x)$ are continuous, then by the Peano existence theorem locally exists a solution.
2. If $\Gamma^h_{\alpha\beta}(x)$ satisfy Lipschitz conditions, then by the Picard-Lindelöf theorem this solution is unique.

In the neighbourhood of p we have constructed a vector field $\lambda^h(x) \neq 0$ which is tangent to the considered geodesics.

In addition, by more detailed analysis it can be shown that $\lambda^h(x) \in C^r$ if $\Gamma^h_{ij} \in C^r$ and moreover

$$\varphi(x^2, \ldots, x^n) \in C^{r+1} \quad \text{and} \quad \Lambda^h(x^2, \ldots, x^n) \in C^r.$$

Note that from the decreasing of the degree of differentiability of the functions φ and Λ^h follows the decreasing of the degree of differentiability of $\lambda^h(x)$.

As an example, we can take initial conditions (2.41) in the form:

$$x^h(0) = (0, x_0^2, \ldots, x_0^n) \quad \text{and} \quad \lambda^h(0) = \delta_1^h. \tag{2.42}$$

Lemma 2.1, p. 75, ensures the existence of a coordinate system x' in which $\lambda'^h(x') = \delta_1^h$. So, this system x' is semi-geodesic, according to Remark 2.4, p. 81. there also exists a transformation $x' = x'(x) \in C^{r+1}$.

Unfortunately, the existence of a solution $\lambda(x) \in C^0$ (if $\Gamma_{ij}^h \in C^0$) does not ensure the existence of a transformation $x' = x'(x) \in C^2$, which leads to the solution of our problem. In this case, the conditions for the transformations of connection are not fulfilled. □

Remark. Evidently, we can prove:

$$\Gamma_{\alpha\beta}^h(x) \in C^r, C^\infty, C^\omega \quad \Rightarrow \quad \Gamma'^h_{\alpha\beta}(x') \in C^{r-1}, C^\infty, C^\omega.$$

On number of components of affine connection, which determined $\mathbb{A}_n$. In paper [386] was proved the first result is the description of this class of connections using $n(n^2-1)$ functions of n variables modulo $2n$ functions of $n-1$ variables. For this purpose, we use the existence of the system of pre-semigeodesic coordinates. For the proof was used the Cauchy-Kowalewska theorem of partial differential equations [95].

A well known fact from Riemannian geometry is that a Riemannian connection has symmetric Ricci form. Our next aim is to determine how big is the class of all real analytic affine connections with skew-symmetric Ricci form (again, in dimension n) and those with symmetric Ricci form. For this purpose, a direct approach using the Cauchy-Kowalevski Theorem can be used. Surprisingly, for the torsion-free connections with a symmetric Ricci form, another method was necessary, see [387] for the details.

The authors of [387] prove that this class of real analytic connections with torsion and with skew-symmetric Ricci form depends on $n(2n^2-n-3)/2$ functions of n variables and $n(n+1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables. They prove further that the class of real analytic connections with symmetric Ricci form depends on $n(2n^2-n-1)/2$ functions of n variables and $n(n-1)/2$ functions of $n-1$ variables, modulo $2n$ functions of $n-1$ variables.

In our work we prove that the pre-semigeodesic chart may exist even when the components Γ_{ij}^h are smooth, and always exists for $\Gamma_{ij}^h \in C^1$. In the pre-semigeodesic chart where the differentiability of components Γ_{ij}^h can be more reduced by not more than one unit.

In the case that there are pre-semigeodesic charts (U, x) and $\Gamma_{ij}^h(x) \in C^1$, then other results obtained in the paper [386], are by identical methods transferred to this case.

2.4.4 Reconstuction of connection

H.H. Hacisalihoglu, A.Kh. Amirov [448, 449], in a neighbourhood of a (positive definite) Riemannian space in which special (semigeodesic) coordinates are given, the metric tensor is calculated from its values on a suitable hypersurface and some of the components of the curvature tensor of type $(1,3)$ in the coordinate domain.

In the present section, we consider the more general situation of non-degenerate metrics in arbitrary signature. We have alredy introduced above special pre-semigeodesic charts characterized both geometrically and in terms of the connection. Let us formulate a version of the Peano's-Picard's-Cauchy-like theorem on existence and uniqueness of solutions of the initial value problems for systems of first-order ordinary differential equations. Given a weak pre-semigeodesic chart $(U,(x^i))$ in a manifold (M,∇) with linear symmetric connection, we are able to reconstruct the connection in some neighbourhood from the knowledge of "initial conditions": the restriction of the connection to a fixed $(n-1)$-dimensional surface S and some of the components of the curvature tensor R in the "volume" (coordinate domain).

The problem of finding a Riemannian metric from this or that information is of interest from both theoretical and practical points of view. Papers by many authors are devoted to the possibility of finding the metric from the curvature tensor, [554, p. 135-136], or prove the existence of metrics with the prescribed Ricci tensor, [377, 901] etc. In general, to solve the problem means to solve a relatively complicated system of non-linear differential equations with partial derivatives, the coefficients of which are expressed through components of the Riemannian curvature tensor. One possibility how to simplify the situation is to find a convenient coordinate system with respect to which the system is simplified considerably. Our aim is to present such preferable coordinates.

Hence a pre-semigeodesic chart is fully characterized by the condition that the curves $x^1 = s$, $x^i = \text{const}$, $i = 2, \dots n$ belong to geodesics of the connection in the coordinate neighbourhood. The definition domain U of such a chart is "tubular", a tube along geodesics.

Our aim is to show that a symmetric linear connection in a pre-semigeodesic coordinate domain U (related to x^1) can be uniquely constructed, or retrieved, in some subdomain of U if we know the restrictions $\tilde{\Gamma}^h_{ij}(\tilde{x})$, $\tilde{x} \in S$ of the connection to the surface S defined by $x^1 = 0$ and prescribed components R^h_{i1k} of the curvature tensor in the given tubular domain U. First recall that the components R^h_{ijk} of the curvature tensor are related to the components Γ^h_{ij} of the connection by the classical formula (2.16):

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\Gamma^h_{mj} - \Gamma^m_{ij}\Gamma^h_{mk}.$$

Now suppose that $\Gamma^h_{11}(x) = 0$ is satisfied. Particularly, setting $i = j = 1$ in (2.16) we get under this assumption

$$\frac{\partial}{\partial x^1}\Gamma^h_{1k} + \sum_m \Gamma^m_{1k}\Gamma^h_{1m} - R^h_{11k} = 0 \tag{2.43}$$

and plugging $j = 1$ in (2.16) we get, for each of the indices $i = 2, \dots, n$, the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{ik} + \sum_m \Gamma^m_{ik}\Gamma^h_{m1} - \sum_m \Gamma^m_{i1}\Gamma^h_{mk} - \frac{\partial}{\partial x^k}\Gamma^h_{i1} - R^h_{i1k} = 0. \tag{2.44}$$

In $\mathbb{R}^n$ with standard coordinates $(x^1, x^2, \dots, x^n)$, let us identify the linear subspace (hypersurface) defined by $x^1 = 0$ with the space $\mathbb{R}^{n-1}$, i.e. $(\tilde{x}) = (x^2, \dots, x^n)$ are standard coordinates in $\mathbb{R}^{n-1}$. Let $\mathcal{J} = (0, 1)$ be the open unit interval and denote by $K_m = \mathcal{J}^m$ the open standard m-cube. Denote

$$D_n(\delta) = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid 0 \le x^1 \le \delta,\ 0 < x^i < 1,\ i = 2, \dots, n\}.$$

Hereafter we will deal with the pre-semigeodesic coordinate system in the domain $D_n(\delta)$. The open $(n-1)$-cube $K_{n-1} = \mathcal{J}^{n-1}$, viewed as

$$K_{n-1} = \{\tilde{x} = (x^2, \dots, x^n) \in \mathbb{R}^{n-1} \mid 0 < x^i < 1,\ i = 2, \dots, n\} \subset \mathbb{R}^{n-1},$$

will be identified with a hypersurface S in $D_n(\delta)$ determined by $x^1 = 0$.

So in what follows let S be a hypersurface in $D_n(\delta)$ defined by $x^1 = 0$. Now let us modify for our purpose the Theorem on existence and uniqueness of solutions of systems of ODEs :

Theorem 2.3 *Let $\tilde{\nabla}$ be a torsion-free linear connection in S (of the class at least C^2) with components $\tilde{\Gamma}^h_{ij}(\tilde{x})$, $\tilde{x} \in S$, $h, i, j \in \{2, \dots, n\}$, let $\tilde{\Gamma}^h_{1j}(\tilde{x})$ be functions in S (at least C^2), $h, j \in \{1, \dots, n\}$, where $\tilde{\Gamma}^h_{11}(\tilde{x}) = 0$ for $h \in \{1, \dots, n\}$. Let A^h_{ij}, $h, i, j \in \{1, \dots, n\}$ be functions (at least C^0) in $D_n(\delta)$ such that each A^h_{1k} is at least C^1 in each of the variables $x^2, \dots, x^n$ and at least C^0 in x^1. Moreover let the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ satisfy $\tilde{R}^h_{i1k}(\tilde{x}) = A^h_{ik}(\tilde{x})$ in S. Then there exists a real number $\hat{\delta}$, $0 < \hat{\delta} \le \delta$ and there is a unique torsion-free linear connection ∇ in the neighborhood $D_n(\hat{\delta})$ with components Γ^h_{ij} such that the following holds: $\Gamma^h_{11}(x) = 0$ in $D_n(\hat{\delta})$ for $h = 1, \dots, n$ (i.e. the coordinates are pre-semigeodesic), $\Gamma^h_{ij}(0, \tilde{x}) = \tilde{\Gamma}^h_{ij}(\tilde{x})$ for $\tilde{x} \in S$, $h, i, j \in \{1, \dots, n\}$ (hence $\nabla | S = \tilde{\nabla}$) and $R^h_{i1k} = A^h_{ik}$ for all $x \in D_n(\hat{\delta})$ where $h, i, k = 1, \dots, n$.*

Proof. Let the assumptions be satisfied. Analyzing the system (2.16) and the consequences mentioned above we find that we can proceed step by step. In three main steps, we find functions Γ^h_{ij} in a certain subdomain of $D_n(\delta)$ such that $\Gamma^h_{ij} = \Gamma^h_{ji}$ and the conclusion of the Theorem 2.3 is satisfied.
Step (1) Let us define $\Gamma^h_{11}(x) = 0$ for $x \in D_n(\delta)$, $h = 1, \dots, n$.
Step (2) Let us solve the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{1k}(x) = -\sum_m \Gamma^m_{1k}(x)\Gamma^h_{1m}(x) + A^h_{1k}(x) \tag{2.45}$$

for unknown functions Γ^h_{1k}, $h = 1, \dots, n$, $k = 2, \dots, n$ which we assume as a system of ordinary differential equations of one variable x^1 (while the remaining

coordinates $(\tilde{x}) = (x^2, \dots, x^n) \in K_{n-1} = S$ are considered as parameters) for the initial data

$$\Gamma^h_{1k}(0, x^2, \dots, x^n) = \tilde{\Gamma}^h_{1k}(x^2, \dots, x^n) \quad \text{for} \quad (x^2, \dots, x^n) \in S.$$

According to the theory, there exists δ_1, $0 < \delta_1 \leq \delta$ and there are uniquely determined functions $\Gamma^h_{1k}(x^1, \dots, x^n)$ of the class at least C^1 in the domain $D_n(\delta_1)$ such that

$$\Gamma^h_{1k}(0, \tilde{x}) = \tilde{\Gamma}^h_{1k}(\tilde{x}), \qquad \tilde{x} \in S. \tag{2.46}$$

These functions together with their derivatives will be used in what follows.
Step (3) Now consider the system

$$\frac{\partial}{\partial x^1}\Gamma^h_{ik} = -\sum_m \Gamma^m_{ik}\Gamma^h_{m1} + \sum_m \Gamma^m_{i1}\Gamma^h_{mk} + \frac{\partial}{\partial x^k}\Gamma^h_{i1} + A^h_{ik} = 0 \tag{2.47}$$

where we plugged for Γ^h_{i1} from the above; $h = 1, \dots, n,\ i, k = 2, \dots, n$. We have again a system of ordinary differential equations of one variable x^1. According to the existence and uniqueness theorem on systems of ODEs there is $\hat{\delta}$, $0 < \hat{\delta} \leq \delta_1$ and there are uniquely determined functions $\Gamma^h_{ik}(x^1, \dots, x^n)$ of the class at least C^1, $i \neq 1 \neq k$ in the domain $D_n(\hat{\delta})$ which satisfy the initial conditions

$$\Gamma^h_{ik}(0, x^2, \dots, x^n) = \tilde{\Gamma}^h_{ik}(x^2, \dots, x^n), \qquad (x^2, \dots, x^n) \in S. \tag{2.48}$$

Moreover, comparing (2.43), (2.45) and (2.47) we can see that

$$R^h_{i1k}(x) = A^h_{ik}(x), \qquad x \in D_n(\hat{\delta}), \quad h, i, k = 1, \dots, n \tag{2.49}$$

holds as required, and Γ^h_{ik} are components of a connection of the above properties. □

As a consequence, if we use prolongation of the solution, we obtain:

Theorem 2.4 *Let $(U, \varphi = (x^1, \dots, x^n))$ be a chart in M. Let $S \subset U$ be a submanifold in U defined by $x^1 = 0$. Let $\tilde{\nabla}$ be a torsion-free linear connection in S of the class at least C^2 with the components $\tilde{\Gamma}^h_{ij}$ and the curvature tensor $\tilde{R}$, and let A^h_{ij} be functions in U such that $\tilde{R}^h_{i1k} = A^h_{ik}$ in S, A^h_{ik}, $i = 2, \dots, n$ are of the class at least C^0, A^h_{1k} are continuous in x^1, and A^h_{1k} are at least C^1 in the remaining variables $x^2, \dots, x^n$. Then there is a unique symmetric linear connection ∇ in U with components satisfying $\Gamma^h_{11} = 0$ for $h = 1, \dots, n$ (i.e. the given chart is pre-semigeodesic w.r.t. ∇) such that $\nabla|S = \tilde{\nabla}$, and $R^h_{j1k} = A^h_{jk}$ in U.*

2.5 On systems of partial differential equations of Cauchy type

Diffeomorphisms and automorphisms of generalized geometric spaces constitute one of contemporary actual directions in differential geometry. A large number of works is devoted to isometric, homothetic, conformal, affine, geodesic, quasigeodesic, holomorphically projective, almost geodesic, F-planar and other mappings, transformations and deformations.

Obviously the existence of a solution of fundamental equations imply the existence of the mentioned mappings, transformations and deformations. These fundamental equations were found in several forms. Among these forms there is a particularly important form of a *system of partial differential equations of Cauchy type*. In the linear case, the question of solvability of the system can be answered by algebraic methods. Investigation of such systems has many aspects concerning existence and unicity of a solution, differentiability of functions under consideration, local and global properties of solutions.

2.5.1 Systems of PDEs of Cauchy type in $\mathbb{R}^n$

Here we introduce the basic notions of the theory of systems of differential equations of Cauchy type. We restrict ourselves to the local theory.

Assume a convex domain $D \subset \mathbb{R}^n$ with coordinates $x = (x^1, x^2, \ldots, x^n)$ and functions $F_i^A(x, y)$, $i = 1, \ldots, n$; $A = 1, \ldots, N$, on $\tilde{D} \subset D \times R^N$. Suppose the functions $F_i^A(x, y)$ are continuous with respect to x and differentiable with respect to y in the domain $\tilde{D}$.

A system of *differential equations of Cauchy type* has the form

$$\frac{\partial y^A(x)}{\partial x^i} = F_i^A(x, y(x)), \qquad A\,,\ B = 1, \ldots, N,\ \ i = 1, \ldots, n, \tag{2.50}$$

where $y(x) = (y^1(x), \ldots, y^N(x))$ are unknown functions.

For *initial data* (= *initial Cauchy conditions*)

$$y^A(x_0) \ = \ y_0^A, \quad A = 1, \ldots, N, \tag{2.51}$$

where $x_0 \in D$ and $(x_0, y_0^A) \in \tilde{D}$, the system (2.50) has at most one solution

$$y^A = y^A(x^1, \ldots, x^n) \tag{2.52}$$

in the class C^1 such that $(x, y(x)) \in \tilde{D}$. For this reason the general solution of the system (2.50) depends on $r \leq N$ real parameters.

Let us suppose that $F_i^A(x, y) \in C^1(\tilde{D})$ and that a solution we are looking for satisfies $y^A(x) \in C^2(D)$. Then the integrability conditions of (2.50) read

$$\partial_{jk} y^A(x) = \partial_{kj} y^A(x) \tag{2.53}$$

and according to (2.50) $\partial_k(F_j^A(x, y(x))) = \partial_j(F_k^A(x, y(x)))$ which we can write as

$$\partial_k F_j^A(x, y) + \partial_B F_j^A(x, y)\, \partial_k y^B - \partial_j F_k^A(x, y) - \partial_B F_k^A(x, y)\, \partial_j y^B = 0.$$

If we apply (2.50) we get

$$\partial_k F_j^A + \partial_B F_j^A \, F_k^B - \partial_j F_k^A - \partial_B F_k^A \, F_j^B = 0. \tag{2.54}$$

Here we denoted $\partial_k F_j^A = \dfrac{\partial F_j^A}{\partial x^k}$ and $\partial_B F_j^A = \dfrac{\partial F_j^A}{\partial y^B}$.

For any solution (2.52) of (2.50), the conditions (2.54) are satisfied identically for $x \in D$. Among others, these conditions must be satisfied for the initial data (2.51).

The conditions (2.54) are called *integrability conditions* of the system (2.50).

If the conditions (2.54) are identically satisfied for $x \in D$ then the system (2.50) is called *completely integrable*. In this case, the system has solution for any initial data (2.51), i.e. the general solution of (2.50) depends on N real parameters.

2.5.2 On mixed systems of PDEs of Cauchy type in $\mathbb{R}^n$

Suppose that the functions $y^A(x)$ satisfy, besides (2.50), also the additional equations

$$f^p(x^1, \ldots, x^n, y^1, \ldots, y^N) = 0, \quad p = 1, \ldots, m. \tag{2.55}$$

The functions $f^p(x, y)$ are defined in the domain $\tilde{D}$.

The system (2.50) and (2.55) is called *a mixed system of PDEs of Cauchy type*. If we study such a system we investigate the integrability conditions of (2.54) and (2.55) together, and denote them by (B) in short.

Let $F_i^A(x, y) \in C^{r+2}(\tilde{D})$ and $f^p(x, y) \in C^{r+1}(\tilde{D})$. Then differentiating step by step we get a system of differential prolongations (B_1), (B_2), …, (B_r). Denote $(B_0) \equiv (B)$. Now (B_{k+1}) is obtained from the system of conditions (B_k) by means of differentiating all equations by ∂_i, $i = 1, \ldots, n$. The conditions (B_0), (B_1), …, (B_r) must hold for the initial data (2.51).

The following theorem can be proved (Eisenhart [52], Finnikov [57], Rashevskij [150], Sinyukov [170] – here the theorem is formulated for analytic solutions)

Theorem 2.5 *In a neighbourhood of a point $x_0 = (x_0^i)$, a mixed system of PDEs (2.50) and (2.55) of Cauchy type has a unique solution (2.52) of the class C^{r+1}, which satisfies the initial conditions (2.51) if and only if the conditions (B_0), (B_1), …, (B_r) hold at the point (x_0, y_0), and r is the least integer for which (B_{r+1}) is a consequence of the system of all preceding prolongations.*

The system (2.50) may be written in terms of covariant derivatives. A fundamental investigation of (2.50) consists in checking of the integrability conditions, which are essentially algebraic equations for the unknown variables y^A. In the case when they are identically fulfilled, we have $r = N$.

2.5.3 On a mixed linear system of PDEs of Cauchy type in $\mathbb{R}^n$

The systems mentioned in the previous subsection are of particular importance in case they are linear. In such a case, the equations (2.50) and the conditions (2.55) read

$$\frac{\partial y^A(x)}{\partial x^i} = F^A_{iB}(x) \cdot y^B(x) + F^A_i(x), \tag{2.56}$$

$$f^p_B(x) \cdot y^B + f^p(x) = 0, \tag{2.57}$$

where $F^A_{iB}(x)$, $F^A_i(x)$, $f^p_B(x)$, $f^p(x)$ are functions on D.

The integrability conditions of (2.56) are linear algebraic equations with respect to y^A and obviously, they have the form of the equations (2.57):

$$\begin{aligned}&(\partial_k F^A_{jB} - \partial_j F^A_{kB} F^A_{jC} F^C_{kB} - F^A_{kC} F^C_{jB})\, y^B \\ &+ \partial_k F^A_j - \partial_j F^A_k + F^A_{jC} F^C_k - F^A_{kC} F^C_j = 0.\end{aligned} \tag{2.58}$$

When $F^A_{iB}, F^A_i \in C^{r+2}(D)$ and $f^p_B, f^p \in C^{r+1}(D)$ there exist conditions $(B_0) \equiv (B)$ and their differential prolongations $(B_1), \ldots, (B_r), (B_{r+1})$ that all are linear algebraic equations in the unknown functions $y^A(x)$, with coefficients being certain functions of the variable $x \in D$.

Obviously, Theorem 2.5 is satisfied, and consequently, the problem of solvability of the linear system (2.56) and (2.57) can be decided by analysis of the linear algebraic equations (B_0), (B_1), ...

2.5.4 Mixed PDEs in tensor form

In applications of the theory of PDEs' the equations (2.50) and (2.56) are often written in tensor form.

Let $D \subset \mathbb{R}^n$ be a coordinate domain of $\mathbb{A}_n$ with a linear connection ∇. A system of PDEs of Cauchy type in covariant derivatives (with respect to the affine connection ∇) of m unknown tensor fields $\underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma}}(x)$, $\sigma = 1, \ldots, m$, of type (p_σ, q_σ) takes the form

$$\begin{aligned}&\underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma}, k}(x) = \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} k}(x, \underset{1}{Y}, \ldots, \underset{m}{Y}), \\ &i_1, i_2, \ldots, i_{p_\sigma}, j_1, j_2, \ldots, j_{q_\sigma}, k = 1, 2, \ldots, n.\end{aligned} \tag{2.59}$$

On the right-hand side of (2.59) there are tensor functions of type (p_σ, q_σ) constructed in a certain way by means of a finite number of tensor operations from unknown tensor fields $\underset{\sigma}{Y}$ and also from components of certain known objects, including the linear connection ∇. The integrability conditions of (2.59) follow from the Ricci identities for tensors $\underset{\sigma}{Y}$ which follow by the conditions (2.52):

$$\begin{aligned}&\underset{\sigma}{Y}{}^{i_1 \cdots i_{p_\sigma}}_{j_1 \cdots j_{q_\sigma}, [lm]} \equiv \underset{\sigma}{Y}{}^{\alpha i_2 \cdots i_{p_\sigma}}_{j_1 \cdots j_{q_\sigma}} R^{i_1}_{\alpha lm} + \cdots + \underset{\sigma}{Y}{}^{i_1 \cdots i_{p_\sigma - 1} \alpha}_{j_1 \cdots j_{q_\sigma}} R^{i_{p_\sigma}}_{\alpha lm} \\ &- \underset{\sigma}{Y}{}^{i_1 \cdots i_{p_\sigma}}_{\alpha j_2 \cdots j_{q_\sigma}} R^{\alpha}_{j_1 lm} - \cdots - \underset{\sigma}{Y}{}^{i_1 \cdots i_{p_\sigma}}_{j_1 \cdots j_{q_\sigma - 1} \alpha} R^{\alpha}_{j_{q_\sigma} lm} - \underset{\sigma}{Y}{}^{i_1 \ldots i_{p_\sigma}}_{j_1 \ldots j_{q_\sigma}, \alpha} S^{\alpha}_{lm} \\ &= \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} l, m} - \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} m, l}.\end{aligned} \tag{2.60}$$

Hence the integrability conditions are written in tensor form.

2.5.5 On systems of PDEs of Cauchy type in manifolds

It is a question which of the results being formulated for PDEs in $\mathbb{R}^n$ remain valid, or can be generalized, for an arbitrary n-dimensional manifold M_n.

The theorem on existence of a solution is of local character, there are examples of equations of type (2.50) that are defined on the whole M_n which admit a local solution in a coordinate neighbourhood of each point, but no global solution exists. Such counterexamples to global metrizability can be found in [354] and [701] where, among others, the aim is to find a global solution of the equation $\nabla g = 0$, that is,

$$\partial_k g_{ij} = g_{i\alpha}\Gamma^{\alpha}_{jk} + g_{j\alpha}\Gamma^{\alpha}_{ik},$$

where Γ^h_{ij} are components of a linear connection and g_{ij} are components of a metric we are searching for.

Of course, uniqueness properties are guaranteed even in this general case.

Reformulate the problem on PDEs on a manifold M_n:
Let us give an n-dimensional manifold M_n and a geometric object $y \in C^1$ defined on it, which is given by a set of N functions $y^A(x)$, $A = 1, \ldots, N$ in each local chart $(U_\alpha, \varphi_\alpha)$. A first order system of PDEs of Cauchy type on M_n with respect to a geometric object y can be given in the following way. In any coordinate domain, a system of differential equations (2.50) is given, where on the right-hand sides, there are expressions depending on chart coordinates x as well as on coordinates of the geometric object y. Of course, on each chart, the differential system can be completed by additional requirements of type (2.54), and a mixed system arises. By step-by-step integration, accounting uniqueness conditions in each coordinate chart $(U_\alpha, \varphi_\alpha)$ (that is, demanding differentiability with respect to x and y^A), we check that there exists at most one global solution.

There might exist local solutions in each chart $(U_\alpha, \varphi_\alpha)$ while no global solution exists. In principle, on an overlapping $U \cap V$ of coordinate domains, there are two possibilities.

Either a solution existing in U might coincide with some solution found in V; then we can extend the solution to $U \cup V$. Of course, a solution which can be step-by step extended (in a obvious way) onto each overlap coordinate neighbourhood, has a global character.

Or else, there exists U such that none of the possible solutions found in U can be "glued together" with some solutions being found on overlapping neighbourhoods to represent a solution on the correpsonding union domain. Then the solution cannot exist globally.

A global solution y of the linear system has the usual properties.

2.5.6 Application

Many problems of differential geometry have been successfully solved by means of homogeneous systems of linearly differential equations of Cauchy type, for example:

- isometric and homothetic transformations of Riemannian spaces,
- affine and projective transformations of Riemannian spaces and manifolds with affine connection,
- holomorphically projective transformations of Kähler spaces,
- affine mappings of Riemannian spaces and manifolds with affine connection.

The above results were found during the years 1900 – 1960 and presented in many monographs and research papers by Aminova [8, 11, 254, 261], É. Cartan [30], do Carmo [28, 29], I.P. Egorov [45, 46], Eisenhart[50–52], Fecko [56], Fubini [422], Helgason [69], Kobayashi, Nomizu [90], Mikeš [118, 119], Mikeš, Kiosak, Vanžurová [121], Mikeš, Vanžurová, Hinterleitner [122], Norden [135], Penrose, Rindler [137], Petrov [139, 140], Radulović, Mikeš, Gavrilchenko [149], Rashevskij [152], Rund [156], Schouten, Struik [163], Sinyukov [170, 171], Vranceanu [933], K. Yano [197], K. Yano, Bochner [200] etc.

Now we want to introduce new results which were obtained in the last 40 years and are connected whith the systems of Cauchy type. This means that for the mentioned types of geometrical problems, regular methods of solution were found for:

- geodesic mappings of Riemannian spaces
  (Sinyukov [817], 1966, see [118, 149, 170]; Sec. 8.1, p. 297),
- geodesic mappings of manifolds with affine (projective) connection onto Riemannian spaces (Berezovsky and Mikeš [651], 1989, see [118]; Hinterleitner, Mikeš [480, 482], 2008; Sec. 7.1.2, 7.2, pp. 275-279),[43)]
- geodesic mappings of Kähler manifolds
  (Mikeš [226, 227, 630, 631, 680], 1979–2002 see [118]; Sec. 7.3, pp. 340-343,
- geodesic mappings of hyperbolic and parabolic Kähler manifolds
  (Mikeš, Shiha, Starko, etc. [227, 235, 680, 693, 802], 1989–2002, see [118]; Sec. 7.3, pp. 139-140),
- geodesic mappings of weakly Berwald spaces and Berwald spaces onto Riemannian spaces (Mikeš, Bácsó, Berezovski [648], 2008),
- geodesic deformation of Riemannian hypersurfaces in Riemannian spaces (Gavrilchenko, Mikeš, etc. [227, 429], see [121]),

[43)] The result by Eastwood and V.M. Matveev [389] follows from previously mentioned result using normal connection by È. Cartan [322] and J.M. Thomas [891], see [51, p. 105].

- conformal mappings of Riemannian spaces onto Einstein spaces (Gavrilchenko, Gladysheva and Mikeš [660, 661], 1992, see [119]),
- holomorphically projective mappings of Kähler spaces (Domashev and Mikeš [382, 628], 1976, 1980, see [119, 170]),
- holomorphically projective mappings onto Kähler spaces (Chodorová (Škodová), Mikeš and Pokorná [882], 2005),
- holomorphically projective mappings of hyperbolical Kähler spaces (Kurbatova [569], 1980, see [119, 173]),
- holomorphically projective mappings of parabolical Kähler spaces (Shiha [799], 1992, see [119, 235]),
- almost geodesic mappings of type π_1 (Berezovsky, Mikeš, Vanžurová [301]),
- almost geodesic mappings of type π_2 (Chudá (Vavříková), Mikeš, etc. [911]),
- F-planar mappings of manifolds with affine connection onto Riemannian spaces (J. Mikeš [644, 646], 1994, 1999, see [119]),
- infinitesimal F-planar transformations of manifolds with affine connection onto Riemannian spaces (Hinterleitner, Mikeš, Stránská [489, 632], 2008).

3 RIEMANNIAN AND KÄHLER MANIFOLDS

3.1 Riemannian manifolds $\mathbb{V}_n$, i.e. Riemannian and pseudo-Riemannian manifolds

3.1.1 Riemannian metric

What is now called a Riemannian geometry, was a natural development of the differential geometry of surfaces in $\mathbb{R}^3$. If $S \subset \mathbb{R}^3$ is a surface we have a natural way how to measure length of tangent vectors to S: the inner product of two tangent vectors is their inner product in $\mathbb{R}^3$, usually calculated with respect to a basis of the 2-dimensional tangent space; the inner product at each point $p \in S$ of the surface yields a quadratic form, the so-called first fundamental form of S at p, which works on the tangent space of S at p. The progress was based, among others, on important observations made by Gauss[44)] and published 1827, [61], concerning curvature of surfaces (roughly, Gauss curvature of a surface measures how much S deviates, at a given point, from the tangent plane in it).

In 1854, B. Riemann[45)] delivered his lecture in Göttingen which opened a new area in Geometry. He not only founded a new field of geometry, called Riemannian geometry nowadays, and contributed to the theory of high-dimensional spaces, but also set the stage for Einstein's general relativity.[46)] The idea of Riemannian spaces found wide applications in mechanics, electrodynamics, theoretical physics etc.

[44)] Johann Karl Friedrich Gauss, 1777-1855, was a German scientist and mathematician who contributed to many fields, including analysis, statistics (Gaussian distribution), geodesy and differential geometry (Gaussian curvature, Theorema Egregium), number theory (an author of Disquisitiones Arithmeticae, finished 1798, published 1801), optics, astronomy, electrostatics.

[45)] Georg Friedrich Bernhard Riemann, 1826-1866, was a German mathematician who made important contributions to differential geometry (Riemannian spaces), analysis (Riemann integral) and number theory (Riemann zeta function). In 1853, Gauss asked his student Riemann to prepare a Habilitation Thesis on the foundations of geometry. It was entitled *Über die Hypothesen welche der Geometrie zu Grunde liegen.* Riemann found the correct way how to extend into n dimensions the differential geometry of surfaces, which also Gauss was interested in (Theorema Egregium). The fundamental object is the so-called Riemannian curvature tensor. For the case of surfaces, this can be reduced to a scalar (number), positive, negative or zero; the non-zero constant cases serve as models of non-Euclidean geometries. Riemann held his first lectures in 1854. In 1857, there was an attempt to promote Riemann to extraordinary professor status at the University of Göttingen, which failed. But his Habilitation Thesis was a starting point of a new approach to geometry, among others, to non-Euclidean geometries. A generalization of Riemannian spaces was later used by Einstein for investigations of the space-time and development of general relativity.

[46)] Albert Einstein, 1879-1955, a German-born theoretical physicist, known e.g. for his theory of relativity, mass-energy equivalence $E = mc^2$ and other discoveries (the law of photoelectronic effect). He became Professor at Zurich 1909, at Prague 1911, since 1914 he was appointed Director of the Kaiser Wilhelm Physical Institute and Professor at the University of Berlin. He was awarded the 1921 Nobel Prize in Physics. In 1933, he emigrated to the USA to take the position of a Professor of Theoretical Physics at Princeton.

Definition 3.1 A *metric tensor* g on a (differentiable) manifold M_n is a symmetric nondegenerate $(0,2)$ tensor field on M. A pair $\mathbb{V}_n = (M, g)$ formed by an n-dimensional manifold and a metric tensor on it is called a *Riemannian manifold*, or a *Riemannian space*.

If g is positive definite, we say that $\mathbb{V}_n$ is a *classical Riemannian space* or *proper Riemannian space*, if g is indefinite we call $\mathbb{V}_n$ *pseudo-Riemannian*.

We say that the Riemannian space $\mathbb{V}_n$ is *of the class* C^r, $\mathbb{V}_n \in C^r$, if $g \in C^r$.

In local coordinates (x^i), a metric tensor g is given by its components $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$; $g = g_{ij}\, dx^i \otimes dx^j$, or, in the classical notation, a *metric form* has the following shape:[47)]

$$ds^2 = g_{ij}(x)\, dx^i dx^j.$$

If a point $p \in M$ is fixed, then the metric g defines a symmetric bilinear form g_p on the tangent space T_pM; the nondegeneracy condition means that for each $p \in M$, if $g_p(X, Y) = 0$ for all $Y \in T_pM$ then $X = 0$. As well known, a symmetric bilinear form g_p is nondegenerate if and only if its matrix relative to (any) basis of T_pM, $p \in M$ is invertible (i.e. has non-vanishing determinant). Symmetry means $g_p(X, Y) = g_p(Y, X)$ for all $X, Y \in T_pM$, $p \in M$. The *index* of g_p is the largest integer q that is the dimension of a subspace $W \subset T_pM$ on which the restriction $g_{p|W}$ is negative definite. Note that the index of a metric is constant (q is the same for all g_p, $p \in M$).

A metric g is positive definite if and only if $q = 0$ for all $p \in M$.

The quadratic form g_2 on M corresponding to g is given by $(g_2)_p \colon X \mapsto g_p(X, X)$, $X \in T_pM$, $p \in M$. Around any point, there exist coordinates on M such that (with respect to the coordinate frame) the quadratic form $(g_2)_p$ is (just in this fixed point p) given by the classical formula

$$ds^2 = (dx^1)^2 + \ldots + (dx^{n-q})^2 - (dx^{n-q+1})^2 - \ldots - (dx^n)^2. \qquad (3.1)$$

Remark 3.1 In the n-dimensional *Euclidean space* the metric has the form $ds^2 = {dx^1}^2 + {dx^2}^2 + \ldots + {dx^n}^2$; a metric form of the shape (3.1) characterizes *pseudo-Euclidean spaces*; particulary, in the case $q = 1$, *Minkowski spaces*, or *Lorentzian spaces*.

If (3.1) is satisfied in a neighbourhood U of any point $p \in \mathbb{V}_n$, then the space is called *locally Euclidean, pseudo-Euclidean,* or *Minkowski (Lorentzian)*, respectively. A coordinate system in which (3.1) holds is usually called *Cartesian*. As we show later on, these spaces are flat.

The question whether there is a Riemannian metric g on any manifold M is answered by the following Theorem.

Theorem 3.1 (On the existence of a metric on manifolds) *On any differentiable manifold M there exists a metric tensor g.*

[47)] Correctly: $ds^2 = e\, g_{ij}(x)\, dx^i dx^j$, where $e = \pm 1$ and the sign is chosen so that $ds^2 \geq 0$.

Proof. For each point $x \in M$ we choose one of the charts and denote it by $(U_x, \varphi_x = (x^i))$. In the domain of this chart we choose a Euclidean metric $\tilde{g}_x$, that is, the coordinates (x^i) are "Cartesian". Thus in each coordinate domain U_x a Euclidean metric $\tilde{g}_x$ is defined. By means of the unit decomposition described on page 50 we "glue" the above locally defined metrics together using the fact that any separable topological space is paracompact and normal [64]. As a consequence of the paracompactness, for any open covering there is a locally finite open subcovering. Hence we can suppose that the domains U_x already belong to a locally finite atlas. Hence the system $\{U_x\colon\ x \in M\}$ is a locally finite covering of the normal manifold M.

So there exists a system of functions $\{f_x\colon\ x \in M\}$ such that $\operatorname{supp} f_x \in U_x$, and they satisfy the conditions for a unit decomposition. We define $g_x(y) = f_x(y)\,\tilde{g}_x(y)$ for $y \in U_x$ and $g_x = 0$ othewise on M. Finally, we set $g = \sum\limits_{x\in M} g_x$. Obviously, g is a metric on M.

If $M \in C^r$ then we can achieve $g \in C^{r-1}$, i.e. the resulting Riemannian space $\mathbb{V}_n \in C^{r-1}$. □

3.1.2 Length of vector and arc, angle and volume

For $X, Y \in T_pM$, the value $g(X,Y)$ is a *scalar product* of tangent vectors. A *length* of a vector X is defined by $\|X\| = \sqrt{|g(X,X)|}$. A non-zero vector of a zero length is called *isotropic* or *null*. Isotropic vectors exist in pseudo-Riemannian spaces only.

Two vectors X, Y are *orthogonal* whenever $g(X,Y) = 0$. The expression $\cos\varphi$ for non-isotropic vectors X and Y is defined by the formula

$$\cos\varphi = \frac{g(X,Y)}{\|X\|\,\|Y\|} \tag{3.2}$$

and in the classical Riemannian case, it satisfies the inequality $-1 \le \cos\varphi \le 1$ (if the manifold is positive Riemannian and $\varphi \in \langle 0, 2\pi)$ we call φ the *angle* between the vectors X and Y).

The length of the arc of a (differentiable) curve $\ell\colon\ x^h = x^h(t)$, $t \in \langle a, b\rangle$ is given by the Riemann integral

$$|\ell| = \int_a^b \left\| \frac{dx^h(t)}{dt} \right\| dt. \tag{3.3}$$

The arc length ℓ is independent of its regular parametrization.

Let $\mathcal{D} \subset U_\alpha$ be an open domain in $\mathbb{V}_n$. Under a *volume of the domain* $\mathcal{D}$ we understand

$$W(\mathcal{D}) = \int_{\mathcal{D}} \cdots \int \sqrt{|\det \|g_{ij}\| \,|}\; dx^1 \cdots dx^n. \tag{3.4}$$

3.1.3 Isometric diffeomorphisms

For the purpose of investigation and classification of mathematical structures we need a criterion deciding when the structures are considered "the same".

Definition 3.2 Let $\mathbb{V}_n = (M_n, g)$ and $\overline{\mathbb{V}}_n = (\overline{M}_n, \overline{g})$ be Riemannian spaces. A diffeomorphism $f\colon M \to \overline{M}$ is called an *isometry* if the lenghts of all arcs are preserved under f.

A diffeomorphism $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ is an isometry if and only if it preserves the metric tensor, $f^*(\overline{g}) = g$, that is, iff

$$g_p(X,Y) = \overline{g}_{f(p)}(Tf_pX, Tf_pY) \quad \text{for all} \quad p \in M,\ X, Y \in T_pM. \tag{3.5}$$

If this condition is satisfied in some neighbourhood of p we speak of *local isometry* at p.

If $f\colon x \mapsto \overline{x}$ is a diffeomorphism of a coordinate neighbourhood $U \subset M_n$ onto a coordinate neighborhood $\overline{U} \subset \overline{M}_n$ such that the points x and $\overline{x}$ have the same coordinates, i.e. $\overline{x}^h = x^h$, then (3.5) takes the form

$$\overline{g}_{ij}(x) = g_{ij}(x). \tag{3.6}$$

Note that the formulae (3.5), (3.6), as well as their modifications, hold up to a $\pm$ sign. That is, locally isometric Riemannian spaces have "equal" metrics, $d\overline{s}^{\,2}(x) = ds^2(x)$. We do not distinguish between two isometric Riemannian spaces.

3.1.4 Levi-Civita connection and Riemannian tensor

A symmetric connection ∇ on $\mathbb{V}_n$ is called *Levi-Civita* if $\nabla g = 0$. This connection is also called *natural*.

For any $\mathbb{V}_n \in C^1$ there exists a unique Levi-Civita affine connection the components of which on every coordinate neighborhood are $\Gamma^h_{ij} = g^{hk}\Gamma_{ijk}$, the so-called Christoffel symbols of the second type, where

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \tag{3.7}$$

are the so-called Christoffel symbols of the first type, and (g^{ij}) is the inverse of the matrix (g_{ij}), $g^{is}g_{sj} = \delta^i_j$. That is why the Riemannian space $\mathbb{V}_n$ can be viewed as a particular manifold with affine connection, endowed with a natural linear connection. In terms of a natural connection, the curvature tensor and the Ricci tensor are given by the formulae (2.15) and (2.18), respectively. In a Riemannian manifold, the curvature tensor is more often called the *Riemannian tensor* or *Riemannian curvature tensor*.

In the Riemannian space $\mathbb{V}_n$, a type $(0,4)$ tensor field is introduced by:

$$R(V,Z,X,Y) = g\big(V, R(X,Y)Z\big),$$

for any $V, X, Y, Z \in \mathcal{X}(\mathbb{V}_n)$. In the coordinate form,

$$R_{hijk} = g_{h\alpha}R^{\alpha}_{ijk}. \tag{3.8}$$

This tensor field is also called *Riemannian (of* 1^{st} *type)* and has the following properties:

$$\begin{aligned}
&\text{(a)} \qquad R_{hijk} + R_{ihjk} = 0, \quad R_{hijk} + R_{hikj} = 0, \quad R_{hijk} = R_{jkhi}\,,\\
&\text{(b)} \qquad R_{hijk} + R_{hjki} + R_{hkij} = 0,\\
&\text{(c)} \qquad R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0,\\
&\text{(d)} \qquad R_{hijk,[lm]} + R_{jklm,[hi]} + R_{lmhi,[jk]} = 0.
\end{aligned} \tag{3.9}$$

The formula (c) is the Bianchi identity and (d) is the Walker identity.

In Riemannian spaces, the so-called *Ricci tensor (Ricci form)* is expressed by the formula

$$\mathrm{Ric}\,(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i),$$

where $\{e_1,\dots,e_n\}$ is an arbitrary orthonormal basis in $\mathcal{X}(\mathbb{V}_n)$. In a coordinate form, the Ricci form is given by the formulae $R_{ij} = R^{\alpha}_{i\alpha j} = R_{\alpha i\beta j}g^{\alpha\beta}$.

This form is symmetric in $\mathbb{V}_n$. Hence a Riemannian space is equiaffine, and satisfies the so-called Voss–Weyl formula derived below, (3.11).

Let us mention a preliminary lemma which is useful in further considerations.

Lemma 3.1 *If* $\mathcal{G} = \det(g_{ij})$ *is a (functional) determinant of a tensor field* g *of the type* $(0,2)$ *then the following formula holds for the partial derivative:*

$$\partial_k \mathcal{G} = \mathcal{G}\ g^{ij}\partial_k g_{ij} \tag{3.10}$$

where g^{ij} *are components of the inverse matrix to* g_{ij}*.*

Proof. For $n = 2, 3$, the verification is immediate. For arbitrary $n \geq 2$, we proceed by induction. □

That is, a partial derivative of a (functional) determinant of n-th order is equal to the sum of n determinants of n-th order each of which is obtained from the original one by exchange of one particular row by an analogous row formed by partial derivatives of the original entries. Accounting $\Gamma_{ikj} + \Gamma_{jki} = \partial_k g_{ij}$, (3.10) reads

$$\frac{1}{\mathcal{G}} \cdot \partial_k \mathcal{G} = g^{ij}\,(\Gamma_{ikj} + \Gamma_{jki}) = 2\,\Gamma^{i}_{ik}.$$

The formula is often written in the form

$$\Gamma^{\alpha}_{\alpha k} = \partial_k \ln \sqrt{|\mathcal{G}|}\,, \tag{3.11}$$

which is known as *the formula of Voss–Weyl*.

In a Riemannian manifold $\mathbb{V}_n$ we define a *scalar curvature* R by

$$R = R_{\alpha\beta}g^{\alpha\beta}. \tag{3.12}$$

Note that in the classical sources the scalar curvature is usually denoted R, and we will follow this convention here. Although we use the symbol R also for the

Riemannian tensor, there is no danger of confusion since their roles in formulae differ significantly.

Contraction of the Bianchi identity (3.9) with g^{hl} followed by contraction with g^{jk} gives subsequently

$$(a)\quad R^{\alpha}_{ijk,\alpha} = R_{ik,j} - R_{ij,k} \quad \text{and} \quad (b)\quad R^{\alpha}_{i,\alpha} = \frac{1}{2} R_{,i}. \tag{3.13}$$

Recall that in Riemannian manifolds $\mathbb{V}_n$, besides the already mentioned Riemannian and Ricci tensors, we introduce by means of the metric g also the so-called Weyl tensor of projective curvature W, see p. 266, the Weyl tensor of conformal curvature C, see p. 239, and the Yano tensor of concircular curvature Y, see p. 248. These tensors play the central role in many problems of Riemannian geometry, particularly in connection with geodesic, conformal and concircular mappings, respectively.

3.1.5 Parallel transport and geodesics

In Riemannian manifolds $\mathbb{V}_n$ the concept of parallel transport of vector and tensor fields is introduced in the way described in Sections 2.4.5, p. 86, as parallel propagation with respect to the natural (Levi-Civita) connection.

The metric tensor g is absolutely parallel in $\mathbb{V}_n$. That is why the scalar product of two parallel vector fields along a fixed curve is constant. This property implies that the length of a parallel vector field is constant, and the cosine of the angle of two parallel vector fields is preserved.

Geodesics in $\mathbb{V}_n$ are defined by Definition 2.16, p. 88. In Riemannian spaces, geodesics are endowed with further characteristic properties. The most important is fact that a geodesic is an extremal curve of the arclength functional, detailed see in Section 3.4, p. 122.

If the parameter on a geodesic is the arc length s then the tangent vector λ along the curve satisfies $g(\lambda, \lambda) = e = \pm 1$. The equations of geodesics parametrized by arclength parameter s read $\nabla_\lambda \lambda = 0$. The fact that (locally) the length of a geodesic is minimal is proved in the Theorem 3.2, p. 119.

Based on the properties of the parallel transport fact follows that the tangent vector of geodesic γ is isotropic ($g(\lambda, \lambda) = 0$) along the geodesic curve. That means if tangent vector is isotropic at a single point of γ, then the tangent vectors of γ are isotropic at every point of γ. In this case we call γ an *isotropic geodesic*.

3.2 Special Riemannian manifolds

3.2.1 Subspaces of Riemannian spaces

There are situations when an n-dimensional manifold N_n is imbedded into an m-dimensional Riemannian manifold $\mathbb{V}_m = (M_m, g)$, that is, there is a differentiable mapping $\Phi\colon N_n \to M_m$ such that $T\Phi_x\colon T_xN \to T_{\Phi(x)}M$ is injective for all $x \in N$; $m - n$ is called the *codimension* of N_n.

In local coordinates (x^i) of a local chart $(U, \varphi = (x^i))$ about a point $p \in N_n$ and (y^I) of a local chart $(V, \psi = (y^I))$ about the image $\Phi(p) \in N_n$, we can express the submanifold N_n by (2.1): $y^I = y^I(x^1, \dots, x^n)$, $I = 1, \dots, m$, $\det(\partial y^I/\partial x^i) \neq 0$. Let us set

$$\tilde{g}_{ij}(x) = g_{IJ}(y(x))\frac{\partial y^I}{\partial x^i}\frac{\partial y^J}{\partial x^j}, \quad i, j = 1, \dots, m \tag{3.14}$$

where g_{IJ} are components of g. In this way, a geometric object $\tilde{g}$ is correctly defined in N_n. We can easily check that $\tilde{g}$ is a type $(0, 2)$ tensor field.

In what follows let us suppose that $\tilde{g}$ is regular; note that in classical (positive) Riemannian spaces, this property automatically follows.

If we choose $\tilde{g}$ to be a metric then N_m turns out to be a Riemannian space $\tilde{\mathbb{V}}_n = (N_n, \tilde{g})$ with the metric form

$$ds^2 = \tilde{g}_{ij}(x)dx^i dx^j.$$

We can prove elementarily that the length of an arc $\ell \subset N_n \subset M_m$ is the same in $\tilde{\mathbb{V}}_n$ as well as in $\mathbb{V}_m$, [28]. From this condition, the formulae (3.14) follow as necessary as well as sufficient ones.

Because of this property, the metric $\tilde{g}$ in N_n is called *induced*, or *natural*. In this case, $\tilde{\mathbb{V}}_n = (N_n, \tilde{g})$ will be called a *surface* of M_m, and we use the notation $\tilde{\mathbb{V}}_n \subset \mathbb{V}_m$.

As a consequence of (3.14), the angle of the curves ℓ_1 and ℓ_2 in N_n is the same in $\tilde{\mathbb{V}}_n$ and in $\mathbb{V}_m$. It follows that a geodesic $\ell \subset N_n$ with respect to $\mathbb{V}_m$ is a geodesic also in $\tilde{\mathbb{V}}_n$. The converse is not true in general.

A surface $\tilde{\mathbb{V}}_n = (N_n, \tilde{g})$ is called *geodesic* in a point $p \in N_n$ if any geodesic of $\tilde{\mathbb{V}}_n$ passing through p is a geodesic in $\mathbb{V}_m$, and a surface $\tilde{\mathbb{V}}_n$ is called *totally geodesic* if any geodesic of $\tilde{\mathbb{V}}_n$ is a geodesic in $\mathbb{V}_m$.

It is known that any Riemannian manifold $\tilde{\mathbb{V}}_n = (N_n, \tilde{g})$ can be realized as an n-dimensional surface in an m-dimensional (pseudo-) Euclidean space where $m \leq \frac{n(n+1)}{2}$. In this case, the difference $m-n$ is called the *class* of a Riemannian manifold.

It follows as a consequence that $\tilde{\mathbb{V}}_2$ can be realized as a surface $\tilde{\mathbb{S}}_2$ of a 3-dimensional Euclidean space. The Gaussian curvature $\tilde{K}$ of $\tilde{\mathbb{S}}_2$ is given by

$$\tilde{K} = \frac{\tilde{R}_{1212}}{\tilde{g}_{11}\tilde{g}_{22} - \tilde{g}_{12}^2}$$

where $\tilde{R}_{1212}$ are the componets of the Riemannian tensor (3.8) calculated with respect to the metric $\tilde{g}$.

3.2.2 Sectional curvature and Spaces of constant curvature

In a Riemanian space $\mathbb{V}_n$ we introduce the concept of sectional curvature.

Given a two-dimensional subspace $\sigma \in T_pM$ at $p \in M$, the *sectional curvature* at a point p with respect to the two-dimensional direction σ is the Gauss curvature at the point p with respect to the two-dimensional surface $\mathbb{S}^2$ which is geodesic at p and tangent to σ at p.

For a pair of linearly independent vectors[48] $X, Y \in \sigma$

$$K_p(\sigma) = K_p(X,Y) = \frac{R(X,Y,X,Y)}{g(X,X)\cdot g(Y,Y) - g(X,Y)^2}. \tag{3.15}$$

Its importance comes from the fact that the knowledge of $K_p(\sigma)$, for all σ, determines the curvature R completely.

Definition 3.3 A space $\mathbb{V}_n$ such that the sectional curvature $K_p(\sigma)$ is independent of the choice of the two-dimensional space σ as well as of the point p is called *a space of constant curvature*.

Remark that Riemann started to investigate spaces of constant curvature in 1854, [154].

Spaces of constant curvature K are characterized by the conditions on curvature

$$R(Y,Z)X = K\cdot\big(g(X,Z)\cdot Y - g(X,Y)\cdot Z\big) \quad \forall X,Y,Z \in \mathcal{X}(V_n), \tag{3.16}$$

in local notation

$$R^h_{ijk} = K\,(g_{ik}\delta^h_j - g_{ij}\delta^h_k). \tag{3.17}$$

Contracting the formula (3.17) with respect to the indices h and k we get for the Ricci tensor $R_{ij} = \varrho\, g_{ij}$, $\varrho = K(n-1)$. This formula characterizes the so-called *Einstein spaces* (p. 115). Hence spaces of constant curvature are Einstein spaces. The scalar curvature $R = K\,n(n-1)$.

Using the covariant derivative of formula (3.17) we get $\nabla R = 0$, i.e. spaces of constant curvature are symmetric (in the *sense of É. Cartan*), see p. 87.

A space of constant curvature with $K = 0$ has zero Riemannian tensor, $R = 0$, hence it is flat. Calculations in a local Cartesian coordinate system (see Remark 3.1, p. 108) of a locally (pseudo-) Euclidean space show that $R = 0$, i.e. the Riemannian tensor vanishes. On the other hand, any flat manifold is a locally (pseudo-) Euclidean space.

As it is shown e.g. in [354], there are examples of a globally defined flat connection for which no globally defined metric can be found.

Two Riemannian manifolds $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ of constant curvatures K and $\overline{K}$, respectively, are locally isometric if and only if $K = \overline{K}$, and their metrics have the same signature, see Theorem 4.8, p. 195.

[48] It can be checked that the definition is correct, i.e. independent of a particular choice of a basis in σ.

The class of Riemannian manifolds, see p. 113, for a (pseudo-) Euclidean space $\mathbb{E}_n$ is equal to 0, and for spaces of constant curvature it is equal to 1. The spaces mentioned above are realized by spheres $\mathbb{S}_n$, or by pseudospheres, respectively, in (pseudo-) Euclidean spaces $\mathbb{E}_{n+1}$.

Let us mention here one of the well-known forms of the metric of spaces with constant curvature K, namely

$$ds^2 = \frac{e_1\, dx^{1\,2} + \cdots + e_n\, dx^{n\,2}}{\left[1 + \frac{K}{4}\,(e_1\, x^{1\,2} + \cdots + e_n\, x^{n\,2})\right]^2}, \quad e_i = \pm 1. \tag{3.18}$$

Remark 3.2 Note that in the case that K is a (non-constant) function the formula (3.16) holds in $\mathbb{V}_2$. In the case of general $\mathbb{V}_n$, $n > 2$, the Schur Theorem tells that (3.16) is a necessary and sufficient condition for a Riemannian manifold to be of constant curvature, in this case K is constant.

3.2.3 Einstein spaces

Definition 3.4 A Riemannian space with the Ricci tensor proportional to the metric tensor is known as *Einstein space*.

Hence on an Einstein space $\mathbb{V}_n$ the condition $Ric = \varrho \cdot g$ holds, i.e.

$$\mathrm{Ric}\,(X, Y) = \varrho\, g(X, Y) \quad \forall X, Y \in \mathcal{X}(\mathbb{V}_n) \tag{3.19}$$

or in local transcription, $R_{ij} = \varrho\, g_{ij}$.

After contraction of (3.19) we obtain $\varrho = \dfrac{R}{n}$ and moreover, if a dimension $n > 2$, the scalar curvature R is constant.

Note that the formula (3.19) is always satisfied on $\mathbb{V}_2$. The spaces with constant sectional curvature are Einstein spaces, see p. 114. In other hand 3-dimensional Einstein spaces $\mathbb{V}_3$ are spaces with constant sectional curvature.

Many properties of Einstein spaces appear when $\mathbb{V}_n \in C^3$ and $n > 3$. Moreover, it is known (D.M. DeTurck and J.L. Kazdan [378]) that an Einstein space $\mathbb{V}_n$ belongs to C^ω, i.e., for all points of $\mathbb{V}_n$, there exists a local coordinate system x for which $g_{ij}(x) \in C^\omega$ (analytic coordinate system).

Einstein spaces provide simple, highly symmetric cosmological models. In general relativity the Einstein equations relate the curvature of space-time to the energy and momentum of all the matter present in space in the following way

$$R_{ik} - \frac{1}{2}\, R\, g_{ik} + \Lambda\, g_{ik} = G\, T_{ik}. \tag{3.20}$$

The first two expressions of the left-hand form the so-called Einstein tensor, According to a wide-spread interpretation representing space-time curvature, Λ is the cosmological constant, on the right-hand side the energy-momentum tensor of matter T_{ik}, multiplied by the gravitational constant G, describes the source of the gravitational field.

After insertion of (3.19) and $R = n\,\varrho$ we obtain

$$\left[\left(1 - \frac{n}{2}\right)\varrho + \Lambda\right] g_{ik} = G\,T_{ik}. \tag{3.21}$$

For $T_{ik} = 0$ this describes vacuum solutions with a nonzero cosmological constant. In this case, for $n > 2$, the factor of proportionality ϱ is determined by

$$\varrho = \frac{2\,\Lambda}{n-2}.$$

Here we make a remark about the cosmological constant: representing a repulsive vacuum energy density of the universe, it plays a crucial role in contemporary cosmology, for its value seems to determine the fate of our universe – eternal expansion or recollaps. Present observations indicate an accelerated expansion that would be in accordance with $\Lambda \approx +10^{-29} g/cm^3$.

3.2.4 Hypersurfaces

In many situations $(m-1)$-dimensional submanifolds S embedded into m-dimensional manifolds M are of interest. Such submanifolds are called hypersurfaces. The most common examples are three-dimensional spatial hypersurfaces of four-dimensional space-time. The tangent space of any point of M can be decomposed into an $(m-1)$-dimensional subspace tangent to S and one vector ν orthogonal to it, unique up to a sign and a normalizing factor. In a chart we can choose *adapted coordinates*, such that $\nu = c\,dx^1$, where c is a constant, and $(x^2, \ldots, x^n)$ are coordinates on S.

If g is a metric on M, the embedding induces a metric h on S, the so-called *first fundamental form* of S. According to the sign of $g_{ij}\nu^i\nu^j = \pm 1$ the components of the induced metric are

$$h_{ij} = g_{ij} \mp \nu_i\nu_j. \tag{3.22}$$

Obviously h_{ij} is a projection operator on the hypersurfaces, because $h_{ij}\nu^i\nu^j = 0$ and $h_{ik}h_{lj}g^{kl} = h_{ij}$. This is an interior quantity of S. The normal vector ν, on the contrary, characterizes the embedding of S in M.

When ν is extended in an arbitrary smooth way to a small neighborhood of S in M, this extension $\overline{\nu}$ can be covariantly differentiated. The projection of the covariant derivative of $\overline{\nu}$ to S,

$$K_{ij} = h_i^k\; h_j^l\; \overline{\nu}_{k,l}, \tag{3.23}$$

is called the *extrinsic curvature* or the *second fundamental form* of S.

In analogy to the manifold M, on the hypersurface S one can construct the Levi-Civita connection from the induced metric and, in the sequel, the Riemann tensor ${R'}^h_{ijk}$, the Ricci tensor ${R'}_{ij}$, and the scalar curvature R', describing the intrinsic curvature of S.

Between the curvature tensors of M, those of S, and the extrinsic curvature of S the following relations hold (*Gauss theorem*)

$$R'^{h}_{ijk} = R^{p}_{qrs}\, h^{h}_{p}\, h^{q}_{i}\, h^{r}_{j}\, h^{s}_{k} \pm K^{h}_{j}\, K_{ik} \mp K^{h}_{k}\, K_{ij}$$

and

$$R' = R \mp 2R_{ij}\nu^{i}\nu^{j} \pm (K^{i}_{i})^{2} \mp K^{ij}K_{ij}$$

with the same sign convention (dependence) as in (3.22). Another relation between the extrinsic curvature and the Ricci tensor of M is

$$K^{h}_{i;h} - K^{h}_{h;i} = R_{kl}\, \nu^{l} h^{k}_{i}, \tag{3.24}$$

where the semicolon denotes the covariant derivative on the hypersurface S. This relation is known as *Codazzi's equation*. For more details on this subject, see [67].

3.3 Special coordinates in Riemannian spaces

3.3.1 Normal coordinates

Riemannian coordinates (y^h) with the reference point x_0 can be introduced for a manifold with affine connection, see p. 93. By a choice of a suitable linear transformation $y^{h'} = a^{h'}_{h}\, y^{h}$ of a Riemannian manifold $\mathbb{V}_n$ with metric form $ds^2 = g_{ij}(y)\, dy^i dy^j$ we are able to manage that the metric form at the point x_0 is

$$ds^2|_{x_0} = e_1\,(dy^{1'})^2 + e_2\,(dy^{2'})^2 + \cdots + e_n\,(dy^{n'})^2; \quad e_i = \pm 1. \tag{3.25}$$

Riemannian coordinates $(y^{h'})$ in which the metric form reads (3.25) are called *normal coordinates* in x_0 for $\mathbb{V}_n$.

3.3.2 Coordinates generated by a system of orthogonal hypersurfaces

Consider a hypersurface Σ and a neighborhood of it defined by the equation $f(x^1, \ldots, x^n) = C$, $C = \text{const}$, in a distinguished chart (U, φ). The gradient $\nabla_i f \equiv \partial f/\partial x^i$ can be interpreted as normal vector to Σ; indeed, $\nabla_\alpha \partial x^\alpha\, dx^i = 0$ holds for any point of Σ.

We say that the hypersurface Σ is *isotropic* if its normal vector (which is a gradient vector $\nabla_i f$) is a null-vector (i.e. isotropic vector).

If we examine locally two hypersurfaces $\Sigma_1\colon f^1 = C^1$ and $\Sigma_2\colon f^2 = C^2$ we say that they are *orthogonal* if their normal vectors are orthogonal, that is, $g^{ij}\nabla_i f^1 \nabla_j f^2 = 0$. Particularly interesting is the case of a pair of coordinate hypersurfaces

$$\Sigma_k\colon x^k = C^k \quad \text{and} \quad \Sigma_l\colon x^l = C^l, \quad k \neq l, \quad k, l \text{ are fixed indices.}$$

Since their normals are $\dfrac{\partial x^k}{\partial x^i} = (\delta^k_i)$ and $\dfrac{\partial x^l}{\partial x^i} = (\delta^l_i)$ the orthogonality condition $g^{\alpha\beta}\delta^k_\alpha \delta^l_\beta = 0$ yields $g^{kl} = 0$ for all $k \neq l$.

Let us show that the given non-isotropic hypersurfaces Σ can be always considered to be coordinate surfaces, and moreover, orthogonal to any other coordinate hypersurfaces. For this purpose consider the orthogonality condition

$$g^{ij}\nabla_i f \nabla_j \tilde{f} = 0. \tag{3.26}$$

For the given function f we have a linear partial differential equation in the unknown function $\tilde{f}$:

$$\frac{\partial \tilde{f}}{\partial x^\alpha}\,(g^{\alpha i}\nabla_i f) = 0.$$

It is known that the given equation has $n-1$ independent solutions $\tilde{f}^2(x)$, $\tilde{f}^3(x), \ldots, \tilde{f}^n(x)$. Now we apply a transformation of local coordinates

$$y^1 = f(x),\ y^2 = \tilde{f}^2(x),\ y^3 = \tilde{f}^3(x), \ldots,\ y^n = \tilde{f}^n(x).$$

Then Σ is given by the equation $y^1 = C$, i.e. it is just the coordinate hypersurfaces Σ^1, and due to (3.26), it is orthogonal to the remaining coordinate hypersurfaces Σ^α, $y^\alpha = C^\alpha$, $\alpha = 2, 3, \ldots, n$.

That is, $g'^{1\alpha}(y) = 0$, $\alpha > 1$, whence the inverse matrix $(g'_{ij}(y))$ has an analogous form. The metric form in local coordinates $y^1, y^2, \ldots, y^n$ reads

$$ds^2 = g'_{11}(y)\,(dy^1)^2 + g'_{\alpha\beta}(y)\,dy^\alpha\,dy^\beta; \quad \alpha, \beta > 1. \tag{3.27}$$

A special case of the above is the system of n mutually orthogonal hyperplanes $\Sigma^k \perp \Sigma^l$ for any $k \neq l$. If such a system exists then the metric reads

$$ds^2 = g'_{11}(y)\,(dy^1)^2 + g'_{22}(y)\,(dy^2)^2 + \cdots + g'_{nn}(y)\,(dy^n)^2.$$

A system of n orthogonal surfaces always exists for 2- and 3-dimensional Riemannian spaces. For a Riemannian manifold $\mathbb{V}_n$ with $n > 3$, it can happen that there is no such system.

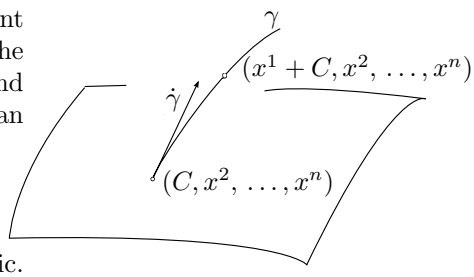
3.3.3 Semigeodesic coordinates

For any Riemannian manifold $\mathbb{V}_n$ let us introduce semigeodesic coordinates, which can be considered as a particular case of coordinates based on a system of hypersurfaces mentioned above. On the other hand these coordinates are pre-semigeodesic coordinates, see p. 94.

Definition 3.5 Let us consider a non-isotropic coordinate hypersurface $\Sigma \equiv \Sigma^1$: $x^1 = C$. Let us fix some point $(C, x^2, \ldots, x^n)$ on Σ and construct the geodesic γ passing through the point and tangent to the unit normal of Σ; γ is an x^1-curve, it is parametrized by

$$\gamma(x^1) = (x^1 + C, x^2, \ldots, x^n)$$

and x^1 is the arc length on the geodesic.



Coordinates introduced in this way are called *semigeodesic coordinates* in $\mathbb{V}_n$.

Let Σ be a non-isotropic hypersurface. Evidently on Σ, $g_{11} = e = \pm 1$ and $g_{1a} = 0$, $a > 1$, hold. When a coordinate system is pre-semigeodesic the following is satisfied (2.36): $\Gamma^h_{11}(x) = 0$. For the Christoffel symbols of the first type it follows $\Gamma_{11k} = 0$, and we get $2\partial_1 g_{1k} - \partial_k g_{11} = 0$. Integrating along x^1-curves we find for $h = 1$ that $g_{11} = e$ and for $h = a > 1$: $g_{1a} = 0$, in the entire coordinate domain. As a result, we obtain the following form for the metric in semi-geodesic coordinates:

$$ds^2 = e\,(dx^1)^2 + g_{ab}(x)\,dx^a\,dx^b, \quad a, b > 1,\ e = \pm 1. \tag{3.28}$$

On the other hand this coordinate form of the metric is a sufficient condition for the coordinate system to be semigeodesic.

Note that coordinate hyperplanes defined by $x^1 = \text{const}$ are orthogonal to the distinguished system of geodesics.

The geometric interpretation is as follows, [139, p. 55]. Classical semigeodesic coordinates can be introduced in a sufficiently small neighborhood of any point of an arbitrary (positive) Riemannian manifold, and is fully characterized by the coordinate form of the metric: (3.28) with $e = 1$. On a cylinder, semigeodesic coordinates can be introduced globally.

Advantages of such coordinates are known since Gauss (*Geodätische Parallelkoordinaten*, [97, p. 201]), and are widely used in the two-dimensional case, particularly in applications, [127] and the references therein, [927] etc. Note that geodesic polar coordinates (*Geodätische Polarkoordinaten*, [97, pp. 197-204] can be also interpreted as a "limit case" of semigeodesic coordinates (all geodesic coordinate lines $\varphi = x^2 = \text{const}$ pass through one point called the pole, corresponding to $r = x^1 = 0$ while $r = x^1 = \text{const}$ are geodesic circles).

One of the important applications of semigeodesic coordinates is the length minimazing property of geodesics.

Theorem 3.2 *In (proper) Riemannian manifolds geodesics have minimal length between two sufficiently near points.*

Proof. Let $\mathbb{V}_n$ be a classical (proper) Riemannian space. Consider some geodesic γ connecting two sufficiently close points x_0 and x_1. Let us consider a hypersurface Σ passing through the point x_0 and orthogonal in x_0 to this geodesic γ. Starting with this hypersurface construct a semigeodesic coordinate system (x^i).

Let ℓ: $x = (s, x^2(s), \ldots, x^n(s))$ be a curve for which $x_0 = (s_0, 0, \ldots, 0)$ and $x_1 = (s_1, 0, \ldots, 0)$. The length of the curve segment of ℓ: $x = x(t)$ between the points x_0 and x_1 is calculated as follows:

$$s = \int_{s_0}^{s_1} \sqrt{1 + g_{ab}(x(s))\,\frac{dx^a(s)}{ds}\,\frac{dx^b(s)}{ds}}\ ds, \quad a, b > 1.$$

Since g is positive definite the minimal value s of this integral is reached if and only if $dx^a(s)/ds = 0$, or equivalently $x^a = \text{const}$. But this means equivalently that the minimum is reached for a curve $\ell = (s, 0, 0 \ldots, 0)$ which is just the original geodesic segment γ. □

3.3.4 Reconstruction of the metric in semigeodesic coordinates

We will continue with the reconstruction process being started in a pre-semigeodesic coordinate system in $\mathbb{A}_n$, p. 94. In semigeodesic coordinates, we can reconstruct a metric of $\mathbb{V}_n$ by the method already explained.

A semigeodesic chart is fully characterized by the condition that the curves $x^1 = s$, $x^i = \text{const}$, $i = 2, \dots n$ belong to geodesics of the connection in the coordinate neighborhood. The definition domain U of such a chart is "tubular", a tube along geodesics.

First let us modify for our purpose the Theorem on existence and uniqueness of solutions of systems of differential equations. In $\mathbb{R}^n$ with standard coordinates $(x^1, x^2, \dots, x^n)$, let us identify the linear subspace (hypersurface) characterized by $x^1 = 0$ with $\mathbb{R}^{n-1}$, i.e. $(\tilde{x}) = (x^2, \dots, x^n)$ are standard coordinates in $\mathbb{R}^{n-1}$. Let $\mathcal{J} = (0, 1)$ be the open unit interval and denote by $K_m = \mathcal{J}^m$ the open standard m-cube. Denote

$$D_n(\delta) = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n : \ 0 \le x^1 \le \delta, \ 0 < x^i < 1, \ i = 2, \dots, n\}.$$

The open $(n-1)$-cube $K_{n-1} = \mathcal{J}^{n-1}$, viewed as

$$K_{n-1} = \{\tilde{x} = (x^2, \dots, x^n) \in \mathbb{R}^{n-1}, \ 0 < x^i < 1, \ i = 2, \dots, n\} \subset \mathbb{R}^{n-1},$$

can be identified with a hypersurface S in $D_n(\delta)$ determined by $x^1 = 0$.

Theorem 3.3 *Let a_{ij} be (at least) continuous functions in $D_n(\delta)$, let $\tilde{g}_{ij}$ be functions of the class (at least) C^2 in K_{n-1} and $\tilde{G}_{ij}$ functions of the class (at least) C^1 in K_{n-1}, $i, j = 2, \dots, n$ such that the matrices $(\tilde{g}_{ij})$ and $(\tilde{G}_{ij})$ are symmetric*[49] *and $\det(\tilde{g}_{ij}) \neq 0$ in K_{n-1}. Fix an element $e \in \{-1, 1\}$. Then there is $\hat{\delta}$, $0 < \hat{\delta} < \delta$ and there exists exactly one non-degenerate metric tensor*[50] *g of the class (at least) C^2 in $D_n(\hat{\delta})$ with components $g_{11} = e$, $g_{1j} = 0$, $j = 2, \dots, n$ such that for $i, j = 2, \dots, n$,*

$$g_{ij}(0, \tilde{x}) = \tilde{g}_{ij}(\tilde{x}), \quad \frac{\partial}{\partial x^1} g_{ij}(0, \tilde{x}) = \tilde{G}_{ij}(\tilde{x}), \quad \tilde{x} \in D_n(\hat{\delta}) \tag{3.29}$$

and

$$a_{ij}(x) = R_{1ij1}(x), \qquad x \in D_n(\hat{\delta}). \tag{3.30}$$

Proof. Components of the curvature tensor R (of type $(0, 4)$) of the semi-Riemannian manifold $V_n = (M, g)$ are related to components of the metric by

$$R_{hijk} = \frac{1}{2}(\partial_{ij} g_{hk} + \partial_{hk} g_{ij} - \partial_{ik} g_{hj} - \partial_{ij} g_{hk}) + g^{rs}(\Gamma_{hkr}\Gamma_{ijs} - \Gamma_{hjr}\Gamma_{kjs}) \tag{3.31}$$

where $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are Christoffel symbols of first type in V_n, and g^{rs} are components of the dual tensor to g. Hence g^{ij} are functions rational in components g_{ij} of the metric[51].

[49] $\tilde{g}_{ji} = \tilde{g}_{ij}$, $\tilde{G}_{ji} = \tilde{G}_{ij}$

[50] $\det(g_{ij}) \neq 0$ in $D_n(\hat{\delta})$

[51] $g^{ij} = 1/\det(g_{ij}) \cdot A_{ji}$ where A_{ji} is the algebraic complement of the matrix element g_{ji}

Setting $h = k = 1$ and using the assumptions $g_{11} = e$, $g_{1j} = 0$ we obtain from (3.31)

$$R_{1ij1} = \frac{1}{2}\partial_{11}g_{ij} - \frac{1}{4}g^{rs}\partial_1 g_{ir}\partial_1 g_{js}. \tag{3.32}$$

Here we can suppose that the indices satisfy $i, j, r, s > 1$. Let us substitute

$$G_{ij} = \partial_1 g_{ij}. \tag{3.33}$$

Then we can write (3.32) as

$$R_{1ij1} = \frac{1}{2}\partial_1 G_{ij} - \frac{1}{4}g^{rs}G_{ir}G_{js}. \tag{3.34}$$

Denote $R_{1ij1}(x) = a_{ij}(x)$. Hence we should solve the system

$$\begin{aligned} \partial_1 g_{ij} &= G_{ij}, \\ \partial_1 G_{ij} &= \tfrac{1}{2}g^{rs}G_{ir}G_{js} + 2a_{ij} \end{aligned} \tag{3.35}$$

with the initial values

$$g_{ij}(0, \tilde{x}) = \tilde{g}_{ij}(\tilde{x}), \quad \frac{\partial}{\partial x^1}g_{ij}(0, \tilde{x}) = \tilde{G}_{ij}(\tilde{x}), \quad \tilde{x} \in K_{n-1},\ i, j = 2, \dots, n. \tag{3.36}$$

Note that since the determinant as well as the algebraic complements are continuous functions in the entries g_{ij}, and since we demand $\det(\tilde{g}_{ij})(0, \tilde{x}) = \det(\tilde{g}_{ij})(\tilde{x}) \neq 0$, it is guaranteed that g^{rs} will be well-defined and well-behaved functions of g_{ij}, similarly as in [449]. So (3.35) can be considered as a system of first-order ordinary differential equations in the variable x^1 for the unknown functions g_{ij} and G_{ij} with initial values (3.36); the remaining coordinates $x^2, \dots, x^n \in K_{n-1}$ are supposed to be parameters. The right sides in (3.35) satisfy the conditions of the existence and uniqueness theorem [38, p. 263] in the domain $D_n(\tilde{\delta})$ and have continuous derivatives with respect to g_{ij} and G_{ij}. The initial value problem (3.35) and (3.36) has precisely one solution $g_{ij}(x)$. The functions g_{ij} are components of a metric tensor in $D_n(\tilde{\delta})$, and comparing (3.35) and (3.34) we find easily that components of its curvature tensor satisfy $R_{1ij1}(x) = a_{ij}(x)$ as required. □

Since the matrices (g_{ij}) and (G_{ij}) are symmetric we may assume $i \leq j$ in (2.25).

Provided $a_{ij}(x) = R_{1ij1}(x)$ the solution of the system (3.35) solves the problem of finding metrics with the prescribed components $R_{1ij1}(x)$ of the $(0,4)$-Riemannian curvature tensor. Substituting the obtained components of the metric we get the relationship to the components of the $(1,3)$-curvature as follows:

$$R_{1ij1} = eR^1_{ij1} = -eR^1_{i1j} = g_{im}R^m_{11j} = -g_{im}R^m_{1j1}. \tag{3.37}$$

Hence our results generalize the results of [448] and [449].

3.4 Variational properties in Riemannian spaces

The definition of geodesics introduced above is the most general one, based exclusively on purely geometric considerations, see Subsection 2.4.6 and 2.4.7.

As we have mentioned in Subsection 2.4.7, Johann I. Bernoulli in his letter to Leibniz from 1697, posed a problem: *find a curve on a given surface which passes through a pair of given points and has the property that its segment connecting the given two points is shorter (or the same) as any other curve segment connecting them.*

Let us now explain the solution of the problem of Bernoulli, but in the more general setting, and more adapted to the language of modern differential geometry, [50].

3.4.1 Variational problem

Let M_n be an n-dimensional manifold and let $\ell\colon I \to M_n$ be a regular curve on an open interval $I \subset \mathbb{R}$ defined in (local coordinates) by

$$x(t) = (x^1(t), \dots, x^n(t)), \quad t \in I, \tag{3.38}$$

and let $\lambda(t) = \dot{x}(t) = dx(t)/dt = (\dot{x}^1(t), \dots, \dot{x}^n(t))$ be the corresponding tangent vector field of ℓ, $\dot{x}(t) \neq 0$ for $t \in I$.

Let $A = \ell(t_0)$, $B = \ell(t_1)$ be two points on $\ell(t)$ corresponding to parameters t_0 and $t_1 \in I$, respectively. Let $\omega^i : M \to \mathbb{R}$ be (differentiable) functions such that $\omega^i(A) = \omega^i(B) = 0$ (in local coordinates, $\omega^i(x^1, \dots, x^n)$), and let ε be a parameter. From now on, if not said otherwise, all indices will varry from 1 to n. Then the equations

$$\overline{x}^i(t) = x^i(t) + \varepsilon \cdot \omega^i(x^1(t), \dots, x^n(t))$$

define a new curve $\overline{\ell}$: $\overline{x}(t) = (\overline{x}^1(t), \dots, \overline{x}^n(t))$, for small ε "close" to the original one and passing through the given points A and B as well. Let us consider the integral

$$I(\ell) = \int_{t_0}^{t_1} \mathcal{L}(t, x^1(t), x^2(t), \dots, x^n(t), \dot{x}^1(t), \dot{x}^2(t), \dots, \dot{x}^n(t))\, dt \tag{3.39}$$

where $\mathcal{L}$ is an analytic function of the given arguments, called a *Lagrange function*.

If $\overline{I} = I(\overline{\ell})$ is a similar integral for $\overline{\ell}$ then expanding $\mathcal{L}$ as a Taylor series in powers of ε we get

$$\overline{I} = I + \varepsilon \cdot \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial x^i} \omega^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \frac{\partial \omega^i}{\partial x^j} \dot{x}^j \right) dt + \ \cdots$$

where "dots" stand for the members of order ε^2 and higher. Coefficients at ε, ε^2 etc. in the above expansion are denoted by δI, $\delta^2 I$ etc., and are called the *first*, *second* etc. *variation* of the integral I. Particularly, the first variation

$$\delta I = \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial x^i} \omega^i + \frac{\partial \mathcal{L}}{\partial dot x^i} \frac{\partial \omega^i}{\partial x^j} \dot{x}^j \right) dt$$

is often written in the following form (we integrate in parts in the second summand and use properties of the functions $\omega^i(x)$)

$$\delta I = \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) \right) \omega^i dt. \tag{3.40}$$

The integral (3.39) is called *stationary* if its first variation vanishes, $\delta I = 0$, for an arbitrary choice of functions ω^i. The curve for which it holds is called the *extremal* of the integral under consideration. As it follows from (3.40), the integral (3.39) is stacionary if and only if the following conditions, called *Euler*, or *Euler-Lagrange equations*, are satisfied:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0. \tag{3.41}$$

A solution $x = x(t)$ of the equations (3.41) is an extremal curve of the integral (3.39).

Let us note here that H. Cartan[52)]. precisely formulated this variational problem for Lagrange functions $\mathcal{L} \in C^2$, [31]. In this case, the formula for first variation reads $\delta I = dI(\bar{\ell}\,)/d\varepsilon|_{\varepsilon=0}$, and its extremals $\ell \in C^2$ are solutions of the Euler-Lagrange equations (3.41).

3.4.2 Variational problem of geodesics in Riemannian spaces

First of all, let us pass from the two-dimensional Riemannian surface S of $\mathbb{E}_3$ to the n-dimensional Riemannian manifold $\mathbb{V}_n = (M_n, g)$; that is, no requirements are posed on the signature of the metric form g. The formula (2.10) is substituted by the formula

$$ds^2 = e\, g_{ij}(x) dx^i dx^j, \qquad i, j = 1, \ldots, n \tag{3.42}$$

where $e = \pm 1$, and we choose the sign in such a way that $ds^2 \geq 0$. From now on, if not said otherwise, all indices will varry from 1 to n.

According to (3.42), arc length s of the arc $\widehat{AB}$ of curve ℓ given by the parametrization (3.38) is expressed by the integral

$$s = \int_{t_0}^{t_1} \sqrt{e\, g_{ij}(x(t))\, \dot{x}^i(t)\, \dot{x}^j(t)}\; dt. \tag{3.43}$$

Definition 3.6 Any extremal curve of the integral (3.43) will be called a *geodesic curve* in $\mathbb{V}_n = (M_n, g)$.

We suppose stationarity of the integral (3.43), and try to find its extremals. Since

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^h} = \frac{e\, g_{hj} \dot{x}^j}{\sqrt{e\, g_{ij}\, \dot{x}^i \dot{x}^j}} = \frac{e\, g_{hj} \dot{x}^j}{\dot{s}}, \qquad \frac{\partial \mathcal{L}}{\partial x^h} = \frac{e\, \partial_h g_{ij}\, \dot{x}^i \dot{x}^j}{2\dot{s}}, \qquad \dot{s} = \frac{ds(t)}{dt},$$

[52)] Henri Paul Cartan, 1904–2008, a French mathematician (a son of the famous mathematician Élie Cartan), a member of the group of mathematicians who called themselves "Bourbaki" (founded 1935). The author of *Theorie Elementaire des Fonctions Analytiques D'une ou Plusieurs Variabl*, Hermann, 1975 (in Russian, Kartan A., *Elementarnaja teorija funkcij kompleksnyh peremennyh* etc.

the Euler–Lagrange equations read

$$g_{ih}\,\ddot{x}^i + \Gamma_{ijh}(x)\,\dot{x}^i\dot{x}^j = \sigma(t)\,g_{ih}\dot{x}^i, \quad \sigma(t) = \frac{\ddot{s}}{\dot{s}}.$$

Using the dual tensor g^* to g, with components g^{ij} (i.e., in matrix notation, $g_{ji}g^{ih} = \delta^h_j$ where δ^h_j is the Kronecker symbol) we get finally

$$\frac{d^2x^h}{dt^2} + \Gamma^h_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt} = \sigma(t)\frac{dx^h}{dt}, \qquad h = 1, \dots, n. \tag{3.44}$$

Formally, we obtained the same equations as in (2.11) (but in a more general type of space). Evidently, all integral curves of the equations (3.44) are geodesic curves in $\mathbb{V}_n$.

Using the classical notation for covariant derivative with respect to the metric tensor g[53)] (denoted by comma) of the tangent vector field λ of the curve, the equations (3.44) read

$$\lambda^h_{,i}\lambda^i = \sigma(t)\,\lambda^h. \tag{3.45}$$

Hence a parametrized curve (3.38) is a solution of (3.44) if and only if its tangent vector field λ ($\equiv \dot{x}$) satisfies (2.25):

$$\nabla_\lambda\lambda = \sigma(t)\,\lambda$$

(at any point, the vector of covariant derivative of λ along ℓ is collinear with λ) where ∇ is the Levi-Civita connection of a Riemannian manifold $\mathbb{V}_n$. Recall that the above condition characterizes a recurrent vector field λ along a curve ℓ. Another speaking, ℓ is a geodesic curve if and only if its tangent vector field is recurrent along ℓ.

Obviously, (2.26) is a particular case, with $\sigma(t) \equiv 0$. Note that for any curve $\ell(t)$ in $\mathbb{V}_n$, there exists a reparametrization, by means of a function $s = s(t)$, such that for this special parametrization, just $\sigma(s) \equiv 0$ holds. If this is the case, s is called a *canonical* parameter of the geodesic ℓ in $\mathbb{V}_n$. Among others, arclength parameters of a geodesic are canonical[54)].

If a geodesic curve ℓ in $\mathbb{V}_n$ is parametrized by a canonical parameter s then $\sigma(s) \equiv 0$, and our systems (3.44), (3.45) and (2.25) can be written as follows:

$$\frac{d\lambda^h}{ds} + \Gamma^h_{ij}\lambda^i\lambda^j = 0, \tag{3.46}$$

$$\lambda^h_{,\alpha}\lambda^\alpha = 0, \tag{3.47}$$

$$\nabla_\lambda\lambda = 0, \tag{3.48}$$

where $\lambda(s) = \dot{x}(s)$. The above formulas in fact express the fact that the tangent vector field $\lambda(s)$ is parallel along a geodesic curve.

[53)] i.e. with respect to the Levi-Civita connection ∇ of $V_n = (M, g)$.

[54)] In $\mathbb{V}_n$, canonical parameter of a geodesic differs from a fixed arc length up to a multiple plus a constant.

The formulas (3.46) – (3.48) represent another form of equations for geodesics.

The question, whether a geodesic is at the same time the "shortest way", needs additional investigations: it is closely connected with the fact that the Euler-Lagrange equations express only *necessary* conditions for existence of extremum of the given integral, [78]. We formulate the answer rather cautiously:

Theorem 3.4 *If there exists a curve of minimal length connecting two points of a (pseudo-) Riemannian manifold then this curve is a geodesic.*

If two points are "close enough" then the geodesic connecting them will surely be the shortest one; the details can be found e.g. in [78, 163].

In a monograph Dubrovin, Fomenko, Novikov [42] a geodesic curve is defined as an extremal of the integral

$$\mathcal{I} = \int g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\,dt\,. \tag{3.49}$$

In this case, extremals are just geodesic curves with a canonical parameter only. In this monograph, it is explained that the integral (3.49) defines kinetic energy. Hence geodesic curves with canonical parameter determine motion of ideal particles.

3.4.3 Generalized variational problem for geodesics

In a Riemannian space consider the following more general *variational problem*:

$$I = \int_{t_0}^{t_1} f(e\,g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j)\,d\tau, \tag{3.50}$$

where e takes the values ± 1, and $f(\tau)$ is a differentiable real-valued function (of the class at least two) defined on some open domain $D \subset \mathbb{R}$ which is regular on D in the sense that $f'(\tau) \neq 0$ for all $\tau \in D$.

Obviously, (3.50) is a generalization of the integrals (3.43) and (3.49). As an immediate consequence of the Euler-Lagrange equations for the Lagrange function $\mathcal{L} = f(e\,g_{ij}\dot{x}^i\dot{x}^j)$, it can be checked that the extremals satisfy the equations

$$\ddot{x}^h + \Gamma^h_{ij}(x)\,\dot{x}^i\dot{x}^j = -\frac{\mathrm{d}}{\mathrm{d}t}(\ln|f'(e\,g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta)|)\,\dot{x}^h. \tag{3.51}$$

We can prove the following [667]:

Theorem 3.5 *In Riemannian spaces, geodesics parametrized by a canonical parameter, which satisfy the condition*

$$e\,g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = k \in D,$$

are extremals of the integral (3.50).

Theorem 3.6 *Consider (all) extremals of the integral (3.50) in a Riemannian space. All curves arising under all possible regular reparametrizations of extremal curves belong to extremals, too, if and only if the function f takes the form $f(x) \equiv \alpha\sqrt{x} + \beta$ where α $(\neq 0)$ and β are some constants.*

Theorem 3.7 *All possible extremals of the integral (3.50) are just those geodesics which figure in Theorems 3.5 and 3.6. More precisely, in the particular case* $f(x) \equiv \alpha\sqrt{x} + \beta$, $\alpha = \text{const} \neq 0$, $\beta = \text{const}$, *they are represented by all unparametrized geodesics (i.e. geodesics under all possible regular reparametrizations), while for all other functions* f, *satisfying the above assumptions of the problem (3.50), extremals are represented just by canonically parametrized geodesics only.*

3.4.4 Applications of geodesics

We have just observed an important property of (parametrized) geodesics, namely, we have investigated the velocity field λ, given by the velocity vector at respective points during the motion. It appears that such a view-point is useful and suitable for further generalization of the notion of geodesic even if we pass to an arbitrary affine manifold $\mathbb{A}_n$. Necessity of such a generalization arises quite naturally. The crucial point is that geodesics preserve direction. This fact, expressed by the formula (2.25), allows us to introduce the concept of a geodesic in most general spaces, namely on a manifold endowed with a linear connection (in which case no particular metric is given or even available), and hence to prepare it for wide application in mechanics and physics.

Geodesics in $\mathbb{V}_n$ and $\mathbb{A}_n$ play an analogous role as lines in the Euclidean space $\mathbb{E}_n$. The local behaviour of geodesics in $\mathbb{V}_n$ as well as in $\mathbb{A}_n$ is similar to those of lines in $\mathbb{E}_n$. If a point and a tangent vector are fixed then there is a (unique) geodesic which matches the initial conditions (i.e. passes through the point and has a prescribed tangent vector in it). For any point there is a neighborhood U in which any two points are connected with a unique geodesic segment contained in U. In $\mathbb{V}_n = (M_n, g)$, a geodesic is a locally length-minimizing curve: a sufficiently small geodesic segment is the shortest one of all segments with the same endpoints. Naturally, geodesics depend on the Riemannian metric, which affects the notion of distance as well as acceleration. E.g. on the two-sphere, the geodesics are great circles (like the equator). On the sphere we also find a transparent example that two points which are not "close enough", might be connected by more than one geodesic: for any pair of antipodal points on the sphere, there are infinitely many geodesics passing through them, any "meridian" (a half-circle arising as a segment of a great circle connecting antipodal points) is a geodesic one.

In mechanics, geodesics describe the motion of point particles. A geodesic is the path that a particle with zero acceleration would follow. More precisely, world lines of free particles with a non-zero rest mass are non-isotropic geodesics[55)] of the 4-dimensional space-time continuum of the general theory of relativity. The trajectory of a free particle with a zero rest mass (photon, neutrino) is an isotropic geodesic[56)] of the space-time under consideration. Differential equations of geodesics are just equations of *motion* in the general theory of relativity.

[55)] of non-zero length, i.e. non-constant curves.

[56)] with zero length, i.e. a constant curve.

3.4.5 Isoperimetric extremals of rotation

In this subsection we present results by J. Mikeš, M. Sochor and E. Stepanova [692] of the existence and the uniqueness of isoperimetric extremals of rotation on two-dimensional (pseudo-) Riemannian manifolds $\mathbb{V}_2$ and on surfaces on Euclidean space. We find the new form of their equations which is easier than results by S. G. Leiko.

A rotary diffeomorphism of surfaces $\mathbb{S}_2$ on a three-dimensional Euclidean space $\mathbb{E}_3$ and also of two-dimensional Riemannian manifolds $\mathbb{V}_2$ is studied in papers of S.G. Leiko [581–589]. These results are local and are based on the known fact that a two-dimensional Riemannian manifold $\mathbb{V}_2$ is implemented locally as a surface $\mathbb{S}_2$ on $\mathbb{E}_3$. Therefore we will deal more with the study of $\mathbb{V}_2$, i. e. the inner geometry of $\mathbb{S}_2$.

The *isoperimetric extremal of rotation* is a special curve on $\mathbb{V}_2$ (resp. $\mathbb{S}_2$) which is extremal of a certain variational problem of geodesic curvature (see [581–588] where the existence of these curves was shown for the case $\mathbb{V}_2 \in C^4$, resp. on $\mathbb{S}_2 \in C^5$).

The above curves have a physical meaning as can be interpreted as trajectories of particles with a spin, see [581, 583].

These part is devoted to the proof of the existence of isoperimetric extremal of rotation on $\mathbb{V}_2 \in C^3$, resp. on $\mathbb{S}_2 \in C^4$. Besides we find the fundamental equations of these curves in a more simple form of ordinary differential equation of Cauchy type. From the above the problem of a rotary diffeomorphism can be solved for the surfaces with the lower smoothness class.

Remark. A two-dimensional Riemannian manifold $\mathbb{V}_2$ belongs to the smoothness class C^r if its metric $g_{ij} \in C^r$. We suppose that the differentiability class r is equal to $0, 1, 2, \ldots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions respectively.

Surface $\mathbb{S}_2$: $\mathbf{p} = \mathbf{p}(x^1, x^2)$ belongs to C^{r+1} if the vector function $\mathbf{p}(x^1, x^2) \in C^{r+1}$ and evidently inner two-dimensional Riemannian manifold $\mathbb{V}_2$ belongs to C^r with induced metric $g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C^r$, where $\mathbf{p}_i = \partial_i \mathbf{p}$, $\partial_i = \partial/\partial x^i$. There $x = (x^1, x^2)$ are local coordinates of $\mathbb{V}_2$, resp. $\mathbb{S}_2$.

An immersion $\mathbb{V}_2$ in Euclidean space is studied in detail, for example, in [146, 718]. In the study of surfaces $\mathbb{S}_2$ we use the notation that is used in the books [3, 66, 97, 143, 147].

Let us consider a two-dimensional Riemannian space $\mathbb{V}_2 \in C^3$ with a metric tensor g. Let $g_{ij}(x^1, x^2) \in C^3$ $(i, j = 1, 2)$ be components of g in some local map.

For the curve ℓ: $(t_0, t_1) \to V_2$ with the parametric equation $x^h = x^h(t)$, we construct the tangent vector $\lambda^h = dx^h/dt$ and vectors

$$\lambda_1^h = \nabla_t \lambda^h \quad \text{and} \quad \lambda_2^h = \nabla_t \lambda_1^h.$$

Here ∇_t is an operator of covariant differentiation along ℓ with respect to the

Levi-Civita connection ∇ of metric g, i.e.

$$\lambda_1^h = \nabla_t \lambda^h \equiv \frac{d\lambda^h}{dt} + \lambda^\alpha \Gamma^h_{\alpha\beta}\left(x(t)\right)\lambda^\beta$$

and

$$\lambda_2^h = \nabla_t \lambda_1^h \equiv \frac{d\lambda_1^h}{dt} + \lambda_1^\alpha \Gamma^h_{\alpha\beta}\left(x(t)\right)\lambda^\beta,$$

where Γ^h_{ij} are the Christoffel symbols of $\mathbb{V}_2$, i. e. components of ∇.

It is known that the scalar product of vectors λ, ξ is defined by $\langle \lambda, \xi \rangle = g_{ij}\lambda^i \xi^j$. We denote

$$s[\ell] = \int_{t_0}^{t_1} \sqrt{\langle \lambda, \lambda \rangle}\, \mathrm{d}t \qquad \text{and} \qquad \theta[\ell] = \int_{t_0}^{t_1} k_g(s)\, \mathrm{d}s$$

functionals of length and rotation of the curve ℓ; k_g is the Frenet curvature[57] and s is the arc length. In the case $\mathbb{S}_2 \subset E_3$ the geodesic curvature of the curve is k_g.

Using these functionals we introduce the following definition

Definition 3.7 (Leiko [582]) A curve ℓ is called the *isoperimetric extremal of rotation* if ℓ is extremal of $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends.

It was shown in [582] that in a (not plain) space $\mathbb{V}_2$ a curve is an isoperimetric extremal of rotation only if its Frenet curvature k_g and Gaussian curvature K are proportional:

$$k_g = c \cdot K, \tag{3.52}$$

where $c = \text{const}$.

Above result is coming from the following fact, see [582]. The functional of rotation has the following form

$$\theta[\ell] = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x}, \ddot{x})\, dt, \qquad \mathcal{L} = \frac{e_0\sqrt{e\,G}}{\langle \lambda, \lambda \rangle},$$

where $G = \langle \lambda, \lambda \rangle \langle \lambda_1, \lambda_1 \rangle - \langle \lambda, \lambda_1 \rangle^2$ is the Gramm determinant of vectors λ, λ_1, $e = e_0 e_1$, due to $k_g = \sqrt{e_1 \langle \lambda_1(s), \lambda_1(s) \rangle}$, $e_i = \pm 1$.

It is easily seen the rotary Lagrangian depends on second order derivatives. These theory comes by M.V. Ostrogradski. The extremals of Lagrangian satisfy the higher order Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt}\,\frac{\partial \mathcal{L}}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\,\frac{\partial \mathcal{L}}{\partial \ddot{x}^i} = 0.$$

[57] In the original paper k_g is denoted as k. This fact can lead to confusion between k and the main curvature of the curve.

In [582] it is proved that for a canonical parameter $t = a \cdot s + b$ ($a, b = \text{const}$) the condition (3.52) can be written in the following form

$$\lambda_2 = -\frac{\langle \lambda_1, \lambda_1 \rangle}{\langle \lambda, \lambda \rangle} \cdot \lambda + \frac{\nabla_\alpha K \cdot \lambda^\alpha}{K} \cdot \lambda_1, \tag{3.53}$$

where $\langle \lambda, \lambda_1 \rangle = 0$ and $\nabla_i K = \partial_i K$ is a gradient vector of the Gaussian curvature ($K \neq 0$).

Using these equations for the case of $\mathbb{V}_2 \in C^4$ the uniqueness of the existence of isoperimetric extremals of rotation can be shown for the following initial conditions (see [585]):

$$x(0), \lambda(0), \lambda_1(0) \text{ such that } \langle \lambda(0), \lambda(0) \rangle = 1 \text{ and } \langle \lambda(0), \lambda_1(0) \rangle = 0.$$

3.4.6 On new equations of isoperimetric extremals of rotation

First we recall the basic knowledge of theory of surfaces $\mathbb{S}_2$ and (pseudo-) Riemannian manifolds, see [3, 62, 66, 97, 122, 143, 147].

For simplicity we will consider that a two-dimensional Riemannian manifold $\mathbb{V}_2$ is a subspace of $\mathbb{S}_2 \subset \mathbb{E}_3$ which is given by the equation $\mathbf{p} = \mathbf{p}(x^1, x^2)$. It is known that metric of $\mathbb{S}_2$ is given by the following functions $g_{ij}(x) = \mathbf{p}_i \cdot \mathbf{p}_j \in C^r$, where $\mathbf{p}_i = \partial_i \mathbf{p}$.

The existence of the surface $\mathbb{S}_2$ with metric g on $\mathbb{V}_2$ results from the Bonnet Theorem; components g_{ij} of the first fundamental form belong to the smoothness class C^2 and components b_{ij} of the second fundamental form belong to the smoothness class C^1 both of them satisfy Gauss and Peterson-Codazzi equations.

For the Gaussian curvature K it holds that

$$K = \frac{b_{11} b_{22} - b_{12}^2}{g_{11} g_{22} - g_{12}^2},$$

where $b_{ij} = \partial_{ij} \mathbf{p} \cdot \mathbf{m}$ and $\mathbf{m} = \dfrac{\mathbf{p}_1 \times \mathbf{p}_2}{|\mathbf{p}_1 \times \mathbf{p}_2|}$ is a unit normal vector of the surface $\mathbb{S}_2$. If $\mathbb{S}_2 \in C^3$ then the curvature K is differentiable.

Now we recall the geometry of a (pseudo-) Riemannian manifold $\mathbb{V}_2$ defined by the metric tensor g_{ij}. The Christoffel symbols of the first and the second kind are given by

$$\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \quad \text{and} \quad \Gamma_{ij}^k = \Gamma_{ij\alpha} g^{\alpha k},$$

where g^{ij} are components of the matrix inverse to (g_{ij}).

The Riemannian tensors of the first and the second type are given by

$$R_{hijk} = g_{h\alpha} R_{ijk}^\alpha \quad \text{and} \quad R_{ijk}^h = \partial_j \Gamma_{ik}^h - \partial_k \Gamma_{ij}^h + \Gamma_{ik}^\alpha \Gamma_{\alpha j}^h - \Gamma_{ij\alpha} \Gamma_{\alpha k}^h.$$

Then from Gauss's Theorema Egregium for surfaces $\mathbb{S}_2 \in C^3$ it follows that ([62, § 22.2], [97, p. 145]):

$$K = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2}.$$

This formula defines the curvature K in a (pseudo-) Riemannian manifold $\mathbb{V}_2$.

Finally, we recall the Gauss equations

$$\partial_{ij}\mathbf{p} = \Gamma_{ij}^k \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}. \tag{3.54}$$

Let a curve ℓ: $\mathbf{p} = \mathbf{p}(s)$ be an isoperimetric extremal of rotation on a surface $\mathbb{S}_2$ parametrized by arclength s. On the other hand, because $\ell \subset \mathbb{S}_2$: $\mathbf{p} = \mathbf{p}(x^1, x^2)$ there exist inner equations ℓ: $x^i = x^i(s)$ such that the following is valid

$$\mathbf{p}(s) = \mathbf{p}\left(x(s)\right)$$

for all $s \in I$, where $\mathbf{p}$ on the left side is a vector function describing the curve ℓ and $\mathbf{p}$ on the right side is a vector function describing the surface S. Let us denote d/ds by a dot. Then $\dot{\mathbf{p}}(s)$ is a unit tangent vector of ℓ.

We compute the second order derivative for a vector $\mathbf{p}(s)$:

$$\begin{aligned}
\dot{\mathbf{p}}(s) &= \mathbf{p}_i\left(x(s)\right) \cdot \dot{x}^i(s) \\
\ddot{\mathbf{p}}(s) &= \partial_{ij}\mathbf{p}\left(x(s)\right) \dot{x}^i(s) \cdot \dot{x}^j(s) + \mathbf{p}_k \cdot \ddot{x}^k(s).
\end{aligned}$$

Now we apply the Gauss equation (3.54) and we obtain

$$\ddot{\mathbf{p}}(s) = \left(\ddot{x}^k(s) + \Gamma_{ij}^k \cdot \dot{x}^i(s)\dot{x}^j(s)\right) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m}. \tag{3.55}$$

It is obvious that vector $\ddot{\mathbf{p}}(s)$ splits into two components: into a normal vector $\mathbf{m}$ and a unit vector $\mathbf{n}$ which is orthogonal to a vector $\mathbf{m}$ and $\dot{\mathbf{p}}(s)$. This vector is tangent to a surface $\mathbb{S}_2$, therefore we can write $\mathbf{n} = n^k\ \mathbf{p}_k$, where n^k are components of the vector $\mathbf{n}$.

Therefore from (3.55) it follows that

$$(\ddot{x}^k(s) + \Gamma_{ij}^k(x(s)) \cdot \dot{x}^i(s)\ \dot{x}^j(s)) \cdot \mathbf{p}_k + b_{ij} \cdot \mathbf{m} = k_g \cdot n^k \cdot \mathbf{p}_k + k_n \cdot \mathbf{m},$$

where k_n is a normal curvature of $\mathbb{S}_2$ in the direction of a tangent vector $\lambda = \dot{x}$.

Because vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}$ are linearly independent, the following equation is true

$$\ddot{x}^k(s) + \Gamma_{ij}^k(x(s))\ \dot{x}^i(s)\ \dot{x}^j(s) = k_g \cdot n^k.$$

We can write this equation in the form:

$$\nabla_s \lambda = k_g \cdot n. \tag{3.56}$$

The formula above is an analogue of the Frenet formulas for the flat curves, see [62], [97, SS 12], and for the curves with non-isotropic tangent vector λ ($|\lambda| \neq 0$) on (pseudo-) Riemannian manifolds $\mathbb{V}_2$, see [590, pp. 22-26].

We show efficient construction of a unit vector n which is orthogonal to λ using a discriminant tensor ε and a structure tensor F on $\mathbb{V}_2$ defined by relations

$$\varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|}\ \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad F_i^h = \varepsilon_{ij} \cdot g^{jh}.$$

The tensor ε is skew-symmetric and covariantly constant and tensor F defined on $\mathbb{V}_2$ structure, for which it holds $F^2 = \pm Id$ and $\nabla F = 0$.

It can be easily proved that vector $F\lambda$ is also a unit vector orthogonal to a unit vector λ. Obviously, it holds that $n = \pm F\lambda$. Therefore from (3.52) and (3.55) follows the theorem.

Theorem 3.8 *The equation of isoperimetric extremal of rotation can be written in the form*

$$\nabla_s \lambda = c \cdot K \cdot F\lambda, \tag{3.57}$$

where $c =$ const .

Remark. Further differentiation of the equation (3.57) gives the equation (3.53) by Leiko [581–588]. Note that the equation (3.57) has more simple form than the equation (3.53). If $c = 0$ is satisfied then the curve is geodesic.

3.4.7 On the existence of isoperimetric extremals of rotation

Analysis of the equation (3.57) convinces of the validity of the following theorem which generalizes and refines the results of Leiko [581–588].

Theorem 3.9 *Let $\mathbb{V}_2$ be a (non flat) Riemannian manifold of the smoothness class C^3. Then there is precisely one isoperimetric extremal of rotation going through a point $x_0 \in \mathbb{V}_2$ in a given non-isotropic direction $\lambda_0 \in T\mathbb{V}_2$ and constant c.*

Proof. Let x_0^h be coordinates of a point x_0 at $\mathbb{V}_2 \in C^3$ and λ_0^h $(\neq 0)$ be coordinates of a unit tangent vector λ_0 in a given point x_0.

We will find an isoperimetric extremal of rotation ℓ: $x^h = x^h(s)$, where s is the arc length, on a space $\mathbb{V}_2$ such that $x^h(0) = x_0^h$ and $\dot{x}^h(0) = \lambda_0^h$, i. e. this curve goes through a point x_0 in the direction λ_0.

Let us write equation (3.57) as a system of ordinary differential equations:

$$\begin{aligned} \dot{x}^h(s) &= \lambda^h(s) \\ \dot{\lambda}^h(s) &= -\Gamma_{ij}^h(x(s)) \cdot \lambda^i(s) \cdot \lambda^j(s) + c \cdot K\,(x(s)) \cdot F_i^h(x(s)) \cdot \lambda^i(s). \end{aligned} \tag{3.58}$$

From the theory of differential equations it is known (see [39, 82, 83]) that given the initial conditions $x^h(0) = x_0^h$ and $\lambda^h(0) = \dot{x}^h(0) = \lambda_0^h$ the system (3.58) has only one solution when

$$\Gamma_{ij}^h \in C^1, K \in C^1 \text{ and } F_i^h \in C^1. \tag{3.59}$$

These conditions (3.59) are met on a space $\mathbb{V}_2 \in C^3$ (we consider that $\mathbb{V}_2$ is a metric of some surface $\mathbb{S}_2 \subset \mathbb{E}_3$ of the smoothness class C^4).

Correctness of the solution of (3.58) lies in the fact that the vector $\lambda(s)$ is unit for all s. Evidently, $\langle \lambda, \lambda \rangle$ is constant along ℓ, i.e. $\nabla_s \langle \lambda, \lambda \rangle = 2 \cdot \langle \lambda, \nabla_s \lambda \rangle = 0$, and from $\langle \lambda_0, \lambda_0 \rangle = \pm 1$ it follows $\langle \lambda, \lambda \rangle = \pm 1$. □

Remark. It is possible to substitute the condition (3.59) by the Lipschitz's condition for these functions.

Continuity of these functions is guaranteed by the existence of a solution to (3.58). This is possible when $\mathbb{V}_2 \in C^2$, resp. $\mathbb{S}_2 \in C^3$.

3.5 Kähler manifolds

3.5.1 Definition and basic properties of Kähler manifolds

This section is devoted to basic concepts of the theory of Kähler[58] manifolds (spaces) which will be useful in what follows. Kähler manifolds can be introduced in several equivalent ways. A Kähler manifold carries three structures that are compatible:

$$\mathbb{K}_n = (M_n, g, F),$$

where M_n is a manifold, g is a metric, and $F\,(\neq \alpha\, Id)$ is a structure tensor of $\mathbb{K}_n$.

Definition 3.8 A Riemannian manifold (M_n, g) is called a *Kähler manifold* $\mathbb{K}_n$ if besides the metric tensor g, a tensor field F of type $(1,1)$ (i.e. an affinor) is given on M_n, called a *structure* F, such that the following conditions hold:

$$(a)\ \ F^2 = e\, Id, \quad (b)\ \ g(X, FX) = 0, \quad (c)\ \ \nabla F = 0, \tag{3.60}$$

where $e = \pm 1, 0$, X is an arbitrary tangent vector of M_n, and ∇ denotes the covariant derivative in $\mathbb{K}_n$.

The manifold $\mathbb{K}_n$ is called

- *elliptic Kähler* $(\mathbb{K}_n^-)$ and F is a *complex structure* when $e = -1$,
- *hyperbolic Kähler* $(\mathbb{K}_n^+)$ and F is a *product structure* when $e = 1$,
- *m-parabolic Kähler* $(\mathbb{K}_n^{o(m)})$ and F is a *tangent structure* when $e = 0$ and $\operatorname{rank} F = m \leq \frac{n}{2}$,
- *parabolic Kähler* $(\mathbb{K}_n^o)$ when $e = 0$ and $\operatorname{rank} F = \frac{n}{2}$.

In components, the formulae (3.60) read as follows:

$$(a)\ \ F_\alpha^h F_i^\alpha = e\delta_i^h, \quad (b)\ \ g_{i\alpha}F_j^\alpha + g_{j\alpha}F_i^\alpha = 0, \quad (c)\ \ F_{i,j}^h = 0, \tag{3.61}$$

where "," denotes the covariant derivative in $\mathbb{K}_n$, $h, i, \alpha \in \{1, \cdots, n\}$.

The formula (3.60b) tells that the tensor $F_{ij} = g_{i\alpha}F_j^\alpha$ is skew-symmetric. For $\mathbb{K}_n^\pm$ and $\mathbb{K}_n^o$ we find out that necessarily the dimension is even, $n = 2m$.

According to the third condition (3.60c), the affinor structure F is locally integrable in $\mathbb{K}_n$. It means that each point has a coordinate neighbourhood in which the components F_i^h of the tensor F are constant.

The spaces $\mathbb{K}_n^-$ were first considered by P.A. Shirokov [167], the spaces $\mathbb{K}_n^+$ were considered by P.K. Rashevskij[59] [768], and $\mathbb{K}_n^o$ were considered by V.V. Vishnevskij[60] [931]. In the investigations mentioned these spaces are referred to as *A-spaces*. Independently of P.A. Shirokov the spaces $\mathbb{K}_n^-$ were studied by E. Kähler. In the mathematical literature these spaces are preferably referred to as Kählerian spaces (see, for example, [119, 122, 170]).

[58] Erich Kähler, 1906-2000, was a German mathematician with wide-ranging geometrical interests.

[59] Petr Konstantinovich Rashevskij, 1907-1985, was a Russian mathematician, Moscow State University, who did fundamental work in the theory of differential geometry [150–152].

[60] Vladimir Vladimirovich Vishnevskij, 1929-2008, was a Russian mathematician, Kazan State University, who did fundamental work in the theory of differential geometry [190, 191].

Under the term *almost Hermitian spaces* we understand all Riemannian spaces where the structure F exists, for which the conditions (3.60a) and (3.60b): $F^2 = -Id$ and $g(X, FX) = 0$ for all vectors X tangent to M_n hold. The investigations [119,170,197] were devoted to the geometrical problems of almost Hermitian spaces.

3.5.2 Canonical coordinates on Kähler manifolds

Elliptic (classical) Kähler manifold

Let us examine skew-symmetric relations that take place in the case $e = -1$, i.e. for elliptic Kähler manifolds. According to (3.60), if $e = -1$ each point has a neighbourhood in which the structure F has the following canonical form:

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a, \quad F_b^a = F_{b+m}^{a+m} = 0, \quad a, b = 1, \cdots, m;\ m = \frac{n}{2}; \quad (3.62)$$

we get, as an immediate consequence, that the dimension is even, $m = 2n$. Such a coordinate system will be called *canonical*. Due to the conditions (3.60) and (3.62), the components of the metric tensor in a canonical coordinate system satisfy

$$g_{a+m,b+m} = g_{ab}, \quad g_{ab+m} = -g_{a+mb}. \quad (3.63)$$

Obviously, the coordinate transformation

$$x'^h = x'^h(x) \quad (3.64)$$

preserves a canonical coordinate system if and only if the Jacobi matrix $J = (A_i^h) = (\frac{\partial x'^h}{\partial x^i})$ satisfies

$$A_{b+m}^{a+m} = A_b^a, \qquad A_b^{a+m} = -A_{b+m}^a. \quad (3.65)$$

Let us set $z^a = x^a + ix^{a+m}$, $z'^a = x'^a + ix'^{a+m}$ (where i is the imaginary unit). Then (3.65) can be interpreted as Cauchy-Riemann conditions for complex functions $z'^a = z'^a(z^1, \cdots, z^m)$, and we will call this mapping *analytic*.

It follows by (3.60) that in canonical coordinate system in $\mathbb{K}_n^-$ the components Γ_{ij}^h (Christoffel symbols of second kind) of the Levi-Civita connection satisfy

$$\Gamma_{bc}^a = \Gamma_{b+mc+m}^{a+m} = -\Gamma_{b+mc+m}^a, \qquad \Gamma_{b+mc+m}^{a+m} = \Gamma_{b+mc}^a = -\Gamma_{bc}^{a+m}. \quad (3.66)$$

We can check that Christoffel symbols of first kind are related by

$$\begin{aligned} &\Gamma_{abc} = \Gamma_{ab+mc+m} = -\Gamma_{a+mb+mc}, \\ &\Gamma_{a+mb+mc+m} = \Gamma_{a+mbc} = -\Gamma_{abc+m}. \end{aligned} \quad (3.67)$$

The conditions (3.67) are equivalent to the system of equations

$$\begin{aligned} &\partial_a g_{bc} - \partial_c g_{ab} = \partial_{a+m} g_{bc+m} - \partial_{c+m} g_{ba+m}, \\ &\partial_{a+m} g_{bc} - \partial_{c+m} g_{ab} = \partial_a g_{b+mc} - \partial_c g_{b+ma}. \end{aligned} \quad (3.68)$$

Using (3.63) we can write (3.68) in the form

$$\begin{aligned} &(a)\ \partial_a g_{bc} - \partial_c g_{ba} = \partial_{b+m} g_{ac+m}, \\ &(b)\ \partial_{a+m} g_{bc} - \partial_{c+m} g_{ba} = \partial_b g_{ca+m}. \end{aligned} \tag{3.69}$$

Now we calculate

$$\begin{aligned} \Gamma_{abc} &= \tfrac{1}{2}\,(\partial_b g_{ac} + \partial_{b+m} g_{ac+m}), \\ \Gamma_{a+mb+mc+m} &= \tfrac{1}{2}\,(\partial_{b+m} g_{ac} + \partial_b g_{a+mc}). \end{aligned} \tag{3.70}$$

Note that if in some coordinate neighbourhood, the metric g of a Riemannian manifold satisfies the conditions (3.63) and (3.69), then it is necessary an elliptic Kähler manifold, and the components of its structure tensor F in this system are just given by (3.62).

Any fixed point $x_0 \in M$ of an elliptic Kähler manifold $\mathbb{K}_n^-$ has a coordinate neighbourhood with canonical coordinate system in which the metric tensor satisfies

$$g_{ii}(x_0) = e_i = \pm 1, \qquad g_{ij}(x_0) = 0 \ \text{ for } i \neq j. \tag{3.71}$$

Since $g_{aa} = g_{a+ma+m}$, i.e. $e_a = e_{a+m}$, the metric of an elliptic Kähler manifold has even signature.

Hyperbolic Kähler manifold

Analogously, in a hyperbolic Kähler manifold, there are local coordinate systems, again called *canonical*, in which the structure tensor F has components

$$F_b^a = \delta_b^a, \quad F_B^A = -\delta_B^A, \quad F_A^a = F_a^A = 0, \tag{3.72}$$

where $a, b = 1, \ldots, m$ and $A, B = m+1, \ldots, n$. Using (3.60) we find that components of the metric tensor satisfy $g_{ab} = g_{AB} = 0$. Since $\det(g_{ij}) \neq 0$ it follows that $m = \frac{1}{2}\,n$, the dimension n of a hyperbolic Kähler manifold is always even.

In canonical coordinate system, components of the structure tensor F and of the metric tensor of a hyperbolic Kähler manifold read

$$F_b^a = -F_{b+m}^{a+m} = \delta_b^a, \qquad F_{b+m}^a = F_b^{a+m} = 0, \tag{3.73}$$

$$g_{ab} = g_{a+mb+m} = 0 \tag{3.74}$$

where a, b range over $1, \ldots, m$, and $m = n/2$. Obviously in a hyperbolic Kähler manifold, the structure tensor cannot be a multiple of the identity tensor, $F \neq \alpha\, I$, i.e.

$$F_i^h \neq \alpha\, \delta_i^h. \tag{3.75}$$

The coordinate transformation (3.64) preserves a canonical coordinate system if and only if the following holds:

$$A_{b+m}^a = A_b^{a+m} = 0, \qquad A_i^h = \frac{\partial x'^h}{\partial x^i}. \tag{3.76}$$

It follows from (3.60) that in canonical coordinates, components of the Riemannian connection satisfy

$$\Gamma^a_{bc+m} = \Gamma^a_{b+mc+m} = \Gamma^{a+m}_{bc} = \Gamma^{a+m}_{b+mc} = 0. \tag{3.77}$$

For Christoffels of first kind we get the equalities

$$\Gamma_{abc} = \Gamma_{a+mbc} = \Gamma_{ab+mc+m} = \Gamma_{a+mb+mc+m} = 0. \tag{3.78}$$

The system (3.78) is equivalent to

$$\partial_b g_{a+mb} = \partial_c g_{a+mb}, \qquad \partial_{b+m} g_{ac+m} = \partial_{c+m} g_{ab+m}. \tag{3.79}$$

Consequently, Christoffels of first kind read

$$\Gamma_{abc+m} = \partial_a g_{bc+m}, \qquad \Gamma_{a+mb+mc} = \partial_{a+m} g_{b+mc}. \tag{3.80}$$

Note that if in some coordinate neighbourhood, the metric g of a Riemannian manifold satisfies the conditions (3.74) and (3.79), then it is necessarily a hyperbolic Kähler manifold, and the components of its structure tensor F in this system are just given by (3.73).

Moreover, for each point x_0 of a hyperbolic Kähler manifold there are local canonical coordinates around x_0 such that the components of the metric satisfy

$$g_{aa+m}(x_0) = 1, \quad g_{ab+m}(x_0) = g_{ab}(x_0) = g_{a+mb+m}(x_0) = 0 \ (a \neq b). \tag{3.81}$$

3.5.3 The operation of conjugation

To make the calculations easier and the formulae more homogeneous and nicer, let us introduce an operation of conjugation in an elliptic and hyperbolic Kähler manifold as follows:

$$A_{\dots\bar{i}\dots} = A_{\dots\alpha\dots} F^\alpha_i, \quad B^{\dots\bar{i}\dots} = B^{\dots\alpha\dots} F^i_\alpha.$$

According to the definition and (3.60) this operation has the following properties:

$$\begin{aligned} &A_{\bar{\bar{i}}} = eA_i,\ A_{\bar{\alpha}} B^{\alpha} = A_\alpha B^{\bar{\alpha}},\ \ (A_{\bar{i}})_{,j} = A_{\bar{i},j}, \\ &B^{\bar{\bar{i}}} = eB^i,\ A_{\bar{\alpha}} B^{\bar{\alpha}} = eA_\alpha B^\alpha,\ (B^{\bar{i}})_{,j} = B^{\bar{i}}_{,j} \end{aligned}. \tag{3.82}$$

Note that

$$\delta^h_{\bar{i}} = \delta^{\bar{h}}_i = F^h_i. \tag{3.83}$$

The formulae (3.60b) can be written as follows: $g(FX, Y) + g(X, FY) = 0$, i.e.

$$g_{\bar{i}j} + g_{i\,\bar{j}} = 0. \tag{3.84}$$

Contraction of (3.84) with F^j_k is equivalent to conjugation of (3.84) in j and yields

$$g_{\bar{i}\,\bar{k}} = -e\, g_{ik}. \tag{3.85}$$

The components of the dual tensor to g, usually denoted by g^{ij}, satisfy

$$g^{\bar{i}j} + g^{i\bar{j}} = 0, \quad g^{\bar{i}\bar{j}} = -e\, g^{ij}. \tag{3.86}$$

The first equation is obtained by contraction of (3.84) with $g^{ik}g^{j\ell}$, and the second one is an immediate consequence.

The integrability conditions for the equations (3.60c) give

$$F(R(Y,Z)X) = R(Y,Z)FX, \quad \text{i.e.} \quad F^h_\alpha R^\alpha_{ijk} - F^\alpha_i R^h_{\alpha jk} = 0,$$

or, accounting conjugation,

$$R^{\overline{h}}_{\bar{i}jk} = R^h_{\bar{i}jk}. \tag{3.87}$$

Here R^h_{ijk} are components of the Riemann tensor R of the metric g.

The equations (3.87) can be written equivalently as

$$R^{\overline{h}}_{\bar{i}jk} = eR^h_{ijk}.$$

Lowering the index h we get

$$R_{\overline{h}\,\bar{i}\,jk} = -eR_{hijk}. \tag{3.88}$$

Using properties of the Riemann tensor we obtain

$$R_{hi\bar{j}\,\bar{k}} = -eR_{hijk}, \quad R_{\overline{h}\,\bar{i}\,\bar{j}\,\bar{k}} = R_{hijk}. \tag{3.89}$$

Contraction of the last formula with g^{hk} gives

$$R_{\bar{i}\,\bar{j}} = -eR_{ij}, \tag{3.90}$$

and finally,

$$R_{\bar{i}\,j} + R_{i\,\bar{j}} = 0. \tag{3.91}$$

Formulae (3.88)–(3.91) can be written in the following form

$$R(FV,FX,Y,Z) = R(V,X,FY,FZ) = -e\,R(V,X,Y,Z) = -e\,R(FV,FX,FY,FZ),$$

$$Ric(FX,FY) = -e\,Ric(X,Y), \quad Ric(FX,Y) + Ric(X,FY).$$

Since in Riemannian space, the Bianchi identity holds, $R^h_{ijk}+R^h_{jki}+R^h_{kij} = 0$, we have $R^{\overline{h}}_{ijk} + R^{\overline{h}}_{jki} + R^{\overline{h}}_{kij} = 0$. Contracting in h and i we find

$$R^{\overline{\alpha}}_{\alpha jk} = 2R_{\bar{j}\,k}. \tag{3.92}$$

Recall that in Kähler manifolds $\mathbb{K}_n$, besides the already mentioned Riemannian and Ricci tensors, we introduce by means of the metric g and the structure F also the so-called tensor of holomorphically projective curvature (13.12), see p. 421.

3.5.4 Holomorphic curvature

In a Kähler manifold K_n, besides the classical Riemann curvature, also the so-called *holomorphic curvature* is defined for a two-dimensional subspace of the tangent space at a point $p \in M$: if X is a tangent vector, we consider the two-space given by the pair of vectors X and FX, and set

$$k_p(X) = K_p(X, FX) = \frac{R(X, FX, X, FX)}{-e\, g(X, X)^2}. \tag{3.93}$$

3.5.5 Space of constant holomorphic curvature

Definition 3.9 A Kähler manifold is called a *space of constant holomorphical curvature* when its holomorphical curvature is independent of points as well as of vectors (directions).

We find by (3.93) that K_n, $n \geq 4$, is a space of constant holomorphical curvature k if and only if its curvature tensor R is related to the metric by

$$R = \frac{k}{4}\, G \tag{3.94}$$

where

$$\begin{gathered} G(V, X, Y, Z) = g(V, Y)g(X, Z) - g(V, Z)g(X, Y) \\ -e\, g(V, FY)g(X, FZ) + e\, g(V, FZ)g(X, FY) - 2e\, g(V, FX)g(Y, FZ). \end{gathered} \tag{3.95}$$

In local expression

$$R_{hijk} = \frac{k}{4}\, G_{hijk} \tag{3.96}$$

where $G_{hijk} = g_{hj}g_{ik} - g_{hk}g_{ij} - eg_{h\bar{j}}g_{i\bar{k}} + eg_{h\bar{k}}g_{i\bar{j}} - 2eg_{h\bar{i}}g_{j\bar{k}}$.

Contracting (3.96) with g^{hk} we get easily that the space of constant holomorphical curvature is an Einstein space, that is, the Ricci tensor of which satisfies $Ric = \varrho\, g$, and $\varrho = \frac{k}{4}(n+2)$. Since for Einstein spaces ($n > 2$) the invariant ϱ is constant, k is necessarily constant as well, and consequently by (3.96), the Riemann tensor of the space of constant holomorphical curvature is covariantly constant, that is, the following holds: $\nabla R = 0$, i.e. $\mathbb{K}_n$ is a symmetric space.

For symmetric spaces, P.A. Shirokov obtained the following formula for the components of the metric tensor in a neighbourhood of a fixed point $x_0 \in M$ of the Riemannian manifold:

$$g_{ij} = \overset{o}{g}_{ij} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{(2k+2)!} m_{ij}^{(k)}, \tag{3.97}$$

where $m_{ij}^{(1)} = m_{ij}$, $m_{ij}^{(k+1)} = m_{i\alpha}^{(k)} m_{\beta j} \overset{o}{g}{}^{\alpha\beta}$, $m_{ij} = \overset{o}{R}_{i\alpha i\beta} y^\alpha y^\beta$, $\overset{o}{g}_{ij}$, $\overset{o}{g}{}^{ij}$, $\overset{o}{R}_{i\alpha j\beta}$ are components at x_0 of the metric tensor, its dual, and the Riemann tensor, and y^h are Riemannian coordinates about x_0.

It can be checked that the metric tensor of a space of constant holomorphical curvature has the components

$$\begin{aligned} g_{ij} \;=\; & \frac{2}{ky}\overset{o}{g}_{ij}(1-\cos\sqrt{ky})+\frac{1}{y}y_iy_j(1-\frac{2}{ky}(1-\cos\sqrt{ky}) \\ & +\frac{e}{y}\overline{y}_i\overline{y}_j\frac{1}{2ky}(3+\cos 2\sqrt{ky}-4\cos\sqrt{ky}), \end{aligned} \tag{3.98}$$

where $y = \overset{o}{g}_{ij}y^iy^j$, $y_i = \overset{o}{g}_{i\alpha}y^\alpha$, $\overline{y}_i = y_\alpha\overset{o}{F}{}^\alpha_i$, and $\overset{o}{F}{}^h_i$ are components of the structure tensor F at the point x_0.

Note that if in the formula (3.98), the expressions under the square root are negative we use the formula relating the functions cosine and hyperbolic cosine, namely $\cos i\varphi = \cosh\varphi$. Moreover, if $y = 0$ we define g_{ij} as a limit value of (3.98) for $y \to 0$. Clearly, this limit is always defined.

It follows that if $\mathbb{K}_n$ is a space of constant curvature then it is Euclidean. On the other hand, an even-dimensional Euclidean space of even non-zero signature is an elliptic (in the case of zero signature a hyperbolic) Kähler manifold of constant holomorphical curvature.

3.5.6 Analytic vector fields

A vector field λ in K_n is called *analytic* if it satisfies $\nabla_X(F\lambda) = \nabla_{FX}(\lambda)$, i.e.

$$\lambda^{\overline{h}}_{,i} = \lambda^h_{,\overline{i}}\,. \tag{3.99}$$

Vector fields satisfying this condition play the important role in Kähler manifolds. As well known, for any non-vanishing vector field λ, there are local coordinates about any point of K_n such that $\lambda^h = \delta^h_1$. For an analytic vector field in $\mathbb{K}^-_n$ there are locally canonical coordinates in which the coordinate expression of the components of the field is analogous:

Lemma 3.2 *Given a non-vanishing analytic vector field λ in an elliptic Kähler manifold $\mathbb{K}^-_n$, each point of $\mathbb{K}^-_n$ has a neighbourhood with canonical coordinates such that the components of the vector field are*

$$\lambda^h = \delta^h_1. \tag{3.100}$$

Proof. Let λ be a non-vanishing analytic vector field in an elliptic Kähler manifold $\mathbb{K}^-_n$ and let x be a fixed point in $\mathbb{K}^-_n$. Let us choose canonical coordinates $(x^1,\dots,x^n)$ about x in which the structure tensor F has the canonical form (3.62). In this coordinates, the condition (3.99) reads

$$\partial_a\lambda^b = \partial_{a+m}\lambda^{b+m}, \quad \partial_{a+m}\lambda^b = -\partial_a\lambda^{b+m}.$$

Due to this formulae the functions $\Lambda^a(z) = \lambda^a + i\lambda^{a+m}$, $(i = \sqrt{-1})$ are analytic functions of complex coordinates $(z^1,\dots,z^m)$.

Under an analytic coordinate transformation $z^{'a} = z^{'a}(z)$ that preserves the structure F_i^h as canonical, as we have already seen, the components of the vector field $\Lambda^a(z)$ transform as follows:

$$\Lambda'^a(z') = \Lambda^b(z)\frac{\partial z'^a(z)}{\partial z^b}, \tag{3.101}$$

where

$$\Lambda'^a = \lambda'^a + i\lambda'^{a+m}, \quad z'^a = x'^a + ix'^{a+m}.$$

Let us consider the homogeneous linear partial differential equation of first order with respect to an unknown analytic function $\Phi(z^1, \ldots, z^m)$ in complex variables:

$$\Lambda^a(z)\frac{\partial \Phi(z)}{\partial z^a} = 0.$$

This equation admits $m-1$ functionally independent solutions $\Phi^2(z)$, $\Phi^3(z)$, $\ldots$, $\Phi^m(z)$. Now let us consider the inhomogeneous differential equation

$$\Lambda^a(z)\frac{\partial \Phi(z)}{\partial z^a} = 1.$$

Obviously, as 1 on the right-hand side is an analytic function, this equation has an analytic solution $\Phi^1(z)$ that is functionally independent of the solutions $\Phi^j(z)$, $j = 2, \ldots, m$. Under the coordinate transformation $z'^a = \Phi^a(z)$, the components of the vector field $\Lambda'^a(z')$ are $\Lambda'^a(z') = \delta_1^a$, in accordance with (3.101). But this means that the real components of λ are $\lambda'^a = \delta_1^a$, $\quad \lambda'^{a+m} = 0$ which finishes the proof. □

In a similar way we can prove the following:

Lemma 3.3 *Given a non-vanishing analytic vector field λ on a hyperbolic Kähler manifold $\mathbb{K}_n^+$, there are canonical coordinates about any point of $\mathbb{K}_n^+$ such that the components of the vector field are*

$$\lambda^h = o_1\delta_1^h + o_2\delta_{1+m}^h, \tag{3.102}$$

where $o_1, o_2 \in \{0, 1\}$ and $o_1 + o_2 > 0$.

3.6 Equidistant spaces

3.6.1 Torse-forming and concircular vector fields

Geometric properties of Riemannian manifolds are frequently studied with respect to the existence of certain vector fields. Vector fields can be related to automorphisms on manifolds and used in this way to classify them according to some symmetry properties. For the sake of completeness, we begin here with a rather general class of vector fields and specify it in the sequel to a subclass, which is related to so-called equidistant manifolds, a class of Riemannian spaces including cosmological models. The general class of vector fields to begin with are the so-called torse-forming vector fields, with the special subclasses of recurrent and concircular fields.

Riemannian manifolds in which these vector fields exist have a metric in a special form, called warped product form. Torse-forming and concircular vector fields were introduced by K. Yano [950] in 1944. Special types of these manifolds were studied before: covariantly constant by T. Levi-Civita [107], convergent by P.A. Shirokov [167], and, concircular by H.W. Brinkmann [314].

The manifolds in which concircular vector fields exist are called *equidistant* manifolds; this concept was introduced by N.S. Sinyukov [170], see [118]. In a number of other papers these manifolds are denoted as *almost warped product manifolds* [362]–[376]. See also [79, 119, 149, 170, 568, 685, 843, 935, 949].

First, we define torse-forming fields and introduce some of their properties, which are used to define recurrent and concircular vector fields.

Definition 3.10 Vector fields Φ in $\mathbb{A}_n$ are called *torse-forming*, when:

$$\nabla_X \Phi = \varrho \cdot X + \omega(X) \cdot \Phi, \tag{3.103}$$

where X is an arbitrary vector field of $\mathbb{A}_n$, ϱ is some function on $\mathbb{A}_n$, ω is a linear form on $\mathbb{A}_n$.

Definition 3.11 A torse-forming vector field Φ which is defined by (3.103) is called:

1. *recurrent* if $\varrho \equiv 0$, i.e. $\nabla_X \Phi = \omega(X)\, \Phi$,
2. *concircular* if the form ω is a gradient (or a local gradient), i.e. (locally) there exists a function a that $\omega = d\, a$,
3. *specially concircular*, if $\omega = 0$.
4. *convergent* if Φ is concircular and $\varrho = \text{const} \cdot \exp(a)$.

Any integral curve ℓ of a torse-forming vector field Φ is a geodesic, because from (3.103) we obtain $\nabla_\Phi \Phi = \sigma\, \Phi$.

On a Riemannian manifold $\mathbb{V}_n$ with a metric tensor g we consider a linear form $\varphi(X) = g(X, \Phi)$. Locally this form $\varphi(X)$ is always a gradient.

A torse-forming (including concircular and convergent) vector field Φ will be called *gradient* if the linear form $\omega(X)$ is a gradient, i.e. on $\mathbb{V}_n$ there exists a scalar function ϕ for which $\omega(X) = \nabla_X \phi$.

A form $\omega(X)$ corresponding to torso-forming fields Φ is collinear to a form $\varphi(X)$. Subsequently, on Riemannian manifolds $\mathbb{V}_n$ we can write equations (3.103) for these vector fields as follows:

$$\nabla_X \nabla_Y \phi = \nu\, g(X,Y) + \tau\, \nabla_X \phi\, \nabla_Y \phi \tag{3.104}$$

where ν, τ are functions.

For concircular and torse-forming fields, we naturally require

$$\Phi, \varphi \in C^1; \quad \phi \in C^2; \quad \mathbb{A}_n \in C^0; \quad \mathbb{V}_n \in C^1.$$

See a local expression

$$\Phi^h(x) \in C^1; \quad \varphi_i(x) \in C^1; \quad \phi \in C^2; \quad \Gamma^h_{ij}(x) \in C^0; \quad g_{ij}(x) \in C^1,$$

where Γ^h_{ij} are components of the affine connection ∇ on $\mathbb{A}_n$, g_{ij} are components of a metric tensor on Riemannian manifold $\mathbb{V}_n$, C^r is the differentiability class. Naturally, $\nu \in C^0$ and $\tau \in C^0$.

For a specially concircular vector field Φ, the fundamental equation can be written in the form:

$$\nabla_X \Phi = \varrho \cdot X, \quad \forall X \in \mathcal{X}(\mathbb{V}_n), \tag{3.105}$$

or locally

$$\varphi^h_{,i} = \varrho \cdot \delta^h_i. \tag{3.106}$$

Since after a suitable normalization, any concircular vector field Φ can be written in the form of a specially concircular vector field, we will call this vector field simply **concircular**. In the above terminology and notation, these fields were introduced by K. Yano [949]; independently by Ya.L. Shapiro [786] as **geodesic fields**, and by N.S. Sinyukov [170, 813] as **equidistant fields**.

From this equation some important properties follow immediately.

1) With the index h lowered this becomes

$$\varphi_{h,i} = \varrho \cdot g_{hi}, \tag{3.107}$$

showing the symmetry of the first covariant derivative of the covector field φ_i, $\varphi_{i,h} = \varphi_{h,i}$. In other words, the one-form φ is closed and so locally there is a function ϕ whose exterior derivative is φ. In a coordinate neighbourhood the equations $\phi(x) = \text{const}$ define hypersurfaces in $\mathbb{V}_n$ with the normal form $\varphi = d\phi$.

2) Contracting (3.107) with φ^i we get

$$\varphi^h_{,\alpha}\, \varphi^\alpha = \varrho\, \varphi^h,$$

showing that the vector field φ^h is geodesic. As long as it is not isotropic, its integral curves form a congruence of geodesics transversal to the family of hypersurfaces given by $\phi(x) = \text{const}$.

3) Differentiating the length of the vector field φ,

$$(\varphi_\alpha \varphi^\alpha)_{,k} = 2\,\varphi_\alpha \varphi^\alpha_{,k} = 2\,\varphi_i \varrho\,\delta^i_k = 2\varrho\,\varphi_k,$$

we find that it varies only along φ, i.e. it is constant on a hypersurface. In this sense the distance between two hypersurfaces $\phi = c_1$ and $\phi = c_2$ is everywhere the same, therefore the notion *equidistant*.

4) If $\varrho \equiv$ const then a concircular field is *convergent*.

5) If $\varrho \neq$ const then a concircular field is non-convergent and nonisotropic.

3.6.2 On differentiability of functions with special conditions

First prove the following universal lemmas of differentiability of solution of special equations (I. Hinterleitner, J. Mikeš [482]).

Lemma 3.4 *Let $\lambda^h(x) \in C^1$ be a vector field and $\varrho(x)$ a function. If $\partial_i \lambda^h - \varrho\,\delta^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.*

Validity of Lemma 3.4 follows from the more general lemma.

Lemma 3.5 *Let $\lambda^h(x) \in C^1$ be a vector field, $\varrho(x)$ a function and*

$$D^h_i = \begin{pmatrix} \delta^a_b & 0 \\ 0 & 0 \end{pmatrix}, \quad a, b = 1, \ldots, r, \quad 1 < r \leq n.$$

If $\partial_i \lambda^h - \varrho\, D^h_i \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.

Proof. The condition $\partial_i \lambda^h - \varrho\, D^h_i \in C^1$ can be written in the following form

$$\partial_i \lambda^h - \varrho\, D^h_i = f^h_i(x), \tag{3.108}$$

where $f^h_i(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $i \neq a$, $1 \leq i \leq n$, $1 \leq a \leq r$, we integrate (3.108) with respect to dx^i:

$$\lambda^a = \Lambda^a + \int_{x^i_o}^{x^i} f^a_i(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots, x^n)\, dt,$$

where Λ^a is a function, which does not depend on x^i.

Because of the existence of the partial derivatives of the functions λ^a and the above integrals (see [100, p. 300]), also the derivatives $\partial_a \Lambda^a$ exist; in this proof we don't use Einstein's summation convention. Then we can write (3.108) for $h = i = a$:

$$\varrho = -f^a_a + \partial_a \Lambda^a + \int_{x^i_o}^{x^i} \partial_a f^a_i(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots, x^n)\, dt. \tag{3.109}$$

Because the derivative with respect to x^i of the right-hand side of (3.109) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i \varrho = \partial_h f^h_i - \partial_i f^h_h$, therefore $\varrho \in C^1$ and from (3.108) follows $\lambda^h \in C^2$. □

In a similar way we can prove that for $r = 1$ Lemma 3.5 is not valid.

3.6.3 Fundamental equations of concircular vector fields for minimal differentiable conditions

We can write formula (3.106) in the following form

$$\nabla_j \varphi^i \equiv \frac{\partial \varphi^i}{\partial x^j} + \Gamma^i_{\alpha j}\varphi^\alpha = \varrho \cdot \delta^i_j. \tag{3.110}$$

It is easily seen that formula (3.110) is true when

$$\mathbb{A}_n \in C^0 \text{ (i.e. } \Gamma^h_{ij}(x) \in C^0),\ \varphi^h(x) \in C^1 \text{ and } \varrho(x) \in C^0,$$

or

$$\mathbb{V}_n \in C^1 \text{ (i.e. } g_{ij}(x) \in C^1),\ \varphi^h(x) \in C^1 \text{ and } \varrho(x) \in C^0.$$

If $\mathbb{A}_n \in C^1$ (i.e. $\Gamma^h_{ij}(x) \in C^1$) or $\mathbb{V}_n \in C^2$ (i.e. $g_{ij}(x) \in C^2$) holds, then from formula (3.110) follows $\dfrac{\partial \varphi^i}{\partial x^j} - \varrho \cdot \delta^i_j \in C^1$, and from Lemma 3.5 we get:

$$\varphi^i(x) \in C^2 \quad \text{and} \quad \varrho(x) \in C^1.$$

From this viewpoint we specify and generalize the results involving concircular vector fields below.

After differention (3.106) we have $\varphi^h_{,jk} = \nabla_k \varrho\, \delta^h_j$ and alternation with respect to indicies j and k we obtain the formula $\varphi^h_{,jk} - \varphi^h_{,kj} = \nabla_k \varrho\, \delta^h_j - \nabla_j \varrho\, \delta^h_k$. From the Ricci identity follows the integrability conditions of equation (3.106) read

$$\varphi^\alpha R^h_{\alpha jk} = \nabla_j \varrho\, \delta^h_k - \nabla_k \varrho\, \delta^h_j, \tag{3.111}$$

where R^h_{ijk} are components of curvature tensor.

We contract (3.111) with indices h and k, we get

$$\nabla_j \varrho = -\frac{1}{n-1}\ \varphi^\alpha R_{\alpha j},$$

where R_{ij} are components of Ricci tensor and (3.111) has form

$$\varphi^\alpha \tilde{W}^h_{\alpha jk} = 0, \tag{3.112}$$

where

$$\tilde{W}^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}\ (R_{ij}\,\delta^h_k - R_{ik}\,\delta^h_j). \tag{3.113}$$

Tensor $\tilde{W}$ is a similar to the Weyl tensor of projective curvature W, see [122, p. 133]. In an equiaffine space $\mathbb{A}_n$ (where the Ricci tensor is symmetric, i.e. $R_{ij} = R_{ji}$) tensor $\tilde{W}$ is identical to W.

Moreover, after contraction of (3.112) with indices h and k for $n > 2$ we get

$$\varphi^\alpha (R_{\alpha i} - R_{i\alpha}) = 0. \tag{3.114}$$

The system of equations

$$\begin{aligned} \nabla_j \varphi^h &= \varrho \cdot \delta^h_j, \\ \nabla_j \varrho &= -\frac{1}{n-1}\, \varphi^\alpha R_{\alpha j} \end{aligned} \tag{3.115}$$

is closed.

It is a system of linear differential equations with respect to the covector $\varphi^h(x)$ and function $\varrho(x)$, of Cauchy type, in first order covariant derivatives with coefficients uniquely determined by the connection ∇ of manifold $\mathbb{A}_n \in C^1$ (or metric g of the Riemannian space $\mathbb{V}_n \in C^2$).

For any family of initial values $\varphi^h(x_0) = \varphi^h_\circ$ and $\varrho(x_0) = \varrho_\circ$ of the functions under consideration in the given point x_0, it admits at most one solution. Consequently, the number of free parameters in the general solution of the system is at most $n+1$. For Riemannian manifolds, see [542, 935], and for manifolds with affine connections, see [214, 232].

Since the system is linear, it admits at most $n+1$ linearly independent solutions corresponding to constant coefficients. It is obvious the cardinality of the system of independent (substantial) concircular vector fields of the space $\mathbb{A}_n \in C^1$.

It is known that only projective flat manifolds, see p. 271, admit the maximal number of $n+1$ linearly independent concircular vector fields. This holds locally. This fact follows from the study of the integrability conditions (3.112) and their differential prolongations (clear $\tilde{W}^h_{ijk} = 0$, $R_{ij} = R_{ji}$ and $R_{ij,k} = R_{ik,j}$).

It follows from the analysis of the system of equations (3.115) that if $\mathbb{A}_n \in C^r$, $r \geq 1$, then $\varphi^h \in C^{r+1}$ and $\varrho \in C^r$. It follows that the function φ belongs to C^{r+1}. From this we obtain the following theorem.

Theorem 3.10 *If the manifold $\mathbb{A}_n$ with affine connection ($\mathbb{A}_n \in C^r$, $r \geq 1$) admits a concircular vector field $\varphi^h \in C^1$, then φ^h belongs to C^{r+1}.*

We suppose that the differentiability class r is equal to $2, 3, \ldots, \infty, \omega$, where ∞ and ω denotes infinitely differentiable, and real analytic functions, respectively.

In the Riemannian spaces $\mathbb{V}_n$ were the equation (3.115) modified as follows

$$\begin{aligned} \nabla_j \varphi_i &= \varrho \cdot g_{ij}, \\ \nabla_j \varrho &= B\, \varphi_j, \end{aligned} \tag{3.116}$$

where B is a function and $\varphi_i = \varphi^\alpha g_{\alpha i}$.

Indeed, contracting the integrability condition (3.111) with φ_h, we obtain easily $\nabla_k \varrho = B\, \varphi_k$. Among other things φ_i is locally gradient-like, i.e. $\varphi_i = \partial_i \Phi$, then it implies that $\varrho = \varrho(\Phi)$ and $B = B(\Phi)$.

3.6.4 A space with affine connection which admits at least two linearly independent concircular vector fields

Furthermore, because the initial conditions $\varphi^h(x_0) = 0$ and $\varrho(x_0) = 0$ have only the trivial solution $\varphi^h(x) = 0$ and $\varrho(x) = 0$ on V_n for the system of equations (3.115). Then $\varphi^h(x)$ and $\varrho(x) = 0$ is vanishing on $\mathbb{A}_n$, if $\varphi^h(x) = 0$ for the neighborhood U_{x_o} of the point x_o. Then the following lemma holds.

Lemma 3.6 *The non-vanishing concircular vector field $\varphi^h(x)$ can be equal to zero only on point sets of zero measure.*

By mathematical induction we have the following lemma.

Lemma 3.7 *The set of r $(r < n)$ of linear independent concircular vector fields $\{\overset{1}{\varphi}{}^h, \overset{2}{\varphi}{}^h, \ldots, \overset{r}{\varphi}{}^h\}$ on $\mathbb{A}_n$ can be linearly dependent only on point sets of zero measure.*

Proof. Successively we are able to substitute $r = 1, 2, \ldots, n-1$. Let $\{\overset{1}{\varphi}{}^h, \overset{2}{\varphi}{}^h, \ldots \overset{r}{\varphi}{}^h\}$ be linear independent (excluding at point sets of zero measure) concircular vector fields on V_n, which satisfy the following equations $\overset{s}{\varphi}{}^h_{,j} = \overset{s}{\varrho}\ \delta^h_j$, where $\overset{s}{\varrho}$ are functions on $\mathbb{A}_n = (M, \nabla)$.

Let these vectors be linearly independent at the point $x_0 \in M$, then these are linearly independent at a point x in a certain neighborhood $U_{x_0} \subset M$. Finally, let φ^h be a concircular vector field on M and

$$\varphi^h(x) = \sum_{s=1}^{r} \overset{s}{\alpha}(x) \cdot \overset{s}{\varphi}{}^h(x) \quad \text{for} \quad x \in U_{x_0}\,, \tag{3.117}$$

where $\overset{s}{\alpha}(x)$ are functions on U_{x_o}. Because $\overset{s}{\varphi}{}^h(x) \in C^1$, the functions $\overset{s}{\alpha}(x)$ are differentiable. Covariantly differentiating (3.117) with respect to x^j we find $(\varrho\ - \sum_{s=1}^{r} \overset{s}{\alpha}\ \cdot \overset{s}{\varrho})\ \delta^h_j = \sum_{s=1}^{r} \nabla_j \overset{s}{\alpha}\ \cdot \overset{s}{\varphi}{}^h$. From this follows that $\varrho = \sum_{s=1}^{r} \overset{s}{\alpha}\ \cdot \overset{s}{\varrho}$ and $\nabla_j \overset{s}{\alpha} = 0$ (i.e. $\overset{s}{\alpha} = \text{const}$) on U_{x_o}.

For the initial conditions $\varphi^h(x_o) = \sum_{s=1}^{r} \overset{s}{\alpha}\ \cdot \overset{s}{\varphi}{}^h(x_o)$ and $\varrho(x_o) = \sum_{s=1}^{r} \overset{r}{\alpha}\ \cdot \overset{r}{\varrho}(x_o)$ the equations (3.115) have only one solution: $\varphi^h(x) = \sum_{s=1}^{r} \overset{s}{\alpha}\ \cdot \overset{s}{\varphi}{}^h(x)$ on $\mathbb{A}_n$. □

Theorem 3.11 *There are no manifold with affine connection $\mathbb{A}_n \in C^1$ $(n > 2)$, except equiaffine projective flat spaces, which admit more than $(n-1)$ linearly independent concircular vector fields $\varphi^h(x) \in C^1$ corresponding to constant coefficients (of linearly dependence).*

Proof. Let us suppose the opposite. Let $\mathbb{A}_n$ be a space which is not equiaffine projective flat and yet admits more than $(n-1)$ linearly independent concircular vector fields with constant coefficients. The integrability conditions of first equation (3.115) read (3.112): $\varphi^\alpha \tilde{W}^h_{\alpha jk} = 0$.

We can write the tensor $\tilde{W}^h_{ijk}$ as $\tilde{W}^h_{ijk} = \sum_{s=1}^{m} a_{s|i}\Omega^h_{s|ijk}$ where $a_{s|i}$ are some linearly independent covector fields, and $\Omega^h_{s|jk}$ are linearly independent tensor fields. Since $\mathbb{A}_n$ is not equiaffine projectively flat, then $m \geq 1$.

From the conditions (3.112) we obtain

$$\varphi^\alpha a_{1|\alpha} = 0, \quad \varphi^\alpha a_{2|\alpha} = 0, \quad \dots, \quad \varphi^\alpha a_{m|\alpha} = 0. \tag{3.118}$$

Since $m \geq 1$, among the equations of the system (3.118) there are at least one substantial equation. From the previous facts it follows that there exist less or equal to $n-1$ linearly independent vector fields φ^h, a contradiction. This proves the Theorem 3.11. □

From the Theorem 3.11 the following two Theorems follow

Theorem 3.12 *Let $\mathbb{A}_n \in C^1$, $(n > 2)$, be spaces with affine connection ∇ in which there are $(n-1)$ linearly independent concircular vector fields $\varphi^h(x) \in C^1$. Then the tensor $\tilde{W}$ has the following expression*

$$\tilde{W}^h_{ijk} = a_i\Omega^h_{jk}, \tag{3.119}$$

where a_i and Ω^h_{jk} are non vanishing covector and tensor of type $(1,2)$, respectively.

Theorem 3.13 *The $\mathbb{A}_n \in C^3$ $(n > 2)$ admits $(n-1)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$, if and only if in $\mathbb{A}_n$ the following relations are satisfied*

$$\begin{aligned} \tilde{W}^h_{ijk} &= a_i\Omega^h_{jk}, \\ a_{i,j} &= \mu_j a_i + b_i a_j; \\ b_{i,j} &= \nu_j a_i + b_i b_j - \frac{1}{n-1} R_{ij}, \end{aligned}$$

where a_i is a non-vanishing covector; c_i, μ_i, ν_i are some covectors.

Example 3.1 Assume that $\mathbb{A}_n$ has the components of affine connection ∇ defined in the following way $\Gamma^1_{11} \in C^r$, $r \geq 0$, $\exists i \neq 1$ $\partial_i\Gamma^1_{11} \neq 0$ and the other components of Γ^h_{ij} are vanishing. If $\Gamma^1_{11} \in C^1$ then $R^h_{ijk} = a_i\Omega^h_{jk}$, $R_{ij} = a_i\Omega^h_{jh}$ and (3.119).

We can easily convince ourselves that in $\mathbb{A}_n$ there exist exactly $n-1$ nonlinear covariantly constant vector fields ξ^h, i.e. $\nabla\xi^h = 0$. These fields have following properties: $\xi^h_{s|} = \delta^h_s$, $s = 2, 3, \cdots, n$. This is a general solution of equations (3.106). Condition $\varrho = 0$ is necessitated.

Above solutions is valid even though $\Gamma^1_{11} \in C^0$ and $\Gamma^1_{11} \notin C^1$. In this case the tensor $\tilde{W}^h_{ijk}$ does not exist and therefore formula (3.119) cannot be written.

Let $\tilde{A}_n$ be a space with affine connection $\tilde{\nabla}$ for which $\tilde{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i$, where $\psi_i = \partial_i\Psi$. Vector fields $\varphi^h_{s|} = \exp(-\Psi)\cdot\xi^h_{s|}$ satisfy equations (3.106). When $\tilde{\Gamma}^h_{ij} \in C^1$, then in $\tilde{A}_n$ the formula (3.119) is valid. Notice that $\mathbb{A}_n$ and $\tilde{A}_n$ have common geodesics.

3.6.5 A Riemannian space which admits at least two linearly independent concircular vector fields

We are going to prove the following:

Theorem 3.14 [331] *If a Riemannian space $\mathbb{V}_n \in C^2$ $(n > 2)$ admits at least two linearly independent concircular vector fields $\varphi_i(x) \in C^1$ with constant coefficients, then B is a constant, uniquely determined by the metric of the space $\mathbb{V}_n$.*

Remark. In [542] and [221, p. 88] a similar theorem was published, but the proof was done only for $\mathbb{V}_n \in C^3$, $\varphi_i(x) \in C^3$ and $\varrho(x) \in C^2$, and, moreover, it has local validity. This also concerns the following Theorems 3.11, 3.12 and 3.13. On the basis of Lemmas 3.6 and 3.7 these Theorems are valid globally.

Proof. Assume in $\mathbb{V}_n$ exist at least two linearly independent concircular vector fields with constant coefficients φ_i and $\tilde{\varphi}_i$, with correspondent functions B and $\tilde{B}$, respectively. Then from (3.111) and (3.116) the following is satisfied:

$$(*)\ \ \varphi_\alpha R^\alpha_{ijk} = B(g_{ij}\varphi_k - g_{ik}\varphi_j), \quad \text{and} \quad (**)\ \ \tilde{\varphi}_\alpha R^\alpha_{ijk} = \tilde{B}(g_{ij}\tilde{\varphi}_k - g_{ik}\tilde{\varphi}_j).$$

Multiplying (*) by $\tilde{\varphi}_\alpha g^{\alpha k}$ and contracting over k we get by (**)

$$(B - \tilde{B})(g_{ij}\varphi_\alpha\tilde{\varphi}^\alpha - \tilde{\varphi}_i\varphi_j) = 0.$$

Suppose $B \neq \tilde{B}$. Then $g_{ij}\varphi_\alpha\tilde{\varphi}^\alpha - \tilde{\varphi}_i\varphi_j = 0$. From the last formula we get $\varphi_\alpha\tilde{\varphi}^\alpha = 0$ and $\tilde{\varphi}_i\varphi_j = 0$, a contradiction, since the vector fields are non-zero.

Hence $B = \tilde{B}$ holds. That is, the function B is uniquely defined by the metric of the space $\mathbb{V}_n$ itself. Because φ_k and $\tilde{\varphi}_k$ are gradient-like covector fields ($\varphi_k = \nabla_k\varphi$ and $\tilde{\varphi}_k = \nabla_k\tilde{\varphi}$) from the equality $B = \tilde{B}$ the fact $B(\varphi) = \tilde{B}(\tilde{\varphi})$ follows. Note that φ and $\tilde{\varphi}$ are indenpendent variables, then from this fact follows: B is constant. □

Note that the above theorem is analogous to some results proven earlier under the additional assumptions $\mathbb{V}_n \in C^3$, [535, 542, 935].

Theorem 3.15 [331] *There are no (pseudo-) Riemannian spaces $\mathbb{V}_n \in C^2$, except spaces of constant curvature, which admit more than $(n-2)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$ corresponding to constant coefficients.*

Remark In [221, p. 86], [220, 535, 542], a similar theorem was published but the proof was done only for $\mathbb{V}_n \in C^3$, $\varphi_i(x) \in C^3$ and $\varrho_i(x) \in C^2$.

Proof. Let us suppose the opposite. Let $\mathbb{V}_n$ be a space which is not of constant curvature and yet admits more than $(n-2)$ linearly independent concircular vector fields with constant coefficients. The conditions (3.111) read

$$\varphi_\alpha Z^\alpha_{ijk} = 0, \tag{3.120}$$

where

$$Z^h_{ijk} = R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}).$$

We can write the tensor Z^h_{ijk} as $Z^h_{ijk} = \sum\limits_{s=1}^{m} \underset{s}{b}{}^h \underset{s}{\Omega}_{ijk}$ where $\underset{s}{b}{}^h$ are some linearly independent vectors, and $\underset{s}{\Omega}_{ijk}$ are linearly independent tensors. Since $\mathbb{V}_n$ is not of constant curvature, $m \geq 2$ holds.

From the conditions (3.120) we obtain

$$\varphi_\alpha \underset{1}{b}{}^\alpha = 0, \quad \varphi_\alpha \underset{2}{b}{}^\alpha = 0, \quad \ldots, \quad \varphi_\alpha \underset{m}{b}{}^\alpha = 0. \tag{3.121}$$

Since $m \geq 2$, among the equations of the system (3.121) there are at least two substantial equations. From the previous facts it follows that there exist less or equal to $n-2$ linearly independent vector fields φ_i, a contradiction. This proves the Theorem 3.15. □

From the Theorem 3.15 and results in [542] the following two Theorems follow

Theorem 3.16 [331] *Let $\mathbb{V}_n \in C^2$, $(n > 2)$, be Riemannian spaces in which there are $(n-2)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$. Then the Riemannian tensor has the following expression*

$$R_{hijk} = B\,(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j),$$

where a_i and b_i are non-colinear and pairwise orthogonal covectors, $e = \pm 1$, $B = \text{const}$.

Theorem 3.17 [331] *The Riemannian space $\mathbb{V}_n \in C^3$ $(n > 3)$ admits $(n-2)$ linearly independent concircular vector fields $\varphi_i(x) \in C^1$, if and only if in $\mathbb{V}_n$ the following relations are satisfied* [149]

$$\begin{aligned}
R_{hijk} &= B(g_{hk}g_{ij} - g_{hj}g_{ik}) + e(a_h b_i - a_i b_h)(a_j b_k - a_k b_j),\\
a_{i,j} &= \underset{1}{\xi}_j a_i + \underset{2}{\xi}_j b_i + c_i a_j;\\
b_{i,j} &= \underset{3}{\xi}_j a_i + \underset{4}{\xi}_j b_i + c_i b_j;\\
c_{i,j} &= \underset{5}{\xi}_j a_i + \underset{6}{\xi}_j b_i + c_i c_j - B g_{ij},
\end{aligned}$$

where a_i and b_i are non-colinear and pairwise orthogonal covectors; c_i, $\underset{s}{\xi}_j$ $(s = 1, \ldots, 6)$ are some covectors; $e = \pm 1$, $B = \text{const}$.

Remark. This theorem was proved locally for $V_n \in C^3$, $\varphi_i \in C^3$, $\varrho \in C^2$, in [542]. The detailed local proof is contained in the Ph.D. thesis [220, p. 94-95], [221, p. 88-92].

3.6.6 Concircular and convergent vector fields on compact manifolds with affine connection

An example of a manifold with flat connection (hence locally metrizable not globally metrizable) can be constructed on the n-dimensional torus, see [211, 340, 354, 652].

Let $M_n = S^1 \times S^1 \times \cdots \times S^1$, and $x^1, x^2, \ldots, x^n$ be the corresponding angles on the circles. We have global vector fields $X_1 = \partial_1, X_2 = \partial_2, \ldots, X_n = \partial_n$. We define the connection ∇ by means of its action on these vector fields, as follows:

$$\nabla_{X_i} X_1 = \varrho\, X_i, \quad \nabla_{X_1} X_i = \varrho\, X_i\,,$$

and otherwise: $\nabla_{X_i} X_j = \sum_{k=1}^{n} \omega_{ij}^k X_k$, ϱ and ω_{ij}^k $(= \omega_{ji}^k)$ are functions on M_n.

* $\mathbb{A}_n = (M_n, \nabla)$ is a *compact manifold with torsion-free affine connection* ∇.
* The vector field $\Phi \equiv X_1$ is *concircular*, satisfying $\nabla_X \Phi = \varrho\, X$.
* In case $\varrho = \text{const}$, the vector field Φ is *convergent*.

Lemma 3.8 *There exist compact manifolds $\mathbb{A}_n$ with torsion-free affine connection and with a globally defined concircular, respectively convergent, vector field.*

It is possible to show that if ϱ is a constant and the above coefficients $\omega_{ij}^k = 0$, then the Riemannian tensor on $\mathbb{A}_n$ is zero. If this is the case, A_n is locally Euclidean. If the vector field Φ is convergent with $\varrho = \text{const} \neq 0$, then the manifold $\mathbb{A}_n$ is not globally metrizable. Indeed, consider $\Theta = g(\Phi, \Phi)$ where Φ is convergent with $\varrho = \text{const} \neq 0$; then for the x^1-coordinate circle, the following is satisfied:

$$\nabla_\Phi \Theta = 2\varrho\, \Theta \implies \Theta = \text{const} \cdot \exp(2\varrho\, x^1) \implies \Theta = 0,$$

in contradiction with the fact that Φ is nonisotropic.

So we proved:

Lemma 3.9 *There exist compact locally Euclidean manifolds $\mathbb{A}_n$ with a global defined convergent vector field, which are not globally metrizable.*

3.6.7 Applications of the achieved results

As we show later, many results from the theory of conformal, geodesic and holomorphically projective mappings and transformations are closely related to the theory of equidistant manifolds and concircular vector fields.

Some of the global results in the theory of concircular vector fields and the theory of conformal, geodesic, holomorphically-projective mappings and transformations were formulated for compact manifolds without boundary. We point out that in the mentioned results $\mathbb{V}_n$ stands both for classical Riemannian manifolds and for pseudo-Riemannian manifolds.

The above compactness assumption for Riemannian manifolds $\mathbb{V}_n$ can be made weaker by means of the concept of completeness (geodesic completeness).

It is evident that any compact Riemannian manifold (without boundary) is complete. The reason is that the arc length is a natural parameter.

The problem of conformal groups on Einstein spaces is connected with concircular vector fields. The detailed analysis of these problems is in the survey work by W. Kühnel and H.-B. Rademacher [568].

Among others from here it follows that *complete equidistant Einstein manifolds are isometric with the standard sphere* $\mathbb{S}^n$ *or the (pseudo-) Euclidean space.*

Here, also possible shapes of metrics are presented. In what follows, we restrict ourselves to metrics, for which the concircular vector field is nonisotropic. These metrics have been in fact discovered already by Brinkmann [314].

3.6.8 Equidistant manifolds and special coordinate system

We show that on equidistant Riemannian manifolds $\mathbb{V}_n$, which are generated by a non isotropic concircular vector field Φ, there exists a semigeodesic coordinate system (x) in which the metric form reads as follows:

$$ds^2 = e\,dx^{1\,2} + f(x^1)\,d\tilde{s}^2, \tag{3.122}$$

where $e = \pm 1$, $f \in C^1$ $(f \neq 0)$ is a function, and $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)dx^a \mathrm{d}x^b$; $(a, b = 2, \ldots, n)$ is the metric form of a certain Riemannian manifold $\tilde{\mathbb{V}}_{n-1}$.

After transformation of the coordinates $x^1 \to \overline{x}^1$ the metric (3.122) is of the form

$$ds^2 = a(\overline{x}^1)d\overline{x}^{1\,2} + f(\overline{x}^1)\,d\tilde{s}^2, \tag{3.123}$$

where $a \in C^1$ $(a \neq 0)$ is a function.

We remark, that H.W. Brinkmann [314] in fact found metrics of Riemannian and also pseudo-Riemannian manifolds $\mathbb{V}_n$, in which non-isotropic concircular vector fields exist, in the following form

$$ds^2 = \frac{1}{f(x^1)}(dx^1)^2 + f(x^1)\,d\tilde{s}^2, \tag{3.124}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\,dx^a\,dx^b$; $(a, b = 2, \ldots, n)$ is the metric form of a certain Riemannian manifold $\tilde{V}_{n-1}$.

Let us present a special coordinate system for the case of equidistant manifolds (and also for manifolds which admit torse-forming vector field Φ) in which the vector field Φ is nonisotropic.

Let us start with a semigeodesic coordinate system, see p. 118, the coordinate surfaces x^1 are hypersurfaces $\overset{*}{\varphi} = \text{const}$. A torse-forming (concircular) vector field $\Phi^h = g^{h\alpha}\delta_\alpha \overset{*}{\varphi}$ consists of tangent vectors to geodesics that generate this system. Hence the metric is of the form (3.28):

$$ds^2 = e\,dx^{1\,2} + g_{ab}(x)\,dx^a, dx^b, \quad e = \pm 1, \quad a, b > 1. \tag{3.125}$$

The equations (3.104) read $\dfrac{\partial^2 \overset{*}{\varphi}}{\partial x^i\,\partial x^j} + \partial_\alpha \overset{*}{\varphi} \Gamma^\alpha_{ij} = \nu g_{ij} + \tau \partial_i \overset{*}{\varphi} \partial_j \overset{*}{\varphi}$.
As $\overset{*}{\varphi} = \overset{*}{\varphi}(x^1)$ these equations can be written as

$$\overset{*}{\varphi}{}''\delta_{ij} + \overset{*}{\varphi}{}' e \Gamma_{ij1} = \nu g_{ij} + \tau (\overset{*}{\varphi}{}')^2 \delta_{ij}.$$

If $i = j = 1$ we have $$\overset{*}{\varphi}{}'' = e\nu + \tau\,(\overset{*}{\varphi}{}')^2. \tag{3.126}$$

If $i, j > 0$ we get $$\overset{*}{\varphi}{}' e\,\partial_1 g_{ij} = \nu\, g_{ij}. \tag{3.127}$$

Since $\overset{*}{\varphi}{}' \neq 0$ we integrate (3.127) and get $g_{ab} = f(x^1, x^2, \ldots, x^n)\,\tilde{g}_{ab}(x^2, \ldots, x^n)$. It follows that the metric of the Riemannian manifold with torse-forming vector field $\Phi = \omega(x^1)\delta_1$ has the form

$$ds^2 = e\,dx^{1\,2} + f(x^1, x^2, \ldots, x^n)\,d\tilde{s}^2, \tag{3.128}$$

where $e = \pm 1$, $f \in C^1$ $(f \neq 0)$ is a function, and $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n) dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric form of a certain Riemannian manifold $\tilde{\mathbb{V}}_{n-1}$.

When $\tau = 0$, i.e. the vector Φ is special concircular, it follows from (3.126) that $\nu = \nu(x^1)$, and by integration of (3.127) we find $g_{ab} = f(x^1)\,\tilde{g}_{ab}(x^2, \ldots, x^n)$. Hence any equidistant manifold admits a metric of the form (3.122).

3.6.9 Einstein equidistant manifolds and the theory of relativity

As we already mentioned, H.W. Brinkmann [314] in fact found metrics of Riemannian and also pseudo-Riemannian manifolds $\mathbb{V}_n$, in which non-isotropic concircular vector fields exist, in the form (3.124). Moreover, the following theorem holds, [314], see also the monograph by Petrov [139], p. 265.

Theorem 3.18 (Brinkmann [314]**)** *A manifold $\mathbb{V}_n$ with the metric (3.124): $ds^2 = f^{-1}\,(dx^1)^2 + f\,d\tilde{s}^2$, is an Einstein space $\mathcal{E}_n$ (or space of constant curvature S_n, respectively) if and only if $f = K\,x^{1\,2} + 2a\,x^1 + b$, where K, a and b are constants and $d\tilde{s}^2$ is a metric of the Einstein space $\tilde{\mathcal{E}}_{n-1}$ (respectively the space $\tilde{\mathcal{S}}_{n-1}$ with constant curvature $\tilde{K}$). The constants K and $\tilde{K}$ are related to the scalar curvatures R and $\tilde{R}$ of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_{n-1}$ (respectively S_n and $\tilde{\mathcal{S}}_{n-1}$) by*

$$\tilde{K} = \frac{\tilde{R}}{(n-1)(n-2)} = b\,K - a^2, \quad \textit{where} \;\; K = \frac{R}{n(n-1)}\,. \tag{3.129}$$

It can be shown that for all spaces S_n with constant curvature K there exists always the coordinate system mentioned above, in which the metric has the form (3.124). As we have said above, all $\mathcal{E}_3$ have constant curvature, moreover Einstein spaces $\mathcal{E}_4$ with metric (3.124) have constant curvature as well, [139,314]. For Einstein spaces $\mathcal{E}_n$ $(n > 4)$ this is not the case in general.

Among equidistant Riemannian spaces there are some cosmological models, see for example [67,73]. Modern cosmological models are based on the so-called

Copernican principle (Bondi 1960), stating that no position in the universe is distinguished in any way. A suitable concrete realisation of this principle is the assumption of spatial homogeneity. At a sufficiently large scale the structure of the universe is everywhere essentially the same. Such a space is invariant under translations, the latter ones form an isometry group, i.e. a group of transformations that leaves the metric invariant. A further interpretation of the Copernican principle is isotropy – the universe appears in every direction approximately the same. The associated isometry group is the group of rotations. Isotropy at every point ascertains homogeneity, the reverse is not true.

Homogenous and isotropic models are the simplest cosmological models characterized by constant spatial curvature. They were introduced and studied by Friedmann, Lemaître, Robertson and Walker. The metric has the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - F^2(t)\left[\frac{\mathrm{d}r^2}{1-\tilde{K}r^2} + r^2\mathrm{d}\vartheta + r^2\sin^2\vartheta\mathrm{d}\varphi^2\right]. \tag{3.130}$$

The parameter $\tilde{K}$ represents the constant (negative, zero, or positive) spatial curvature and can be normalized to -1, 0, or $+1$, the function $a(t)$ describes the time evolution of the size of the universe. Spatial homogeneity is expressed by the dependence of the metric components only on t. Given two values t_1 and t_2 of t, the distance between two corresponding points (r, ϑ, φ) on the t_1 and the t_2 hypersurface, repectively, is independent of the location of the points.

To illustrate (3.124) and the statement below it, we choose a Lorentzian signature to keep contact with physical applications and transform it to the form

$$ds^2 = dt^2 - F^2(t)d\tilde{s}^2.$$

This is achieved by setting $dt^2 = \dfrac{(dx^1)^2}{f(x^1)}$, leading to

$$t = \int_{x_0^1}^{x^1} \frac{dx}{\sqrt{Kx^2+2ax+b}}\ .$$

Having calculated this integral we express x^1 in terms of t and insert into f to obtain $F^2(t) = f(x^1(t))$.

For the explicit form of the above integral we have to distinguish four cases

1) $K>0$, $Kb>a^2$: $\quad x^1 = \dfrac{\sqrt{Kb-a^2}}{K}\ \sinh\left(\sqrt{K}(t-t_0)\right) - \dfrac{a}{K}$,

2) $K>0$, $Kb=a^2$, $Kx^1+a>0$: $\quad x^1 = \dfrac{1}{2K}\mathrm{e}^{\sqrt{K}(t-t_0)} - \dfrac{a}{K}$,

3) $K>0$, $Kb=a^2$, $Kx^1+a<0$: $\quad x^1 = -\dfrac{1}{2K}\mathrm{e}^{-\sqrt{K}(t-t_0)} - \dfrac{a}{K}$,

4) $K<0$, $Kb<a^2$, $|Kx^1+a| < \sqrt{a^2-Kb}$:

$$x^1 = -\frac{\sqrt{a^2-Kb}}{K}\ \sin\left(\sqrt{-K}(t-t_0)\right) - \frac{a}{K}.$$

For $F^2(t) = K(x^1(t))^2 + 2ax^1(t) + b$ we obtain

$$F^2(t) = \begin{cases} \dfrac{Kb - a^2}{K} \cosh^2\left(\sqrt{K}\ (t - t_0)\right), & \text{in case 1),} \\ \dfrac{1}{4K} \mathrm{e}^{\pm 2\sqrt{K}(t-t_0)}, & \text{in case 2) and 3),} \\ \dfrac{Kb - a^2}{K} \cos^2\left(\sqrt{-K}(t - t_0)\right), & \text{in case 4).} \end{cases}$$

Taking $d\tilde{s}^2$ from (3.130) we obtain the scalar curvature

$$R = -3\frac{(F^2(t))'' + 2\tilde{K}}{F^2(t)},$$

which equals

$$R = -6K\frac{2(Kb - a^2)\cos^2\left(\sqrt{K}(t - t_0)\right) - Kb + a^2 + \tilde{K}}{(Kb - a^2)\cos^2\left(\sqrt{K}(t - t_0)\right)}$$

in case 4), for example. For $\tilde{K} = bK - a^2$ this reduces to the constant $R = -12K$ in all cases. Case 1) is the de Sitter, case 4) is the anti-de Sitter model of the universe with constant space-time curvature. Cases 2) and 3) describe flat spatial hypersurfaces ($\tilde{K} = 0$) embedded with an exponentially increasing / decreasing scale factor into a space time manifold with constant curvature. Adding one dimension and interchanging one space coordinate with time yields a five-dimensional manifold with a metric of this kind. In this manifold the flat space-like hypersurfaces are replaced by four-dimensional Minkowski space. Such manifolds occur in string cosmology as the Randall-Sundrum model of type two [762].

3.6.10 Equidistant Kähler spaces

As in $\mathbb{K}_n$ the structure F is covariantly constant, from [626] it follows that on a Kähler manifold $\mathbb{K}_n$ admitting a nontrivial geodesic mapping, there exists a nonzero convergent vector field (see [118, 226, 227, 626, 630, 631, 693, 802]), see Corollary 9.1.

In every manifold $\mathbb{K}_n^-$ with a covariantly nonconstant convergent vector field there exists a coordinate system (x) with the following metric and structure (see [118, 226, 227, 626, 630, 631]):

$$g_{ab} = g_{a+m\,b+m} = \partial_{ab} f + \partial_{a+m\,b+m} f; \quad g_{a\,b+m} = \partial_{a\,b+m} f - \partial_{a+m\,b} f; \quad (3.131)$$

$$F_b^{a+m} = -F_{b+m}^a = \delta_b^a; \quad F_b^a = F_{b+m}^{a+m} = 0,$$

where $a, b = 1, \ldots, m$, $m = n/2$,

$$f = \exp(2x^1)\, G(x^2, x^3, ..., x^m, x^{2+m}, x^{3+m}, ..., x^n).$$

If $G \in C^3$, then these formulas generate (provided $|g_{ij}| \neq 0$) the metric of a Kähler manifold $\mathbb{K}_n^-$, where a non-constant convergent vector field exists.

A similar property holds (see [693]) for the hyperbolic Kähler manifolds $\mathbb{K}_n^+$ with metric and structure of the type

$$g_{ab} = g_{a+m\,b+m} = 0; \quad g_{a\,b+m} = \partial_{a\,b+m} f;$$

$$F_b^{a+m} = F_{b+m}^a = 0; \quad F_b^a = -F_{b+m}^{a+m} = \delta_b^a,$$

where $a, b = 1, \ldots, m$, $m = n/2$,

$$f = \exp(x^1 + x^{1+m})\, G(x^2 + x^{2+m}, x^3 + x^{3+m}, ..., x^m + x^n).$$

Metrics of parabolic Kähler manifolds $\mathbb{K}_n^o$, admitting covariantly nonconstant convergent vector field, were found by Mikeš and Shiha [802]. In a canonical coordinate system, where the structure has the form

$$F_b^{a+m} = \delta_b^a; \quad F_b^a = F_{b+m}^a = F_{b+m}^{a+m} = 0, \quad \text{for} \quad a, b = 1, \ldots, m,$$

the nonzero components of this metric ($g_{ij} = g_{ji}$) take the form

$$g_{11} = 4\exp(2x^1)\, G; \qquad g_{1a} = 2\exp(2x^1)\, \partial_a G;$$

$$g_{1\,a+m} = -\, g_{1+m\,a} = 2\exp(2x^1)\, \partial_{a+m} G;$$

$$g_{a\,b+m} = \exp(2x^1)\, (\partial_b G_a - \partial_a G_b);$$

$$g_{ab} = \exp(2x^1)\, \{\partial_{a+m\,b+m} \mathcal{H} - x^{c+m}(\partial_{ac} G_b - \partial_{bc} G_a)\};$$

where $G = G_0 + G_c x^{c+m}$, G_0, G_c ($c = 2, 3, \ldots, m$) are function dependent on $x^2, \ldots, x^m$, and the function $\mathcal{H}$ is independent of x^1, $a, b, c = 2, 3, \ldots, m$, $m = n/2$,

3.6.11 On Sasaki spaces and equidistant Kähler manifolds

Definition 3.12 An odd-dimensional Riemannian space S_n is said to be a Sasaki space if it contains a vector field χ satisfying the following conditions [164]:

$$\chi^i \chi_i = 1, \quad \chi_{,ij}^h = -\chi^h g_{ij} + \delta_j^h \chi_i, \quad \chi_{i,j} + \chi_{j,i} = 0 \tag{3.132}$$

where $\chi_i = g_{ij}\chi^j$, $h, i, j = 1, \ldots, n$.

It is known that the metric of equidistant spaces of basic type in some system of coordinates (y) (we call such a coordinate system canonical) takes the following form (3.122):

$$ds^2 = {dy^1}^2 + \varphi(y^1)\, d\tilde{s}^2, \tag{3.133}$$

where φ is a nonconstant function of y^l, $d\tilde{s}^2 = \tilde{g}_{\alpha\beta}\, dy^\alpha\, dy^\beta$ is the metric form of an $(n-1)$-dimensional Riemannian space $\tilde{\mathbb{V}}_{n-1}$, $\alpha, \beta = 2, 3, \ldots, n$; $\partial_1 \tilde{g}_{\alpha\beta} = 0$.

The metric (3.131) reduces to the form (3.133) by means of the following coordinate transformation:

$$y^1 = 2\, \exp(2x^1)\sqrt{|f|}, \qquad y^a = x^a, \quad a = 2, 3, \ldots, n.$$

It is easy to calculate that in the new coordinate system (y) the function φ and the metric tensor of the Riemannian space $\tilde{g}_{\alpha\beta}(y)$ have the structure

$$\begin{aligned}
&\varphi(y^1) = (y^1)^2,\\
&\tilde{g}_{1+m\,1+m} = 1, \quad \tilde{g}_{1+m\,a} = -\frac{\partial_{a+m} f}{2f}, \quad \tilde{g}_{1+m\,a+m} = \frac{\partial_a f}{2f},\\
&\tilde{g}_{ab} = \frac{1}{4f}\left(\partial_{ab} f + \partial_{a+m\,b+m} f - \frac{\partial_a f \partial_b f}{f}\right),\\
&\tilde{g}_{a+m\,b+m} = \frac{1}{4f}\left(\partial_{ab} f + \partial_{a+m\,b+m} f - \frac{\partial_{a+m} f \partial_{b+m} f}{f}\right),\\
&\tilde{g}_{a\,b+m} = \frac{1}{4f}\left(\partial_{a\,b+m} f - \partial_{a+m\,b} f - \frac{\partial_a f \partial_{b+m} f}{f}\right), \quad a, b = 2, 3, \ldots, m.
\end{aligned} \tag{3.134}$$

It is not difficult to find the components of F in this coordinate system.

We can verify by direct calculation that there is a direct relationship between equidistant Kähler spaces of basic type and Sasaki spaces. Namely, we have the following result.

Theorem 3.19 *An equidistant space of basic type referred to the canonical coordinate system (y) is a Kähler space if and only if $\varphi = (y^1)^2$ and $\tilde{\mathbb{V}}_{n-1}$ is a Sasaki space $\tilde{\mathbb{S}}_{n-1}$.*

The structure F satisfying the conditions (3.66) is determined by means of the relations

$$F_1^1 = 0, \quad F_1^a = (1/y^1)\,\tilde{\chi}^a, \quad F_a^1 = -y^1\,\tilde{\chi}_a, \quad F_b^a = \tilde{\chi}^a_{\wr b},$$

where $\tilde{\chi}^a(y^2, \ldots, y^n)$ is the vector that takes part in the definition of Sasaki space (3.132) and " $\wr$ " denotes the covariant derivative with respect to the connection of $\tilde{\mathbb{S}}_{n-1}$.

Thus, (3.134) gives the structure of the metrics (in some coordinate system) of all Sasaki spaces. (Note: the function φ is defined up to a constant factor.)

These results were published in [118, 631] and analogous results are valid also for hyperbolic and parabolic Kähler spaces, see [118, 693, 802].

3.7 A five-dimensional Riemannian manifold with an irreducible $SO(3)$-structure as a model of statistical manifold

In the present section we consider a five-dimensional Riemannian manifold (M, g) with an irreducible $SO(3)$-structure as a model of a statistical manifold and study properties of this model. In addition, we show the geometrical obstruction to the existence of a nearly integrable $SO(3)$-structure on a compact five-dimensional Riemannian manifold (M, g).

3.7.1 Information geometry

Information geometry is a branch of differential geometry dedicated to provide families of probability distributions with differential geometrical structures. The idea to use differential geometry in the probability theory and statistics is almost seventy years old. Namely, Jeffreys and Rao (see [509] and [764]) introduced a Riemannian metric definite by Fisher information matrix, and exposed the relationship between statistical curvature and the characteristics of inference (see [391]). In turn, Chentsov defined (see [35]) a 1-parameter group of affine connections in the space of statistic distributions. Nagaoka and Amari gave a dual structure to information geometry, a finding that has played a fundamental role in developing more applications of information geometry. Many other authors tried to construct a geometrical theory of statistic. The results of these researches are reflected in the following monographs [5, 6, 84, 111, 130] etc. On the role of differential geometry in the probability theory and statistics anyone can read in [288].

See J. Mikeš and E. Stepanova [697] continue in examining statistical manifolds, the first of which the author began in his papers [871] and [874].

3.7.2 An abstract statistical manifold

Lauritzen gave (see [5, pp.165-215]) a modern differential geometric treatment of statistical problems by introducing the notion of an abstract statistical manifold. In this section, we shall discuss such a manifold.

A *statistical manifold* is a triple (M, g, D), where M is a connected and smooth manifold, g is a Riemannian metric and D is a smooth covariant completely symmetric tensor field of order 3, called the *skewness* of the manifold, i.e. $D \in C^r S^3 M$ for $r \geq 2$. This definition is due to Lauritzen [5, p. 179]. Further, he defined one-parameter group of affine torsion-free α-connections $\overset{\alpha}{\nabla}$:

$$\overset{\alpha}{\nabla}_X Y = \nabla_X Y - \frac{\alpha}{2}\,\overline{D}(X, Y), \tag{3.135}$$

where α is a real parameter, ∇ is the Levi-Civita connection and

$$g(\overline{D}(X, Y), Z) = D(X, Y, Z) \tag{3.136}$$

for any $X, Y, Z \in C^r S^3 M$. After that, Lauritzen proved the identity (see [5, p. 180])

$$(\overset{\alpha}{\nabla}_X g)(Y, Z) = \alpha \cdot D(X, Y, Z) \tag{3.137}$$

and hence $\overset{\alpha}{\nabla} g \in C^{r+1}S^3M$. From the identity (3.137) was obtained the *Codazzi equation* for g with respect to an α-connections $\overset{\alpha}{\nabla}$ (see [168, pp. 56, 68, 142], [134, p. 21]).

It is well known (see [168, p. 53]), that if ∇ is an affine connection and g is a non-degenerate bilinear form, then the *conjugate α-connection* $\overset{\alpha}{\nabla}$ is given by

$$g(\overset{\alpha}{\nabla}_X Y, Z) = X\, g(Y, Z) - g(Y, \nabla_X Z). \tag{3.138}$$

Hence in statistical manifold the conjugate of $\overset{\alpha}{\nabla}$ is $\overset{-\alpha}{\nabla}$ and in particular, $\overset{0}{\nabla} = \nabla$ is self conjugate (see [5, p. 181]).

Next, Lauritzen introduced (see [5, p. 185]) a tensor field F which plays an important role in the statistical manifold. It is defined as follows

$$F(X, Y, Z, V) = (\nabla_X D)(Y, Z, V), \tag{3.139}$$

where D denotes the skewness of the manifold. If F is symmetric, then (M, g, D) is said to be a *conjugate symmetric* statistical manifold. This manifold has the following properties (see [5, p. 185])

$$\overset{\alpha}{R}(X, Y, Z, V) = -\overset{\alpha}{R}(X, Y, V, Z); \quad \overset{\alpha}{R}(X, Y, Z, V) = \overset{\alpha}{R}(Z, V, X, Y), \tag{3.140}$$

where $\overset{\alpha}{R}(X, Y, Z, V) = g(\overset{\alpha}{R}(X, Y)Z, V)$ for an α-curvature tensor $\overset{\alpha}{R}$ of $\overset{\alpha}{\nabla}$. In addition, Takeuchi and Amari showed (see [885]) that the sufficient condition for a statistical manifold to be conjugate symmetric is that its connection $\overset{\alpha}{\nabla}$ is *equiaffine* (see [134, p. 14]).

In the case, when $\alpha = 1$ the curvature $\overset{\alpha}{R}$ and Ricci $\overset{\alpha}{Ric}$ tensors of the connection $\overset{\alpha}{\nabla}$ are called the *curvature* and *Ricci tensors of a statistical manifold* (M, g, D), respectively (see, for example, [571] and [460]). In turn, the scalar invariant $\overset{1}{s} := \mathrm{trace}_g \overset{1}{Ric}$ is called a *scalar curvature of a statistical manifold* (M, g, D). Moreover, (M, g, D) is said to be of a *constant curvature* if $\overset{1}{s}$ is constant and $\overset{1}{R}(X, Y)Z = \dfrac{1}{n(n-1)}\ \overset{1}{s} \cdot (g(Y, Z)X - g(X, Z)Y)$. In particular, a statistical manifold (M, g, D) is said to be *locally flat* if $\overset{1}{R} = 0$.

3.7.3 An irreducible $SO(3)$-structure

In this section we will consider an application for a model of a statistical manifold.

It is well known that a Riemannian metric g on an n-dimensional orientable manifold M determines $SO(n)$-structures, where the tangent space T_xM at each point $x \in M$ behaves as a representation for $SO(n)$. An arbitrary subgroup $G \subset SO(n)$ determines restricted Riemannian G-structure, i.e. the tangent space T_xM must behave as a representation for G. Usually, the tangent space T_xM is regarded as an irreducible representation of a subgroup $G \subset SO(n)$. In the next series of papers [246]; [339] and [309] was considered an irreducible

$SO(3)$-structure on a five-dimensional Riemannian manifold (M, g). In particular, it was shown that an irreducible $SO(3)$-structure on a five-dimensional Riemannian manifold (M, g) is a structure defined by means of a smooth covariant completely symmetric and trace free tensor field $\Upsilon \in C^r S^3 M$ which satisfies on each chart $U(x^1, \ldots, x^5) \subset M$ the following condition:

$$\Upsilon_{ijm}\Upsilon_{kl}{}^{m} + \Upsilon_{imk}\Upsilon_{jl}{}^{m} + \Upsilon_{mjk}\Upsilon_{il}{}^{m} = g_{ij}g_{kl} + g_{ik}g_{jl} + g_{jk}g_{il}, \tag{3.141}$$

where $\Upsilon_{ijk} = \Upsilon(X_i, X_j, X_k)$ and $g_{kl} = g(X_k, X_l)$ for $X_k = \frac{\partial}{\partial x^k}$ are local components of Υ and g, respectively. Moreover, $\Upsilon_{jk}{}^{s} = g^{st}\Upsilon_{jkt}$ is a local expression of $\overline{\Upsilon}$ (see the formula (3.136)) for $(g^{ij}) = (g_{ij})^{-1}$. Identities (3.141) after contraction with g_{ij} and $\Upsilon_{m}{}^{ij}$, respectively, imply

$$\Upsilon_{ikl}\Upsilon_{j}{}^{kl} = \frac{7}{2}\, g_{ij}, \tag{3.142}$$

$$\Upsilon_{il}{}^{m}\Upsilon_{jn}{}^{l}\Upsilon_{km}{}^{n} = -\frac{3}{4}\, \Upsilon_{ijk}. \tag{3.143}$$

A five-dimensional Riemannian manifold (M, g) with an irreducible $SO(3)$-structure was obtained as (M, g, Υ). Further, it was defined a *characteristic* $\overset{\Gamma}{\nabla}$ *connection* of (M, g, Υ) such that $\overset{\Gamma}{\nabla} g = 0$; $\overset{\Gamma}{\nabla}\Upsilon = 0$ and it was proved that an irreducible $SO(3)$-structure (M, g, Υ) admits a characteristic connection $\overset{\Gamma}{\nabla}$ if it is *nearly integrable*, i.e. the tensor field Υ satisfies $(\nabla_X \Upsilon)(X, X, X) = 0$ for all non-zero vector fields X.

3.7.4 The model of a statistical manifold

In this paragraph we will consider a five-dimensional connected Riemannian manifold (M, g) with an irreducible $SO(3)$-structure as a model of a statistical manifold and show properties of this model.

Let (M, g, Υ) be a statistical manifold where Υ is a *skewness* of this manifold. We define 1-parameter group of affine torsion-free connections $\overset{\alpha}{\nabla}$ defined on (M, g) by the equation (3.136) where we suppose $D = \Upsilon$. Then using (3.141) and (3.142) we get

$$\begin{aligned} \overset{\alpha}{\nabla}_{l}\Upsilon_{ijk} = \nabla_l \Upsilon_{ijk} + \tfrac{\alpha}{2}\, (\Upsilon_{ijm}\Upsilon_{kl}{}^{m} + \Upsilon_{imk}\Upsilon_{jl}{}^{m} + \Upsilon_{mjk}\Upsilon_{il}{}^{m}) = \\ \nabla_l \Upsilon_{ijk} + \tfrac{\alpha}{2}\, (g_{ij}g_{kl} + g_{ik}g_{jl} + g_{jk}g_{il}). \end{aligned} \tag{3.144}$$

From (3.144) we conclude that if one of the four vectors X, Y, Z, V is orthogonal to the other three then the equality

$$(\overset{\alpha}{\nabla}_{V}\Upsilon)(X, Y, Z) = (\nabla_V \Upsilon)(X, Y, Z)$$

is true. Next, we can rewrite (3.137) in the following form

$$(\overset{\alpha}{\nabla}_{X} g)(Y, Z) = \alpha \cdot \Upsilon(X, Y, Z). \tag{3.145}$$

Moreover, on each chart $U(x^1,\dots,x^5)\subset M$ the local components of the curvature tensors $\overset{\alpha}{R}$ and R of the connections $\overset{\alpha}{\nabla}$ and ∇ are related by the equations (see [50, p. 33])

$$\overset{\alpha}{R}_{ijk}{}^{l} = R_{ijk}{}^{l} + \frac{\alpha}{2}\,(-\nabla_i \Upsilon_{jk}{}^{l} + \frac{\alpha}{2}\,\Upsilon_{mi}{}^{l}\Upsilon_{jk}{}^{m} - \frac{\alpha}{2}\,\Upsilon_{mj}{}^{l}\Upsilon_{ik}{}^{m}) \tag{3.146}$$

where $\overset{\alpha}{R}(X_i,X_j)X_k = \overset{\alpha}{R}_{ijk}{}^{l}X_l$ and $R(X_i,X_j)X_k = R_{ijk}{}^{l}X_l$ for $X_k = \frac{\partial}{\partial x^k}$. Then from (3.146) we obtain the following identities

$$\overset{\alpha}{R}_{jk} = R_{jk} - \frac{\alpha}{2}\,\nabla_m \Upsilon_{jk}{}^{m} - \frac{7\alpha^2}{2}\cdot g_{jk} \tag{3.147}$$

where $\overset{\alpha}{R}_{jk} = \overset{\alpha}{R}_{ljk}{}^{l}$ and $R_{jk} = R_{ljk}{}^{l}$ are local components of the Ricci tensors $\overset{\alpha}{Ric}$ and Ric of the affine connection $\overset{\alpha}{\nabla}$ and the Levi-Civita connection ∇, respectively. In turn, from (3.147) we conclude that

$$\overset{\alpha}{R}_{jk} = \overset{\alpha}{R}_{kj} \tag{3.148}$$

In this case, the affine connection $\overset{\alpha}{\nabla}$ is called *Ricci-symmetric* (see [168, pp. 57-58]).

Remark 3.3 A torsion-free affine connection $\overset{\alpha}{\nabla}$ is called *equiaffine* (see [134, p. 14]) if $\overset{\alpha}{\nabla}\overset{\alpha}{\omega} = 0$ for the volume form $\overset{\alpha}{\omega}\colon x\in M\to \overset{\alpha}{\omega}_{x} = \overset{\alpha}{\omega}(X_x,Y_x,Z_x,V_x,W_x)$ on the tangent space T_xM at each point $x\in M$. Takeuchi and Amari (see [885]) developed an expression for this α-parallel volume form for the exponential family. In addition, we recall that the necessary and sufficient condition for a torsion-free connection to admit (uniquely up to a constant scalar factor) a parallel volume form is that it has symmetric tensor Ricci (see also [134, p. 14]).

The Christoffel symbols $\overset{\alpha}{\Gamma}{}^{i}_{kj}$ of the Ricci-symmetric connection $\overset{\alpha}{\nabla}$ satisfy the equalities (see [168, pp. 57-59])

$$\overset{\alpha}{\Gamma}{}^{k}_{kj} = X_j\,(\ln \overset{\alpha}{\omega}_{12345}),\quad \overset{\alpha}{\omega}_{12345} = \overset{\alpha}{\omega}(X_1,X_2,X_3,X_4,X_5)$$

where $\overset{\alpha}{\nabla}_{X_k}X_j = \overset{\alpha}{\Gamma}{}_{kj}{}^{l}\,X_l$. On the other hand, the Christoffel symbols Γ^{l}_{jk} of the Levi-Civita connection ∇ satisfy the following condition $\Gamma^{k}_{kj} = X_j(\ln\sqrt{\det g})$ where $\omega = \sqrt{\det g}\;\; dx^1\wedge dx^2\wedge dx^3\wedge dx^4\wedge dx^5$ is the Riemannian volume form. Then using (3.135) we can conclude that $\overset{\alpha}{\omega}_{12345} = C\cdot\sqrt{\det g}$ for an arbitrary real positive constant C.

Next, from (3.147) we obtain $\overset{\alpha}{s} = s - \frac{35}{2}\cdot\alpha^2$ and hence the α-scalar curvature $\overset{\alpha}{s}$ of (M,g,Υ) satisfies the equality $\overset{\alpha}{s}\le s$ for the scalar curvature s of (M,g). Thus, the following theorem is valid.

Theorem 3.20 *Let a five-dimensional connected Riemannian manifold (M,g) with an irreducible $SO(3)$-structure be a model of statistical manifold (M,g,Υ),*

then the one-parameter family of affine α-connections $\overset{\alpha}{\nabla}$ consists of the Ricci-symmetric connections and a volume form $\overset{\alpha}{\omega}$ associated with each connection $\overset{\alpha}{\nabla}$ is defined by the equality $\overset{\alpha}{\omega}_{12345} = C \cdot \sqrt{\det g}$ and does not depend on the parameter α. Moreover, the α-scalar curvature $\overset{\alpha}{s}$ of (M, g, Υ) has the form $\overset{\alpha}{s} = s - \frac{35}{2} \cdot \alpha^2$ and hence $\overset{\alpha}{s} \leq s$ for all α.

3.7.5 On a conjugate symmetric statistical manifold

Let (M, g, Υ) be a five-dimensional connected Riemannian manifold (M, g) with an irreducible $SO(3)$-structure such that $\nabla\Upsilon \in C^r S^4 M$ then (M, g, Υ) is a *conjugate symmetric* statistical manifold. On the other hand, for the Riemannian manifold (M, g) with an irreducible $SO(3)$-structure we have $\Upsilon \in C^r S^3_0 M$ where $S^3_0 M$ is a tensor bundle of covariant symmetric trace-free tensor fields of order three then $\nabla\Upsilon \in C^r S^4 M$ for the case of a *conjugate symmetric* statistical manifold (M, g, T).

From (3.144) we obtain the following identities

$$\overset{\alpha}{\nabla}_l \Upsilon_{ijk} - \overset{\alpha}{\nabla}_i \Upsilon_{ljk} = \nabla_l \Upsilon_{ijk} - \nabla_i \Upsilon_{ljk}. \tag{3.149}$$

In turn, from (3.149) we conclude that a five-dimensional Riemannian manifold (M, g) with an irreducible $SO(3)$-structure is a conjugate symmetric statistical manifold if and only if the condition $\overset{\alpha}{\nabla}\Upsilon \in C^r S^4 M$ holds.

In addition to the above proposition, from (3.145) we conclude that a five-dimensional Riemannian manifold (M, g) with an irreducible $SO(3)$-structure is a conjugate symmetric statistical manifold if and only if the equations $\overset{\alpha}{\nabla}_k \overset{\alpha}{\nabla}_l g_{ij} = \overset{\alpha}{\nabla}_l \overset{\alpha}{\nabla}_k g_{ij}$ are true. In particular, if we suppose that (M, g, Υ) is a statistical manifold of constant curvature then from the Ricci identities (see [50, p. 30])

$$\overset{1}{\nabla}_k \overset{1}{\nabla}_l g_{ij} - \overset{1}{\nabla}_l \overset{1}{\nabla}_k g_{ij} = -g_{im} \overset{1}{R}_{klj}{}^m - g_{mj} \overset{1}{R}_{kli}{}^m \tag{3.150}$$

where $\overset{1}{R}_{kji}{}^l = \frac{1}{20} \overset{1}{s} (\delta^l_k g_{ji} - \delta^j_k g_{ki})$ we obtain the condition $\nabla\Upsilon \in C^r S^4 M$. Moreover, from (3.147) we have $R_{jk} = \left(\frac{1}{4} \overset{1}{s} + \frac{7\alpha^2}{2}\right) \cdot g_{jk}$ and, consequently, (M, g) is an Einstein manifold. Thus we prove the following

Theorem 3.21 *Let (M, g, Υ) be a five-dimensional connected Riemannian manifold (M, g) with an irreducible $SO(3)$-structure and $\overset{\alpha}{\nabla}$ be the one-parameter family of affine α-connections $\overset{\alpha}{\nabla}$ on M. Then (M, g) is a conjugate symmetric statistical manifold if and only if $\overset{\alpha}{\nabla}\Upsilon \in C^r S^4 M$. In particular, if (M, g, Υ) is a manifold of constant curvature then it is a conjugate symmetric statistical manifold and (M, g) is an Einstein manifold.*

Now we introduce a vector field X with local components $\Upsilon^l_{jk}\nabla_l\Upsilon^{ijk} - \Upsilon^i_{jk}\nabla_l\Upsilon^{ljk}$. For this vector field we have

$$\begin{aligned}\mathrm{div}X = (\nabla_i\Upsilon^l_{jk})\nabla_l\Upsilon^{ijk} + \Upsilon^l_{jk}(\nabla_i\nabla_l\Upsilon^{ijk}) - (\nabla_i\Upsilon^i_{jk})\nabla_l\Upsilon^{ljk} - \Upsilon^i_{jk}(\nabla_i\nabla_l\Upsilon^{ljk}) = \\ \frac{7}{2}\,s + 2R_{ijkl}\Upsilon^{ikm}\Upsilon^{jl}_m + \nabla_i\Upsilon^{jkl}\nabla_j\Upsilon^i_{kl} - \nabla_i\Upsilon^{ikl}\nabla_m\Upsilon^m_{kl} = \\ \frac{7}{2}\,s + 2g(\mathcal{R}(\Upsilon),\Upsilon) + \frac{1}{6}\,g(\delta^*\Upsilon,\delta^*\Upsilon) - \frac{1}{3}\,g(\nabla\Upsilon,\nabla\Upsilon) - g(\delta\Upsilon,\delta\Upsilon)\end{aligned} \tag{3.151}$$

where $\mathcal{R}$: $S^2 \to S^2M$ is the *curvature operator of the second kind* which is defined in [728] as an operator acting on a covariant symmetric tensor fields of order two:

$$B = (B_{ij}) \in C^rS^2M \to \mathcal{R}(B) = (R_{ikjl}B^{ij}) \in C^rS^2M.$$

Therefore, we can define $g(\mathcal{R}(\Upsilon),\Upsilon) := R_{jikl}\Upsilon^{jkm}\Upsilon^{il}{}_m$. On the other hand, the operators δ^*: $C^rS^2M \to C^rS^3M$ and δ: $C^rS^3M \to C^rS^2M$ are formal adjoint and have the following local expressions $(\delta^*\Upsilon)_{kijl} = \nabla_k\Upsilon_{ijl} + \nabla_i\Upsilon_{jlk} + \nabla_j\Upsilon_{lki} + \nabla_l\Upsilon_{kij}$ and $(\delta\Upsilon)_{jk} = -\nabla_l\Upsilon^l{}_{jk}$ (see [781], [23, p. 35]). That means $\langle\delta^*B,\Upsilon\rangle = \langle B,\delta,\Upsilon\rangle$ for the global inner product $\langle\Upsilon,\Upsilon'\rangle = \int_M g(\Upsilon,\Upsilon')d\nu$ for an arbitrary $B \in C^rS^2M$ and $\Upsilon,\Upsilon' C^rS^3M$ whose common support is compact.

Let (M,g) be a compact connected manifold, then applying Green's theorem (see [199, p. 11]) to (3.151) we obtain the following integral equation

$$\begin{aligned}s(M) = -\tfrac{4}{7}\,\langle\mathcal{R}(\Upsilon),\Upsilon\rangle - \tfrac{1}{42}\,\langle\delta^*\Upsilon,\delta^*\Upsilon\rangle + \\ \tfrac{2}{21}\,\langle\nabla\Upsilon,\nabla\Upsilon\rangle + \tfrac{2}{7}\,\langle\delta\Upsilon,\delta\Upsilon\rangle = 0\end{aligned} \tag{3.152}$$

where $s(M) = \int_M s d\nu$ is the *total scalar curvature* of a Riemannian manifold (see [23, p. 4, 119]). Note that, in order to apply Green's theorem, it is necessary to assume that M is orientable. If M is not orientable, then we only need to take an orientable double covering.

In the case, when (M,g,Υ) is a conjugate symmetric statistical manifold the following equalities $\delta^*\Upsilon = 4\Upsilon$ and $\delta\Upsilon = 0$ are true and therefore we can rewrite (3.152) in the form

$$s(M) = -\frac{4}{7}\,\langle\mathcal{R}(\Upsilon),\Upsilon\rangle - \frac{2}{21}\,\langle\nabla\Upsilon,\nabla\Upsilon\rangle = 0. \tag{3.153}$$

From this we have

Theorem 3.22 *Let (M,g,Υ) be a five-dimensional connected and compact Riemannian manifold with an irreducible $SO(3)$-structure. If the total scalar curvature $s(M)$ of (M,g) is positive and $\langle\mathcal{R}(\Upsilon),\Upsilon\rangle \geq 0$ then (M,g,Υ) cannot be a conjugate symmetric statistical manifold.*

Using the formula (2.11) we will prove the following theorem.

Theorem 3.23 *Let (M, g, Υ) be a five-dimensional connected and complete Riemannian manifold (M, g) with an irreducible $SO(3)$-structure. If the curvature operator of the second kind $\mathcal{R}$ is bounded below by a positive number then (M, g, Υ) cannot be a conjugate symmetric statistical manifold.*

Proof. We define a quadratic form $\Phi(B, B) = g(\mathcal{R}(B), B)$ for any $B \in C^r S^2 M$. If we set $B = X \otimes Y + Y \otimes X$, where X and Y are two orthonormal unitary vector fields, then

$$\Phi(B, B) = g(\mathcal{R}(B), B) = 2 \cdot g(R(X, Y)Y, X) = 2 \cdot sec(X, Y),$$

where (see [199, p. 15])

$$sec(V, W) = -\frac{R(V, W, W, V, W)}{g(V, V)g(W, W) - g(V, W)^2}$$

is caled the *sectional curvature* of (M, g) with respect to the 2-section $\pi = \text{span}\{V, W\}$. Side by side, we have (see [199, pp. 15-16])

$$s = \text{trace}_g Ric = 2 \sum_{i<j} \text{sec}(e_i, e_j)$$

for an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ for $T_x M$. Therefore, if we suppose that the curvature operator of the second kind $\mathcal{R}$ is positive define and bounded below by a positive number λ^2, i.e. $R_{ijkl} B^{jl} B^{ik} \geq \lambda^2 B_{ik} B^{ik} > 0$ for any $B \in C^r S^2 M$, then the sectional curvature of (M, g) is positive and bounded below by a positive number. Moreover, in this case $g(\mathcal{R}(\Upsilon), \Upsilon) > 0$ and the scalar curvature s must be positive too. The above conditions give rise to a contradiction with the formula (3.153).

Finally, it should be noted that a complete Riemannian manifold (M, g) is compact and oriented if its sectional curvature is bounded below by some positive number (see [64, pp. 212-213]). This completes the proof. □

The following corollary is obvious.

Corollary 3.1 *An irreducible $SO(3)$-structure with the totally symmetric covariant derivative $\nabla\Upsilon$ does not exist on an Euclidean sphere S^5.*

Next, for a conjugate symmetric statistical manifold (M, g, Υ) from (3.146) we obtain

$$\overset{\alpha}{R}_{ijkl} = R_{ijkl} + \frac{\alpha^2}{4} (\Upsilon_{mil} \Upsilon_{jk}{}^m - \Upsilon_{mjl} \Upsilon_{ik}{}^m). \tag{3.154}$$

Then using (3.140), (3.142) and (3.143) from (3.154) we get

$$g(\mathcal{R}(\Upsilon), \Upsilon) = g(\overset{\alpha}{\mathcal{R}}(\Upsilon), \Upsilon) + \frac{35}{16} \alpha^2. \tag{3.155}$$

If we suppose that there is an α-connection $\overset{\alpha}{\nabla}$ such that its curvature operator of the second kind $\overset{\alpha}{\mathcal{R}}$ is positive definite then

$$g(\mathcal{R}(\Upsilon), \Upsilon) = g(\overset{\alpha}{\mathcal{R}}(\Upsilon), \Upsilon) + \frac{35}{16} \alpha^2 \geq \frac{35}{16} \alpha^2 > 0.$$

Thus, as a consequence of Theorem 3, we have

Corollary 3.2 *Let (M, g, Υ) be a five-dimensional connected and complete Riemannian manifold (M, g) with an irreducible $SO(3)$-structure. If there exists an α-connection $\overset{\alpha}{\nabla}$ such that its curvature operator of the second kind $\overset{\alpha}{\mathcal{R}}$ is positive definite then (M, g, Υ) cannot be a conjugate symmetric statistical manifold.*

Remark 3.4 We recall that a symmetric 2-tensor field B on a Riemannian manifold (M, g) is said to be a *Codazzi tensor* (see [23, pp. 435]) if it satisfies the condition $\nabla B \in C^r S^3 M$. In addition, every Codazzi tensor B with constant *trace* on compact (M, g) with non-negative sectional curvature *sec* is parallel. If, moreover, $sec > 0$ at some point, then B is trivial, i.e. B is a constant multiple of g (see [23, p. 436]). We note that the tensor F of a conjugate symmetric statistical manifold (M, g, D) is an analogue of a Codazzi tensor. Moreover, our Theorem 4 and Corollary 2 are analogues of the above "vanishing theorem" for Codazzi tensors.

For a conjugate symmetric statistical manifold (M, g, Υ) the equations $\nabla_m \Upsilon^m_{jk} = \nabla_j \Upsilon^m_{mk} = 0$ are true. In this case, from (3.147) we obtain the following equalities

$$\overset{\alpha}{R}_{jk} = R_{jk} - \frac{7\alpha^2}{2} \cdot g_{jk}.$$

In addition, we suppose that (M, g, Υ) is complete and there exists an α-connection $\overset{\alpha}{\nabla}$ such that the Ricci tensor $\overset{\alpha}{Ric}$ is positive semi-definite. Then $Ric \geq \frac{7\alpha^2}{2} \cdot g > 0$ and by Mayer's theorem (see [106, p. 201]) the manifold (M, g) must be compact with a finite fundamental group.

Proposition 3.1 *Let a five-dimensional connected and complete Riemannian manifold (M, g) with an irreducible $SO(3)$-structure be a conjugate symmetric statistical manifold (M, g, Υ). If there is an α-connection $\overset{\alpha}{\nabla}$ such that its Ricci tensor $\overset{\alpha}{Ric}$ is positive semi-definite then (M, g) is compact and its fundamental group is finite.*

3.7.6 On a nearly integrable $SO(3)$-structure

We consider now a geodesic of (M, g), γ: $x^i = x^i(t)$ for $t \in I \subset \mathbf{R}$ satisfying $\nabla_{\dot{x}} \dot{x} = 0$. If each solution $x^i = x^i(t)$ of the equations $\nabla_{\dot{x}} \dot{x} = 0$ of geodesics satisfies the condition $B(\dot{x}, \ldots, \dot{x}) = \text{const}$ for $B \in C^r S^p M$ and $\dot{x} = \frac{dx^k}{dt} \cdot X_k$, then the equations $\nabla_{\dot{x}} \dot{x} = 0$ are said to admit a *first integral of the p-th order*. The equation $\delta^* B = 0$ serves as a necessary and sufficient condition for this (see [50, pp. 128-129]). In the case, when (M, g, Υ) is a five-dimensional connected Riemannian manifold (M, g) with a nearly integrable $SO(3)$-structure the following equation $\delta^* \Upsilon = 0$ is true. Therefore, $\Upsilon(\dot{x}, \dot{x}, \dot{x}) = \text{const}$ is a first integral of third order of the geodesics on (M, g).

On the other hand, if $\delta^* \Upsilon = 0$, then necessarily $\delta \Upsilon = 0$, as is readily seen, and so $\Delta_{\text{sym}} \Upsilon := \delta^* \delta \Upsilon - \delta \delta^* \Upsilon = 0$, where Δ_{sym} is a Laplacian on symmetric covariant tensor fields (see [781], [23, pp. 52-54]). In this case, the tensor field Υ is called a *harmonic symmetric tensor* (see [781]), because $\Delta_{\text{sym}} \Upsilon = 0$ is the

condition for a free extremal of $E(\Upsilon) = 1/2\ \langle \Upsilon, \Delta_{\rm sym}\Upsilon\rangle$. Thus the following theorem is valid.

Proposition 3.2 *The tensor Υ of a nearly integrable $SO(3)$-structure on a five-dimensional connected Riemannian manifold (M, g) is a harmonic symmetric tensor and this tensor defines a first integral of third order of the equations of geodesics.*

Remark 3.5 For a harmonic symmetric tensor, it was said in [781] that it does not have any geometrical interpretation. It was also pointed out that symmetric tensors, whose covariant derivative vanishes is clearly harmonic, the metric tensor g is the most important example. These tensors are trivial examples of a harmonic symmetric tensor. Now we can say that the tensor Υ of a nearly integrable $SO(3)$-structure on a five-dimensional connected Riemannian manifold (M, g) is a non-trivial example of a harmonic symmetric tensor.

In the case, when (M, g, Υ) is a compact connected five-dimensional Riemannian manifold (M, g) with a nearly integrable $SO(3)$-structure we can rewrite (3.152) in the form

$$s(M) = -\frac{2}{7}\langle \mathcal{R}(\Upsilon), \Upsilon\rangle + \frac{2}{21}\langle \nabla\Upsilon, \nabla\Upsilon\rangle = 0. \tag{3.156}$$

From this we have

Theorem 3.24 *Let (M, g, Υ) be a five-dimensional connected and compact Riemannian manifold with an irreducible $SO(3)$-structure. If the total scalar curvature $s(M)$ of (M, g) is negative and $\langle \mathcal{R}(\Upsilon), \Upsilon\rangle \le 0$ then (M, g, Υ) cannot be a Riemannian manifold with a nearly integrable $SO(3)$-structure.*

Using the formula (3.156) we will prove the following theorem.

Theorem 3.25 *Let (M, g, Υ) be a five-dimensional connected and compact Riemannian manifold (M, g) with an irreducible $SO(3)$-structure. If the curvature operator of the second kind is bounded from above by a negative number then the $SO(3)$-structure is not a nearly integrable structure.*

Proof. If we suppose that the curvature operator of second kind $\mathcal{R}$ is negative definite and bounded from above by a negative number $-\mu^2$, i.e. $R_{ijkl}B^{jl}B^{ik} \le -\mu^2\, B_{ik}B^{ik} < 0$ for any $B = (B_{ij}) \in C^r S^2 M$, then $g(\mathcal{R}(\Upsilon), \Upsilon) < 0$ and the scalar curvature s must be negative too. The above conditions give rise to a contradiction with the formula (3.156). Thus we have proved the "vanishing theorem" for a nearly integrable $SO(3)$-structure. □

In particular, we consider a compact connected Riemannian manifold (M, g) of constant curvature, then $R(X, Y)Z = \frac{s}{20}\ (g(Y, Z)\, X - g(X, Z)\, Y)$ for $s = s_0 = \text{const}$. In this case, the formula (3.153) has the following form

$$2s_0 \mathrm{Vol}\, M = \int_M \|\nabla\Upsilon\|^2\ d\nu \ge 0. \tag{3.157}$$

If $\int_M \|\nabla\Upsilon\|^2 \, d\nu \geq 0$ (in particular, if $\nabla\Upsilon \neq 0$) then from (3.157) we obtain that $s = s_0 > 0$. In this case, (M, g) is a spherical space form (see [196]). In particular, if (M, g) is simply connected then (M, g) is isometric to a Euclidean sphere S^5. On the other hand, if $\int_M \|\nabla\Upsilon\|^2 \, d\nu = 0$ (that is equal to the equation $\nabla\Upsilon = 0$) then $s = s_0 = 0$. In this case, (M, g) is a Euclidean space form (see [196]). Thus we can formulate the following proposition.

Corollary 3.3 *A compact and connected five-dimensional Riemannian manifold (M, g) of constant nonzero sectional curvature with a nearly integrable $SO(3)$-structure is a spherical space form. In particular, if (M, g) is simply connected then (M, g) is isometric to a Euclidean sphere S^5.*

Next, let (M, g, Υ) be a five-dimensional connected and complete Riemannian manifold (M, g) with a nearly integrable $SO(3)$-structure then the equations $\nabla_m \Upsilon^m{}_{jk} = 0$ are true. In this case, we obtain from (3.147) the following equalities $\overset{\alpha}{R}_{jk} = R_{jk} - \frac{7\alpha^2}{2} \cdot g_{jk}$. In addition, we suppose that there exists an α-connection $\overset{\alpha}{\nabla}$ such that the Ricci tensor $\overset{\alpha}{Ric}$ is positive semi-definite. Then $Ric \geq \frac{7\alpha^2}{2} \cdot g > 0$ and hence the complete manifold (M, g) must be compact with a finite fundamental group (see [106, p. 201]).

Proposition 3.3 *Let (M, g, Υ) be a five-dimensional connected and complete Riemannian manifold (M, g) with a nearly integrable $SO(3)$-structure. If there is an α-connection $\overset{\alpha}{\nabla}$ such that its Ricci tensor $\overset{\alpha}{Ric}$ is positive semi-definite then (M, g) is compact and its fundamental group is finite.*

3.8 Traceless decomposition of tensors

3.8.1 Introduction

The notion of traceless decompositions of tensors spaces was introduced by Weyl in [941], where Weyl formulated the traceless decomposition problem as a expression of the the given tensor in the form of a sum of the traceless tensor of the same type and a linear combinations of tensors of certain types.

The theory of traceless decompositions of tensors has been developed by many authors, because it has great importance in a lot of mathematical branches, especially in the tensor calculus.

In this chapter we bring the description of results of traceless decomposition problems for manifolds over real numbers, manifolds with almost complex structure, Riemannian and Ricci tensors and its generalization. In this cases explicit formulas are presented.

3.8.2 On decomposition of tensors on manifolds

As is known, on any n-dimensional manifold M_n, it is always possible to introduce a certain sign definite metric g, which thus equips M_n with a Riemannian space structure.

In what follows, we consider M_n together with this sign-definite metric g.

Let us introduce the following notation. Assume that there is an ordered system of indices $i_1, i_2, \ldots, i_p$ assuming the values $1, 2, \ldots, n$. Among them, let us isolate two indices i_r and i_s where $r < s$.

Then

$$\overset{(r,s)}{M}_{i_1,\ldots i_{r-1}\overset{r}{\curlyvee} i_{r+1}\ldots i_{s-1}\overset{s}{\curlyvee} i_{s+1}\ldots\ i_q}$$

denotes a tensor of type $(0, q-2)$ in which the indices occupying the r-th and s-th places are omitted.

Theorem 3.26 (M. Jukl, L. Juklová, J. Mikeš [76])
Let $T_{i_1\ldots i_q}$ be a tensor of type $(0,q)$ and

$$(i_{k_1}, i_{l_1}),\ (i_{k_2}, i_{l_2}),\ \ \ldots\ ,\ (i_{k_s}, i_{l_s}), \quad \text{where} \quad k_\sigma < l_\sigma, \tag{3.158}$$

be s pairs of indices, $s \leq \frac{1}{2}\, q(q+1)$.

Then there exists a decomposition of tensor $T_{i_1\ldots i_q}$ of the form

$$T_{i_1\,\ldots\,i_q} = \tilde{T}_{i_1\,\ldots\,i_q} + \sum_{\sigma=1}^{s} g_{i_{k_\sigma} i_{l_\sigma}} \cdot \overset{(k_\sigma,l_\sigma)}{M}_{i_1,\ldots i_{r-1}\overset{k_\sigma}{\curlyvee} i_{r+1}\ldots i_{s-1}\overset{l_\sigma}{\curlyvee} i_{s+1}\ldots\ i_q}, \tag{3.159}$$

where the tensor $\tilde{T}$ having the property

$$\tilde{T}_{i_1\ldots\ i_{k_\sigma}\ldots\ i_{l_\sigma}\ldots\ i_q} \cdot g^{i_{k_\sigma} i_{l_\sigma}} = 0 \tag{3.160}$$

for all pairs of indices (3.158) is uniquely defined and $\overset{(k_\sigma,l_\sigma)}{M}$ are some tensors of type $(0, q-2)$.

Proof. On the vector space T_q^0 let us introduce the inner product as follows:

$$\overset{1}{T} \circ \overset{2}{T} \;=\; \overset{1}{T}_{i_1 \cdots i_q} \cdot \overset{2}{T}_{j_1 \cdots j_q}\, g^{i_1 j_1} \,\cdots\, g^{i_q j_q}. \tag{3.161}$$

Furthermore, let us consider the linear subspace T^* of the space T_q^0 generated by all tensors of the form

$$\sum_{\sigma=1}^{s} g_{i_{k_\sigma} i_{l_\sigma}} \cdot \overset{(k_\sigma, l_\sigma)}{M}_{i_1, \ldots \overset{k_\sigma}{\curlyvee} \ldots \overset{l_\sigma}{\curlyvee} \ldots i_q}. \tag{3.162}$$

Obviously, the relations $\tilde{T} \circ T^* = 0$ and $\tilde{T} \cap T^* = \{0\}$ hold for all tensors $\tilde{T}$ from the subspace defined by relation (3.160), and, therefore, this subspace is the orthogonal complement of the subspace T^*. Hence this implies the uniqueness of representation of the tensor $T_{i_1 \cdots i_q}$ in the form of decomposition (3.159). Thus, we have proved Theorem 3.26. □

Let us recall the well-known definition of raising and lowering the indices. Using a tensor of type $(0, p+q)$, according to this rule, one constructs a new tensor of type (p, q) in the following way:

$$T^{i_1 \cdots i_p}_{j_1 \cdots j_q} \overset{\text{def.}}{=} g^{i_1 \alpha_1} g^{i_2 \alpha_2} \cdots\; g^{i_p \alpha_p} T_{\alpha_1 \cdots\, \alpha_p j_1 \cdots\, j_q}. \tag{3.163}$$

A direct application of (3.163) to Theorem 3.26 gives the following theorem.

Theorem 3.27 *Let $T^{j_1 \cdots j_q}_{i_1 \cdots i_p}$ be a tensor of type (p, q) and*

$$(i_{k_1}, j_{l_1});\; (i_{k_2}, j_{l_2});\; \ldots\; (i_{k_s}, j_{l_s}), \quad \textit{where} \quad 1 \le k_\sigma \le p,\; 1 \le l_\sigma \le q, \tag{3.164}$$

be s pairs of indices, $s \le p\,q$.

Then there exists a decomposition of $T_{i_1 \cdots i_p}$ of the form

$$T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \tilde{T}^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \sum_{\sigma=1}^{s} \delta^{i_{k_\sigma}}_{j_{l_\sigma}} \cdot \underset{(l_\sigma)}{\overset{(k_\sigma)}{M}}{}_{j_1, \cdots \overset{l_\sigma}{\curlyvee} \cdots j_q}^{i_1, \cdots \overset{k_\sigma}{\curlyvee} \cdots i_p}, \tag{3.165}$$

where the tensor $\tilde{T}$ having the property

$$\tilde{T}^{i_1 \cdots\; i_{k_\sigma} \cdots\; i_p}_{j_1 \cdots\; j_{l_\sigma} \cdots\; j_q} \cdot \delta^{j_{l_\sigma}}_{i_{k_\sigma}} = 0, \tag{3.166}$$

for all pairs of indices (3.164) *is uniquely defined and $\overset{(k_\sigma)}{\underset{(l_\sigma)}{M}}$ are some tensors of type $(p-1, q-1)$.*

Condition (3.166) is equivalent to the tensor $\tilde{T}$ is *traceless* with the respect to all pairs of indices (3.164). This condition may be written in the form

$$\tilde{T}^{i_1 \cdots\; i_{k_{\sigma-1}} \alpha\; i_{k_{\sigma+1}} \cdots\; i_p}_{j_1 \cdots\; j_{l_{\sigma-1}} \alpha\; j_{l_{\sigma+1}} \cdots\; j_q} = 0. \tag{3.167}$$

Note that Theorem 3.27 holds for any manifold, since the metric tensor, which is necessary for (the proof of) Theorem 3.26, can always be additionally constructed.

The following theorem holds (see [563, 645]).

Theorem 3.28 *Let $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a tensor of type (p, q). If $n+1 \geq p+q$, then there exists a unique decomposition of $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ of the form*

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{min\{p,q\}} \sum_{\oplus} \delta^{i_{\varrho 1}}_{j_{\sigma 1}} \delta^{i_{\varrho 2}}_{j_{\sigma 2}} \cdots \delta^{i_{\varrho t}}_{j_{\sigma t}} \overset{\star}{B}{}^{\cdots}_{\cdots} , \tag{3.168}$$

where

$$\begin{aligned} \oplus &= \begin{cases} \forall \varrho_1, \varrho_2, \ldots, \varrho_t = 1, 2, \ldots, p \ \ (\varrho_1 < \varrho_2 < \cdots < \varrho_t) \\ \forall \sigma_1, \sigma_2, \ldots, \sigma_t = 1, 2, \ldots, q \ \ (\sigma_i \ \textit{are pairwise distinct}) , \end{cases} \\ \star &= \begin{Bmatrix} \varrho_1 & \varrho_2 & \cdots & \varrho_t \\ \sigma_1 & \sigma_2 & \cdots & \sigma_t \end{Bmatrix} \end{aligned} \tag{3.169}$$

and the tensors $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $\overset{\star}{B}{}^{\cdots}_{\cdots}$ are traceless.

Remark. A tensor A is called *traceless*, if all its traces are zero. Decomposition (3.168) is called *traceless decomposition of A.* The fulfillment of this theorem for tensors of type $(1, q)$ follows from results of [642].

Proof. The proof of Theorem 3.28, presented in [563], immediately follows Weyls studies [941]. The possibility of decomposition (3.168) obviously follows from a sequential application of Theorem 3.27 for any dimension of the considered space.

Let us give the proof of the unicity of decomposition (3.168), whose idea goes back to [645]. This proof is constructive, and, in essence, it is based on consideration of linear algebraic equations with respect to the components of the desired tensors $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $\overset{\star}{B}{}^{\cdots}_{\cdots}$.

Without loss of generality, assume that $p \leq q$; then (3.168) can be written in the form

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{p} \sum_{\oplus} \delta^{i_{\varrho 1}}_{j_{\sigma 1}} \delta^{i_{\varrho 2}}_{j_{\sigma 2}} \cdots \delta^{i_{\varrho t}}_{j_{\sigma t}} \overset{\star}{B}{}^{\cdots}_{\cdots} \tag{3.170}$$

where indices $\varrho_1, \varrho_2, \ldots, \varrho_t, \sigma_1, \sigma_2, \ldots, \sigma_t$ have properties (3.169) and tensors $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $\overset{\star}{B}{}^{\cdots}_{\cdots}$ are traceless.

Let us denote

$$Y_1 \equiv n + p + q - 2; \quad Y_t \equiv n Y_{t-1} + p + q - 2t \qquad \text{for } t \geq 2.$$

One may prove that $n+1 \geq p+q$ implies for any $t = 1, 2, \ldots, p$:

$$n^t > Y_t - n^t.$$

a) First, contracting (3.170) with $\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_p}_{i_p}$ we have

$$A^{\alpha_1\alpha_2\cdots\alpha_p}_{\cdots\alpha_1\alpha_2\cdots\alpha_p\cdots} = n^p \overset{\diamond}{B}_{\cdots} + \sum \overset{\square}{B}_{\cdots} , \tag{3.171}$$

where

$$\diamond = \begin{Bmatrix} 1 & 2 & \cdots & p \\ \sigma_1 & \sigma_2 & \cdots & \sigma_p \end{Bmatrix}; \quad \square = \begin{Bmatrix} 1 & 2 & \cdots & p \\ T_1 & T_2 & \cdots & T_p \end{Bmatrix}.$$

The right-hand side of expression (3.171) is equal to the contraction of the tensor A with respect to all p contravariant indices. The sum entering the left-hand side of (3.171) contains no more than $Y_p - n^p$ summands $\overset{\square}{B}$, where $\{T_1\ T_2\ \cdots\ T_p\} \neq \{\sigma_1\ \sigma_2\ \cdots\ \sigma_p\}$ with pairwise different indices $T_i = 1, 2, \ldots, q$.

Relation (3.171) represents a system of linear equations with respect to the unknowns

$$\overset{\diamond}{B}_{i_{p+1}\cdots i_q} \quad (\sigma_i = 1, 2, \ldots, q). \tag{3.172}$$

If $n + 1 \geq p + q$, then $n^p > Y_p - n^p$. In this case the system (3.171) has exactly one solution, therefore all tensors (3.172) are determined uniquely.

b) Furthermore, let us transpose the expression containing tensors $B^{i_1 i_2\cdots i_p}_{j_1 j_2\cdots j_q}$ and $\overset{\star}{B}{}^{\cdots}_{\cdots}$ to the left side of (3.170) and denote it by $\tilde{A}$. Clearly, the left side is defined uniquely. Contracting the obtained equation with $\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\cdots\delta^{j_{p-1}}_{i_{p-1}}$, we obtain the equations

$$\tilde{A}^{\cdots\alpha_1\alpha_2\cdots\alpha_{p-1}\cdots}_{\cdots\alpha_1\alpha_2\cdots\alpha_{p-1}\cdots} = n^{p-1} \overset{\overline{\diamond}}{B}{}^h_{\cdots} + \sum \overset{\square}{B}{}^h_{\cdots} , \tag{3.173}$$

where

$$\overline{\diamond} = \begin{Bmatrix} \varrho_1 & \varrho_2 & \cdots & \varrho_{p-1} \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{p-1} \end{Bmatrix}; \quad \square = \begin{Bmatrix} \nu_1 & \nu_2 & \cdots & \nu_{p-1} \\ T_1 & T_2 & \cdots & T_{p-1} \end{Bmatrix}.$$

The relation (3.173) represents a system of linear equations with respect to the uknowns $\overset{\overline{\diamond}}{B}$. The sums in (3.173) have no more than $Y_{p-1} - n^{p-1}$ summands. Because $n + 1 > p + q$ implies that $n^{p-1} > Y_{p-1} - n^{p-1}$, the system (3.173) has exactly one solution with respect to the tensors $\overset{\overline{\diamond}}{B}$.

After finite number of analogous steps, we obtain that the tensors

$$\overset{\star}{B} \quad \left(\star = \begin{Bmatrix} 1 & 2 & \cdots & p \\ \sigma_1 & \sigma_2 & \cdots & \sigma_p \end{Bmatrix}; \begin{Bmatrix} \varrho_1 & \varrho_2 & \cdots & \varrho_{p-1} \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{p-1} \end{Bmatrix}; \cdots; \begin{Bmatrix} \varrho_1 & \varrho_2 \\ \sigma_1 & \sigma_2 \end{Bmatrix}\right) \tag{3.174}$$

are determined uniquely.

Now, let us describe the final step of this process. Let us transpose all known tensors to the left side of equations (3.170); there are the tensor A and expressions containig tensors (3.174). Denote the obtained left side by $\tilde{A}$. Contracting this expression by $\delta^{j\sigma_1}_{i\varrho_1}$ we obtain the following system of linear equations

$$\tilde{A}^{\cdots\alpha\cdots}_{\cdots\alpha\cdots} = n \overset{\substack{\varrho_1 \\ \sigma_1}}{B}{}^{\cdots}_{\cdots} + \sum \overset{\substack{\nu_1 \\ T_1}}{B}{}^{\cdots}_{\cdots} \tag{3.175}$$

where $\nu \in \{1, 2, \ldots, p\} \setminus \{\varrho_1\}$ $\ T \in \{1, 2, \ldots, q\} \setminus \{\sigma_1\}$.

Tensors $\overset{g}{\underset{\sigma}{B}}{}^{\cdots}_{\cdots}$ are the unknowns in this system. For $n > p+q-2$ this system has exactly one solution.

We see that all tensors B and $\overset{\star}{B}$ are determined uniquely. The theorem has been proved. □

Using Theorem 3.28 and its proof, we may elementary construct traceless decompositions of tensors of types $(1,1)$, $(1,2)$, $(1,3)$, and $(2,2)$, which were done in [562] in detail. Let us note that such decompositions were used previously in concrete calculations, since they are sufficiently trivial [226, 642].

Example 3.2 Let $A \in T^1_1\mathbb{R}^n, A = (A^i_j), n \geq 2$. Then there exists a unique traceless system $B \in T^1_1\mathbb{R}^n$, $B = (B^i_j)$ and unique number $c \in \mathbb{R}$ such that

$$A^i_j = B^i_j + c\delta^i_j,$$

where $c = \frac{1}{n}A^\alpha_\alpha$; $B^i_j = A^i_j - \frac{1}{n}\delta^i_j A^\alpha_\alpha$.

Example 3.3 Let $A \in T^1_2\mathbb{R}^n, A = (A^i_{jk}), n \geq 2$. Then there exist a unique traceless system $B = (B^i_{jk}) \in T^1_2\mathbb{R}^n$ $\overset{1}{B} = (\overset{1}{B}_k)$, $\overset{2}{B} = (\overset{2}{B}_k) \in T^0_1\mathbb{R}^n$ such that

$$A^i_{jk} = B^i_{jk} + \delta^i_j \overset{1}{B}_k + \delta^i_k \overset{2}{B}_j$$

and

$$\overset{1}{B}_k = \tfrac{1}{n^2-1}\left(nA^\alpha_{\alpha k} - A^\alpha_{k\alpha}\right), \ \overset{2}{B}_j = \tfrac{1}{n^2-1}\left(-A^\alpha_{\alpha j} + nA^\alpha_{j\alpha}\right),$$
$$B^i_{jk} = A^i_{jk} - \tfrac{1}{n^2-1}\left(\delta^i_j\left(nA^\alpha_{\alpha k} - A^\alpha_{k\alpha}\right) + \delta^i_k\left(-A^\alpha_{\alpha j} + nA^\alpha_{j\alpha}\right)\right).$$

Example 3.4 Let $A \in T^1_3\mathbb{R}^n, A = (A^i_{jkl}), n \geq 3$. Then there exist a unique traceless system $B \in T^1_3\mathbb{R}^n, B = (B^i_{jkl})$ and unique systems

$$\overset{1}{B} = (\overset{1}{B}_{kl}), \ \overset{2}{B} = (\overset{2}{B}_{jl}), \ \overset{3}{B} = (\overset{3}{B}_{jk}) \in T^0_2\mathbb{R}^n$$

such that

$$A^i_{jkl} = B^i_{jkl} + \delta^i_j \overset{1}{B}_{kl} + \delta^i_k \overset{2}{B}_{jl} + \delta^i_l \overset{3}{B}_{jk}$$

and these systems are defined by

$$\begin{aligned}
\overset{1}{B}_{jk} &= \tfrac{1}{(n^2-1)(n^2-4)}\Big(n\left(n^2-3\right)A^\alpha_{\alpha jk} - \left(n^2-2\right)A^\alpha_{j\alpha k} + nA^\alpha_{jk\alpha} - \\
&\qquad 2A^\alpha_{\alpha kj} + nA^\alpha_{k\alpha j} - \left(n^2-2\right)A^\alpha_{kj\alpha}\Big), \\
\overset{2}{B}_{jk} &= \tfrac{1}{(n^2-1)(n^2-4)}\Big(-\left(n^2-2\right)A^\alpha_{\alpha jk} + n\left(n^2-3\right)A^\alpha_{j\alpha k} - \left(n^2-2\right)A^\alpha_{jk\alpha} + \\
&\qquad nA^\alpha_{\alpha kj} - 2A^\alpha_{k\alpha j} + nA^\alpha_{kj\alpha}\Big), \\
\overset{3}{B}_{jk} &= \tfrac{1}{(n^2-1)(n^2-4)}\Big(nA^\alpha_{\alpha jk} - \left(n^2-2\right)A^\alpha_{j\alpha k} + n\left(n^2-3\right)A^\alpha_{jk\alpha} - \\
&\qquad \left(n^2-2\right)A^\alpha_{\alpha kj} + nA^\alpha_{k\alpha j} - 2A^\alpha_{kj\alpha}\Big),
\end{aligned}$$

and

$$B^i_{jkl} = A^i_{jkl} - \delta^i_j \overset{1}{B}_{kl} - \delta^i_k \overset{2}{B}_{jl} - \delta^i_l \overset{3}{B}_{jk}.$$

3.8.3 F-traceless decomposition

In this section, we present one generalization of the Theorem 3.28. This generalization was formulated by Mikeš in [645]. He introduced s. c. *F-traceless tensor* by the following way:

Let an affinor F_j^i be given ($F_j^i \in E_1^1,\ \ F_\alpha^\alpha = 0$). A tensor $A \in E_q^p$ is called *F-traceless* if the following conditions hold:

$$\forall k = 1, \dots, p; \quad \forall r = 1, \dots, q; \quad A^{\cdots\ i_{k-1}\alpha i_{k+1}\ \cdots}_{\cdots\ j_{r-1}\beta j_{r+1}\cdots} F_\alpha^\beta = 0; \quad A^{\cdots\ i_{k-1}\alpha i_{k+1}\ \cdots}_{\cdots\ j_{r-1}\alpha j_{r+1}\cdots} = 0.$$

The equality (3.176) presented bellow is said to be *F-traceless decomposition* of the tensor A. By *e-structure* is meant an affinor with $F_\alpha^i F_j^\alpha = e\delta_j^i$, where $e = \pm 1$. For $e = -1$ this structure is the almost complex structure, and for $e = 1$ it is the almost product structure [191, 197].

Theorem 3.29 *Let $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ be a tensor of type (p, q). If $n > 2(p + q)$, then there exists a unique decomposition of the tensor A of the form*

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{min\{p,q\}} \sum_{\oplus} \overset{*}{Q}{}^{i_{\varrho_1} i_{\varrho_2} \cdots i_{\varrho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \overset{\diamond}{B}{}^{\cdots}_{\cdots} \tag{3.176}$$

where $* = \{\tau_1, \tau_2 \dots \tau_t\}$;

$$\oplus = \begin{cases} \forall \varrho_1, \varrho_2 \dots \varrho_t = 1, \dots, p \ \ (\varrho_1 < \varrho_2 \ < \cdots < \varrho_t) \\ \forall \sigma_1, \sigma_2 \dots \sigma_t = 1, \dots, q \ (\sigma_i \textit{ are pairwise different}) \\ \tau_1, \tau_2 \dots \tau_t \in \{0, 1\} \end{cases} \qquad \diamond = \begin{Bmatrix} \varrho_1, \ \varrho_2 \ \dots \ \varrho_t \\ \sigma_1, \ \sigma_2 \ \dots \ \sigma_t \\ \tau_1, \ \tau_2 \ \dots \ \tau_t \end{Bmatrix}.$$

Tensors $B^{\cdots}_{\cdots}$ and $\overset{\diamond}{B}{}^{\cdots}_{\cdots}$ are F-traceless and

$$\overset{*}{Q}{}^{i_{\varrho_1} i_{\varrho_2} \dots i_{\varrho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} = \overset{\{\tau_1\}}{F}{}^{i_{\varrho_1}}_{j_{\sigma_1}} \overset{\{\tau_2\}}{F}{}^{i_{\varrho_2}}_{j_{\sigma_2}} \dots \overset{\{\tau_t\}}{F}{}^{i_{\varrho_t}}_{j_{\sigma_t}}; \qquad \overset{\{0\}}{F}{}^{i}_{j} = \delta^i_j; \quad \overset{\{1\}}{F}{}^{i}_{j} = F^i_j.$$

Let us remark that the proof of this theorem si analogous to the proof of Theorem 3.28 and therefore it will be omitted.

Now, we will find F-traceless decompositions of tensors of the type $(1, 1)$, $(1, 2)$, and the decomposition of tensors of the type $(1, 3)$ under special conditions. To derive these formulas we may use the proof of the theorem 3.27; see [574].

The decomposition of tensors of the type $(1, 3)$ was obtained for the case $e = 1$. In the general case it may be obtained due to theorems 3.27 and 3.29 and it reduces to solving a system of 48 linear algebraic equations with 48 unknowns. The determinant of the matrix of this system may be expressed by the form

$$\det(T) = n^{12} (n-2)^{16} (n+2)^{16} (n-4)^2 (n+4)^2.$$

This system has exactly one solution iff $\det(T) \neq 0$. The dimension of the tensor space with e-structures must be even. We can verify that a unique decomposition exists for $n > 4$.

Example 3.5 Let $A = (A^i_j)$ be a tensor of type $(1,1)$, and let F be a traceless e-structure. If $n \geq 2$, then there exist a unique F-traceless system $B = (B^i_j)$ and unique invariants $\overset{1}{c}, \overset{2}{c} \in \mathbb{R}$ such that

$$A^i_j = B^i_j + \overset{1}{c}\,\delta^i_j + \overset{2}{c}\,F^i_j$$

and this system is defined as follows:

$$\overset{1}{c} = \tfrac{1}{n}A^\alpha_\alpha,\ \overset{2}{c} = \tfrac{1}{en}A^\beta_\alpha F^\alpha_\beta \quad \text{and} \quad B^i_j = A^i_j - \overset{1}{c}\,\delta^i_j - \overset{2}{c}\,F^i_j.$$

Example 3.6 Let $A = (A^i_{jk})$ be a tensor of type $(1,2)$ and let F be a traceless e-structure. If $n > 2$, then there exists exactly one F-traceless system $B = (B^i_{jk}) \in T^1_2\mathbb{R}^n$ and exactly one system $\overset{1}{B} = (\overset{1}{B}_k), \overset{2}{B} = (\overset{2}{B}_k), \overset{3}{B} = (\overset{3}{B}_j), \overset{4}{B} = (\overset{4}{B}_j) \in T^0_1\mathbb{R}^n$ such that

$$A^h_{ij} = B^h_{ij} + \delta^h_i\ \overset{1}{B}_j + \delta^h_j\ \overset{2}{B}_i + \overset{3}{B}_i\ F^h_j + \overset{4}{B}_j\ F^h_i,$$

where

$$\begin{aligned}
\overset{1}{B}_j &= \tfrac{1}{n(3e+1-n^2)}\left((2e-n^2)A^\alpha_{\alpha j} + nA^\alpha_{j\alpha} + nA^\alpha_{\gamma\beta}F^\gamma_j F^\beta_\alpha - 2A^\alpha_{\beta\gamma}F^\gamma_j F^\beta_\alpha\right),\\
\overset{2}{B}_j &= \tfrac{1}{n(3e+1-n^2)}\left(nA^\alpha_{\alpha j} + (2e-n^2)A^\alpha_{j\alpha} - 2A^\alpha_{\gamma\beta}F^\gamma_j F^\beta_\alpha + nA^\alpha_{\beta\gamma}F^\gamma_j F^\beta_\alpha\right),\\
\overset{3}{B}_j &= \tfrac{1}{n(3e+1-n^2)}\left((2-n^2)A^\alpha_{j\beta}F^\beta_\alpha + nA^\alpha_{\beta j}F^\beta_\alpha + nA^\alpha_{\alpha\beta}F^\beta_j - 2A^\alpha_{\beta\alpha}F^\beta_j\right),\\
\overset{4}{B}_j &= \tfrac{1}{n(3e+1-n^2)}\left(nA^\alpha_{j\beta}F^\beta_\alpha + (2-n^2)A^\alpha_{\beta j}F^\beta_\alpha - 2A^\alpha_{\alpha\beta}F^\beta_j + nA^\alpha_{\beta\alpha}F^\beta_j\right),\\
B^h_{ij} &= A^h_{ij} - \delta^h_i\ \overset{1}{B}_j - \delta^h_j\ \overset{2}{B}_i - \overset{3}{B}_i\ F^h_j - \overset{4}{B}_j\ F^h_i.
\end{aligned}$$

Example 3.7 Let A be a tensor of type $(1,3)$, let $n = 6$ and F be a traceless e-structure ($e = 1$). Then there exists exactly one F-traceless decomposition of the tensor A in the form

$$A^h_{ijk} = B^h_{ijk} + \delta^h_i C_{jk} + \delta^h_j D_{ik} + \delta^h_k E_{ij} + G_{jk}F^h_i + H_{ik}F^h_j + I_{ij}F^h_k,$$

where

$$\begin{aligned}
C_{jk} = \tfrac{1}{960}\Big[&184A^\alpha_{\alpha jk} - 36\left(A^\alpha_{j\alpha k} + A^\alpha_{kj\alpha} + A^\alpha_{\gamma j\beta}F^\beta_\alpha F^\gamma_k + A^\alpha_{\gamma\beta k}F^\beta_\alpha F^\gamma_j\right) +\\
&9\left(A^\alpha_{jk\alpha} + A^\alpha_{k\alpha j} + A^\alpha_{\gamma\beta j}F^\beta_\alpha F^\gamma_k + A^\alpha_{j\gamma\beta}F^\beta_\alpha F^\gamma_k + A^\alpha_{k\beta\gamma}F^\beta_\alpha F^\gamma_j +\right.\\
&\left.A^\alpha_{\gamma k\beta}F^\beta_\alpha F^\gamma_j + A^\alpha_{\beta\alpha\gamma}F^\gamma_j F^\beta_k + A^\alpha_{\gamma\beta\alpha}F^\gamma_j F^\beta_k\right) - 6\left(A^\alpha_{\alpha kj} + A^\alpha_{\beta\gamma j}F^\beta_\alpha F^\gamma_k +\right.\\
&A^\alpha_{j\beta\gamma}F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta k\gamma}F^\beta_\alpha F^\gamma_j + A^\alpha_{k\gamma\beta}F^\beta_\alpha F^\gamma_j + A^\alpha_{\alpha\beta\gamma}F^\gamma_j F^\beta_k + A^\alpha_{\beta\gamma\alpha}F^\gamma_j F^\beta_k +\\
&\left.A^\alpha_{\gamma\alpha\beta}F^\gamma_j F^\beta_k\right) + 14\left(A^\alpha_{\beta j\gamma}F^\beta_\alpha F^\gamma_k + A^\alpha_{\beta\gamma k}F^\beta_\alpha F^\gamma_j\right) + 4A^\alpha_{\alpha\gamma\beta}F^\beta_k F^\gamma_j\Big],
\end{aligned}$$

$$D_{jk} = \tfrac{1}{960}\Big[184A^{\alpha}_{j\alpha k} - 36\left(A^{\alpha}_{\alpha jk} + A^{\alpha}_{jk\alpha} + A^{\alpha}_{j\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta\gamma k}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + 9\left(A^{\alpha}_{kj\alpha} + A^{\alpha}_{\alpha kj} + A^{\alpha}_{\beta\gamma j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma j\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta k\ \gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{k\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\beta\gamma\alpha}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta\gamma}F^{\gamma}_{j}F^{\beta}_{k}\right) - 6\left(A^{\alpha}_{k\alpha j} + A^{\alpha}_{\beta\alpha\gamma}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\alpha\gamma\beta}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\gamma\beta\alpha}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta j\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma\beta j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{k\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\gamma k\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + 14\left(A^{\alpha}_{j\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma\beta k}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + 4A^{\alpha}_{\gamma\alpha\beta}F^{\gamma}_{j}F^{\beta}_{k}\Big],$$

$$E_{jk} = \tfrac{1}{960}\Big[184A^{\alpha}_{jk\alpha} - 36\left(A^{\alpha}_{j\alpha k} + A^{\alpha}_{\alpha kj} + A^{\alpha}_{j\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta k\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + 9\left(A^{\alpha}_{\alpha jk} + A^{\alpha}_{k\alpha j} + A^{\alpha}_{\gamma\beta j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta j\gamma}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\beta\gamma k}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{k\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\alpha\gamma\beta}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha\gamma}F^{\gamma}_{j}F^{\beta}_{k}\right) - 6\left(A^{\alpha}_{kj\alpha} + A^{\alpha}_{\beta\gamma j}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma j\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{k\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\gamma\beta k}F^{\beta}_{\alpha}F^{\gamma}_{j} + A^{\alpha}_{\alpha\beta\gamma}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\beta\gamma\alpha}F^{\gamma}_{j}F^{\beta}_{k} + A^{\alpha}_{\gamma\alpha\beta}F^{\gamma}_{j}F^{\beta}_{k}\right) + 14\left(A^{\alpha}_{j\gamma\beta}F^{\beta}_{\alpha}F^{\gamma}_{k} + A^{\alpha}_{\gamma k\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}\right) + 4A^{\alpha}_{\gamma\beta\alpha}F^{\gamma}_{j}F^{\beta}_{k}\Big],$$

$$G_{jk} = \tfrac{1}{960}\Big[184A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} - 36\left(A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} + A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j}\right) + 9\left(A^{\alpha}_{jk\beta}F^{\beta}_{\alpha} + A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta k\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) - 6\left(A^{\alpha}_{\beta kj}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) + 14\left(A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j}\right) + 4A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big],$$

$$H_{jk} = \tfrac{1}{960}\Big[184A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} - 36\left(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + A^{\alpha}_{jk\beta}F^{\beta}_{\alpha}\right) + 9\left(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) - 6\left(A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta k\alpha}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) + 14\left(A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j}\right) + 4A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big],$$

$$I_{jk} = \tfrac{1}{960}\Big[184A^{\alpha}_{jk\beta}F^{\beta}_{\alpha} - 36\left(A^{\alpha}_{j\beta k}F^{\beta}_{\alpha} + A^{\alpha}_{j\alpha\beta}F^{\beta}_{k} + A^{\alpha}_{\alpha k\beta}F^{\beta}_{j} + A^{\alpha}_{\beta kj}F^{\beta}_{\alpha}\right) + 9\left(A^{\alpha}_{\beta jk}F^{\beta}_{\alpha} + A^{\alpha}_{k\beta j}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha j\beta}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha j}F^{\beta}_{k} + A^{\alpha}_{\alpha\beta k}F^{\beta}_{j} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j} + A^{\alpha}_{\beta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\beta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) - 6\left(A^{\alpha}_{kj\beta}F^{\beta}_{\alpha} + A^{\alpha}_{\alpha\beta j}F^{\beta}_{k} + A^{\alpha}_{\beta j\alpha}F^{\beta}_{k} + A^{\alpha}_{\beta\alpha k}F^{\beta}_{j} + A^{\alpha}_{k\beta\alpha}F^{\beta}_{j} + A^{\alpha}_{\gamma\beta\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\beta\delta\gamma}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k} + A^{\alpha}_{\delta\gamma\delta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\right) + 14\left(A^{\alpha}_{j\beta\alpha}F^{\beta}_{k} + A^{\alpha}_{k\alpha\beta}F^{\beta}_{j}\right) + 4A^{\alpha}_{\gamma\delta\beta}F^{\beta}_{\alpha}F^{\gamma}_{j}F^{\delta}_{k}\Big].$$

Now let us present F-decompositions of tensors of types $(2,2)$ and $(1,3)$ in the case of the tensor space with almost complex structure.

The explicit expression of F-traceless tensors B in the general case is unknown to us. The explicit expression was found for tensors of the type $(1,3)$ and $(2,2)$ with algebraic properties of Riemannian tensor of Kählerian spaces in [515, 516, 518] and [645].

Example 3.8 Let A be a tensor of the type $(2,2)$ with following properties

$$A^h_{\cdot\, ij\, \cdot}{}^k = A^k_{\cdot\, ji\, \cdot}{}^h; \quad A^\alpha_{\cdot\, \alpha j\, \cdot}{}^k = A^h_{\cdot\, i\alpha\, \cdot}{}^\alpha = 0; \quad A^h_{\cdot\, ij\, \cdot}{}^k = A^\alpha_{\cdot\, \beta\gamma\, \cdot}{}^\delta F^h_\alpha F^\beta_i F^\gamma_j F^k_\delta \,.$$

Let F be an almost complex structure. If $n > 4$ then there exists a unique F-decomposition of the tensor A in the form

$$\begin{aligned} A^h_{\cdot\, ij\, \cdot}{}^k = {} & B^h_{\cdot\, ij\, \cdot}{}^k + \delta^h_i C^k_j + \delta^k_j C^h_i + \delta^h_j D^k_i + \delta^k_i D^h_j + F^h_i G^k_j + F^k_j G^h_i F^h_j H^k_i + \\ & + F^k_i H^h_j + a\delta^h_i \delta^k_j + b\delta^h_j \delta^k_i + cF^h_i F^k_j + dF^h_j F^k_i + g\left(\delta^h_i F^k_j + F^h_i \delta^k_j\right) \end{aligned}$$

where

$$\begin{aligned} a &= \tfrac{1}{n(n^2-4)}(-nA^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + 2A^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + 2A^\alpha_\alpha); \\ b &= \tfrac{1}{n(n^2-4)}(2A^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta - nA^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta - nA^\alpha_\alpha); \\ c &= \tfrac{1}{n(n^2-4)}(nA^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + n(2-n^2)A^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta - 2A^\alpha_\alpha); \\ d &= \tfrac{1}{n^2-4}((2-n^2)A^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + A^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + A^\alpha_\alpha); \\ f &= 0; \quad g = \tfrac{1}{2n} A^\alpha_\beta F^\beta_\alpha; \quad A^h_j \equiv -A^h_{\cdot\, \alpha j\, \cdot}{}^\alpha, \end{aligned}$$

and B, C, D, G, H are F-traceless tensors with the following properties

$$\begin{aligned} C^k_j = {} & \tfrac{1}{n(n^2-16)}(-2nA^k_j - 8A^\alpha_{\cdot\, \beta\gamma\, \cdot}{}^k F^\beta_\alpha F^\gamma_j + 2A^\alpha_{\cdot\, \gamma\beta\, \cdot}{}^k F^\beta_\alpha F^\gamma_j) + \\ & \tfrac{1}{n^2(n^2-4)(n^2-16)}((-n^4+n^3-8n^2-4n)\delta^k_j A^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + \\ & (6n^3-4n^2+16)\delta^k_j A^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + (2n^3-4n^2+16n+16)\delta^k_j A^\alpha_\alpha) + \\ & \tfrac{1}{2n} F^k_j A^\beta_\alpha F^\alpha_\beta; \\ D^k_j = {} & \tfrac{1}{(n^2-16)}\,(\tfrac{n^4-12n^2-32}{n(n^2-4)} A^k_j + 2A^\alpha_{\cdot\, \beta\gamma\, \cdot}{}^k F^\beta_\alpha F^\gamma_j - 8A^\alpha_{\cdot\, \gamma\beta\, \cdot}{}^k F^\beta_\alpha F^\gamma_j) + \\ & \tfrac{1}{n^2(n^2-4)(n^2-16)}((n^3-12n^2-4n)\delta^k_j A^\alpha_\alpha + \\ & (6n^3+4n^2-16)\delta^k_j A^\beta_{\cdot\, \alpha\gamma\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta + \\ & (-n^4+n^3-8n^2-4)\delta^k_j A^\beta_{\cdot\, \gamma\alpha\, \cdot}{}^\sigma F^\gamma_\sigma F^\alpha_\beta); \end{aligned}$$

$$
\begin{aligned}
G_j^k &= \tfrac{1}{n^2-16}\big(\tfrac{2(n^4-4)}{n^2(n^2-4)}A_j^k - (n^2-8)A_{\cdot\ \beta\gamma\cdot}^{\alpha\ \ \ k}F_\alpha^\beta F_j^\gamma + 2A_{\cdot\ \gamma\beta\cdot}^{\alpha\ \ \ k}F_\alpha^\beta F_j^\gamma\big)+\\
&\tfrac{1}{n^2(n^2-4)(n^2-16)}\big((2(n^3-n^2+8n-8))\delta_j^k A_\alpha^\alpha+\\
&(-n^4-8n^2+12n)\delta_j^k A_{\cdot\ \alpha\gamma\cdot}^{\beta\ \ \ \sigma}F_\sigma^\gamma F_\beta^\alpha+\\
&(6n^3-2n^2-16)\delta_j^k A_{\cdot\ \gamma\alpha\cdot}^{\beta\ \ \ \sigma}F_\sigma^\gamma F_\beta^\alpha\big);\\
H_j^k &= \tfrac{1}{n(n^2-16)}\big(8A_j^k + 2nA_{\cdot\ \beta\gamma\cdot}^{\alpha\ \ \ k}F_\alpha^\beta F_j^\gamma - (n^2-8)A_{\cdot\ \gamma\beta\cdot}^{\alpha\ \ \ k}F_\alpha^\beta F_j^\gamma+\\
&\tfrac{1}{n^2(n^2-4)(n^2-16)}\big((12n-12n^2)\delta_j^k A_\alpha^\alpha-\\
&(6n^3-2n^2-16)\delta_j^k A_{\cdot\ \alpha\gamma\cdot}^{\beta\ \ \ \sigma}F_\sigma^\gamma F_\beta^\alpha + (-n^4-8n^2+12n)A_{\cdot\ \gamma\alpha\cdot}^{\beta\ \ \ \sigma}F_\sigma^\gamma F_\beta^\alpha\big);\\
B_{\cdot\ ij\ \cdot}^{h\ \ \ k} &= A_{\cdot\ ij\ \cdot}^{h\ \ \ k} - \delta_i^h C_j^k - \delta_j^k C_i^h - \delta_j^h D_i^k - \delta_i^k D_j^h - F_i^h G_j^k + F_j^k G_i^h - F_j^h H_i^k+\\
&F_i^k H_j^h - a\delta_i^h\delta_j^k - b\delta_j^h\delta_i^k - cF_i^h F_j^k - dF_j^h F_i^k - g\left(\delta_i^h F_j^k + F_i^h\delta_j^k\right).
\end{aligned}
$$

Now we will compute F-decompositon of tensors of the type $(1,3)$ for e-structures with $e=-1$. This structure is called an almost complex structure. As we mentioned above, it was proved in [575] that we get 48 algebraic equations in 48 unknowns in general and this system is not easily solvable, therefore the explicit formulas were not obtained there.

In explicit form, the following result was obtained for tensors of type $(1,3)$ which have following properties

$$A_{ijk}^h + A_{ikj}^h = 0;\ A_{ijk}^h + A_{jki}^h + A_{kij}^h = 0;\ A_{\alpha jk}^\alpha = 0;\ A_{i\alpha\beta}^h F_j^\alpha F_k^\beta = A_{ijk}^h. \quad (3.177)$$

For example, the Riemannian tensors of Kählerian space, K-space and CR-space have these properties [119, 170, 191, 197]. We have next decomposition.

Example 3.9 Let A be a tensor of the type $(1,3)$ with properties (3.177). Let F be an almost complex structure. If $n>4$ then there exists a unique F-decomposition of the tensor A in the form

$$A_{ijk}^h = B_{ijk}^h + \delta_i^h C_{jk} + \delta_j^h D_{ik} + \delta_k^h E_{ij} + F_i^h G_{jk} + F_j^h H_{ik} + F_k^h I_{ij},$$

where the tensors $B_{ijk}^h, C_{jk}, D_{jk}, E_{jk}, G_{jk}, H_{jk}, I_{jk}$ are defined by the following conditions

$$
\begin{aligned}
&C_{jk}=0;\ E_{jk}=-D_{jk}=\tfrac{1}{n+2}A_{jk};\ A_{jk}=A_{jk\alpha}^\alpha\\
&\tfrac{1}{2}G_{jk}=H_{jk}=-I_{jk}=-\tfrac{1}{n+2}\overline{A}_{jk};\ \overline{A}_{jk}=A_{j\alpha}F_k^\alpha;\\
&B_{ijk}^h = A_{ijk}^h + \tfrac{1}{n+2}\left(\delta_j^h A_{ik} - \delta_k^h A_{ij} + 2F_i^h\overline{A}_{jk} + F_j^h\overline{A}_{ik} - F_k^h\overline{A}_{ij}\right).
\end{aligned}
\quad (3.178)
$$

and the tensor B is F-traceless.

3.8.4 Quaternionic trace decomposition

Now we will generalize the traceless decomposition and F-decomposition for spaces with quaternionic structure.

Let $\overset{\{1\}}{F}{}^i_j$, $\overset{\{2\}}{F}{}^i_j$ be affinor structures fulfilling the next relations.

$$\overset{\{1\}}{F}{}^i_\alpha \ \overset{\{1\}}{F}{}^\alpha_j = -\delta^i_j, \quad \overset{\{2\}}{F}{}^i_\alpha \ \overset{\{2\}}{F}{}^\alpha_j = -\delta^i_j, \quad \overset{\{1\}}{F}{}^i_\alpha \ \overset{\{2\}}{F}{}^\alpha_j + \overset{\{2\}}{F}{}^i_\alpha \ \overset{\{1\}}{F}{}^\alpha_j = 0 \tag{3.179}$$

Denote

$$\overset{\{0\}}{F}{}^i_j = \delta^i_j, \quad \overset{\{3\}}{F}{}^i_j = \overset{\{1\}}{F}{}^i_\alpha \ \overset{\{2\}}{F}{}^\alpha_j = - \overset{\{2\}}{F}{}^i_\alpha \ \overset{\{1\}}{F}{}^\alpha_j . \tag{3.180}$$

A space is called a *space with almost quaternionic structure* if conditions (3.179) hold.

A tensor $A \in E^p_q$ is called *quaternionic traceless (Q-traceless)* if the next equation is true:

$$\text{for any } k = 1, \cdots, p, \ r = 1, \cdots, q; \tau_i = 0, 1, 2, 3 : \ A^{\cdots i_{k-1} \alpha i_{k+1} \cdots}_{\cdots j_{r-1} \beta j_{r+1} \cdots} \ \overset{\{\tau_i\}}{F}{}^\beta_\alpha = 0.$$

The Theorem 3.28 and the Theorem 3.29 can be generalized for spaces with quaternionic structure. Since the proof may be analogous to the proof of the Theorem 3.28 it will be omitted (it may be found in [574]).

The formula (3.181) in the following theorem is called *quaternionic traceless decomposition* (*Q-decomposition*) *of tensor A*.

Theorem 3.30 *Let A be a tensor of type (p, q). If $n > 4(p + q - 2)$ then there exists a unique decomposition of A in the form*

$$A^{i_1 i_2 \cdots \ i_p}_{j_1 j_2 \cdots \ j_q} = B^{i_1 i_2 \cdots \ i_p}_{j_1 j_2 \cdots \ j_q} + \sum_{t=1}^{min\{p,q\}} \sum_{\bigoplus} \overset{\star}{Q}{}^{i_{\varrho_1} i_{\varrho_2} \cdots \ i_{\varrho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots \ j_{\sigma_t}} \overset{\diamond}{B}{}^{\cdots}_{\cdots} \tag{3.181}$$

where

$$\bigoplus = \begin{cases} \varrho_1, \varrho_2, \dots \ , \varrho_t = 1, \cdots, p & (\varrho_1 < \varrho_2 < \dots \ < \varrho_t) \\ \sigma_1, \sigma_2, \dots \ , \sigma_t = 1, \cdots, q & (\sigma_i \quad \textit{are different to each other)} \\ \tau_1, \tau_2, \dots \ , \tau_t \in 0, 1, 2, 3 \end{cases} \tag{3.182}$$

$$\star = \{\tau_1, \tau_2 \dots \ , \tau_t\}, \qquad \diamond = \left\{ \begin{matrix} \varrho_1, \varrho_2 \cdots \ , \varrho_t \\ \sigma_1, \sigma_2, \dots \ \sigma_t \\ \tau_1, \tau_2, \dots \ , \tau_t \end{matrix} \right\}.$$

The tensors $B^{\cdots}_{\cdots}$ and $\overset{\diamond}{B}$ are quaternionic traceless and

$$\overset{\star}{Q}{}^{i_{\varrho_1} i_{\varrho_2} \cdots \ i_{\varrho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots \ j_{\sigma_t}} \equiv \overset{\{\tau_1\}}{F}{}^{i_{\varrho_1}}_{j_{\sigma_1}} \ \overset{\{\tau_2\}}{F}{}^{i_{\varrho_2}}_{j_{\sigma_2}} \ \dots \ \overset{\{\tau_t\}}{F}{}^{i_{\varrho_t}}_{j_{\sigma_t}}.$$

Next example gives quaternionic traceless decomposition of tensors of types $(1, 1)$ and $(1, 2)$.

Example 3.10 Let A be a tensor of the type $(1,1)$. Then there exists a unique quaternionic traceless system $B=(B^i_j)$ and unique numbers $\overset{1}{c},\overset{2}{c},\overset{3}{c},\overset{4}{c}\in\mathbb{R}$ such that

$$A^i_j = B^i_j + \overset{1}{c}\,\delta^i_j + \overset{2}{c}\overset{1}{F}{}^i_j + \overset{3}{c}\overset{2}{F}{}^i_j + \overset{4}{c}\overset{3}{F}{}^i_j$$

and these systems are defined by

$$\overset{1}{c} = \tfrac{1}{n}A^\alpha_\alpha,\ \overset{2}{c} = -\tfrac{1}{n}A^\beta_\alpha\,\overset{1}{F}{}^\alpha_\beta,\ \overset{3}{c} = -\tfrac{1}{n}A^\beta_\alpha\,\overset{2}{F}{}^\alpha_\beta,\ \overset{4}{c} = -\tfrac{1}{n}A^\beta_\alpha\,\overset{3}{F}{}^\alpha_\beta,\ \text{and}$$

$$B^i_j = A^i_j - \overset{1}{c}\,\delta^i_j - \overset{2}{c}\overset{1}{F}{}^i_j - \overset{3}{c}\overset{2}{F}{}^i_j - \overset{4}{c}\overset{3}{F}{}^i_j.$$

Example 3.11 Let A be a symmetric tensor of the type $(1,2)$. Let $\overset{\{0\}}{F}{}^i_j$, $\overset{\{1\}}{F}{}^i_j$, $\overset{\{2\}}{F}{}^i_j$, $\overset{\{3\}}{F}{}^i_j$ be quaternionic structures. If $n>6$ then there exists a unique Q-decomposition of the tensor A in the form

$$A^h_{ij} = B^h_{ij} + \overset{\{0\}}{F}{}^h_{(i}C_{j)} + \overset{\{1\}}{F}{}^h_{(i}D_{j)} + \overset{\{2\}}{F}{}^h_{(i}G_{j)} + \overset{\{3\}}{F}{}^h_{(i}H_{j)}$$

where

$$C_j = \tfrac{1}{n^3+2n^2-2n-4}\Big(n(n+1)A^\alpha_{\alpha j} + nA^\beta_{\alpha\gamma}\,\overset{\{1\}}{F}{}^\alpha_\beta\,\overset{\{1\}}{F}{}^\gamma_j + (n+2)A^\beta_{\alpha\gamma}\,\overset{\{2\}}{F}{}^\alpha_\beta\,\overset{\{2\}}{F}{}^\gamma_j + (n+2)A^\beta_{\alpha\gamma}\,\overset{\{3\}}{F}{}^\alpha_\beta\,\overset{\{3\}}{F}{}^\gamma_j\Big),$$

$$D_j = \tfrac{1}{n^2(n+2)}\Big(-(n+2)A^\alpha_{\alpha\gamma}\,\overset{\{1\}}{F}{}^\gamma_j + n(n+1)A^\beta_{\alpha j}\,\overset{\{1\}}{F}{}^\alpha_\beta + nA^\beta_{\alpha\gamma}\,\overset{\{2\}}{F}{}^\alpha_\beta\,\overset{\{3\}}{F}{}^\gamma_j - (n+2)A^\beta_{\alpha\gamma}\,\overset{\{3\}}{F}{}^\alpha_\beta\,\overset{\{2\}}{F}{}^\gamma_j\Big),$$

$$G_j = \tfrac{1}{n^2-2}\Big(-A^j_{\alpha\gamma}\,\overset{\{2\}}{F}{}^\gamma_j - A^\beta_{\alpha\gamma}\,\overset{\{1\}}{F}{}^\alpha_\beta\,\overset{\{3\}}{F}{}^\gamma_j + (n+1)A^\beta_{\alpha j}\,\overset{\{2\}}{F}{}^\alpha_\beta + A^\beta_{\alpha\gamma}\,\overset{\{3\}}{F}{}^\alpha_\beta\,\overset{\{1\}}{F}{}^\gamma_j\Big),$$

$$H_j = \tfrac{1}{n^3+2n^2-2n-4}\Big(-(n+2)A^j_{\alpha\gamma}\,\overset{\{3\}}{F}{}^\gamma_j + nA^\beta_{\alpha\gamma}\,\overset{\{1\}}{F}{}^\alpha_\beta\,\overset{\{2\}}{F}{}^\gamma_j - (n+2)A^\beta_{\alpha\gamma}\,\overset{\{2\}}{F}{}^\alpha_\beta\,\overset{\{1\}}{F}{}^\gamma_j + n(n+1)A^\beta_{\alpha j}\,\overset{\{3\}}{F}{}^\alpha_\beta\Big),$$

and tensor B^h_{ij} is Q-traceless and

$$B^h_{ij} = A^h_{ij} - \big(\,\overset{\{0\}}{F}{}^h_{(i}C_{j)} + \overset{\{1\}}{F}{}^h_{(i}D_{j)} + \overset{\{2\}}{F}{}^h_{(i}G_{j)} + \overset{\{3\}}{F}{}^h_{(i}H_{j)}\big)\,.$$

3.8.5 Generalized decomposition problem for Ricci and Riemannian tensors

In the Riemannian geometry and in the geometry of spaces of affine connection, an important role is played by Riemannian tensor, Ricci tensor, Weyl tensor of conformal curvature and Weyl tensor of projective curvature.

Let us present the main concepts of the theory of spaces of affine connection, Riemannian and Kähler spaces.

Spaces of affine connection. Then the Weyl tensor of the projective curvature is defined by the following way:

$$W^h_{ijk} \equiv R^h_{ijk} - \tfrac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \tfrac{1}{n+1}\left(\delta^h_i R_{[jk]} - \tfrac{1}{n-1}\left(\delta^h_k R_{[ji]} - \delta^h_j R_{[ki]}\right)\right),$$

where $\partial_i \equiv \partial/\partial x^i$, $[i,j]$ denotes an skew-symmetrization without division.

Because the Weyl tensor of projective curvature is traceless, its definition gives the traceless decomposition of the Riemannian tensor R^h_{ijk}, immediately.

Riemannian spaces. Using g_{ij} and g^{ij}, in V_n, we introduce the operation of lowering and rising indices, for example: $R_{hijk} \equiv g_{h\alpha}R^\alpha_{ijk}$; $R^{h\ k}_{.ij.} \equiv g^{k\alpha}R^h_{ij\alpha}$; $R^h_i \equiv g^{h\alpha}R_{\alpha i}$. Together with the Riemann, Ricci, and the projective Weyl tensor (the latter is simplified in V_n by $W^h_{ijk} \equiv R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})$), in V_n, we introduce the scalar curvature $R \equiv R_{\alpha\beta}g^{\alpha\beta}$ and the Einstein and Brinkmann tensors and Weyl tensor of conformal curvature:

$$E_{ij} \equiv R_{ij} - \frac{R}{n}g_{ij}; \quad L_{ij} \equiv \frac{1}{n-2}(R_{ij} - \frac{R}{2\,(n-1)}g_{ij});$$

$$C_{hijk} \equiv R_{hijk} + g_{hj}L_{ik} - g_{hk}L_{ij} - g_{ij}L_{hk} + g_{ik}L_{hj}.$$

- Because $E^\alpha_\alpha = 0$ ($E^h_i = E_{i\alpha}g^{h\alpha}$), the Einstein tensor is traceless. Therefore we have the traceless decomposition of the Ricci tensor R^h_i, immediately:

$$R^i_j = E^i_j + \frac{R}{n}\delta^i_j\,.$$

- The Weyl tensors of conformal curvature C^h_{ijk} and $C^{h\ \ k}_{.\ ij\ .}$ are also traceless.

- Therefore we obtain traceless decomposition of $R^{h\ \ k}_{.\ ij\ .}$ in the form

$$R^{h\ \ k}_{.\ ij\ .} = K^{h\ \ k}_{.\ ij\ .} + \frac{2}{n^2-4}(\delta^h_i E^k_j + \delta^k_j E^h_i) - \frac{n}{n^2-4}(\delta^h_j E^k_i + \delta^k_i E^h_j) +$$

$$\frac{R}{n(n^2-1)}\delta^h_i\delta^k_j - \frac{R}{n^2-1}\delta^h_j\delta^k_i;$$

where $K^{h\ \ k}_{.\ ij\ .}$ is a traceless tensor, which may be expressed from the formula above, explicitly.

Kähler spaces. In these spaces, holomorphic projective curvature tensor and Bochner tensor play an important role.

- The holomorphic projectice curvature tensor is introduced by the following way:

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(\delta^h_k R_{ij} - \delta^h_j R_{ik} - eF^h_k R_{i\alpha}F^\alpha_j + eF^h_j R_{i\alpha}F^\alpha_k - 2eF^h_i R_{j\alpha}F^\alpha_k).$$

  Because this tensor is F-traceless we obtain (from this definition) the F-traceless decomposition of the Riemannian tensor R^h_{ijk}, immediately.

- The Bochner tensor is defined by

$$B^h_{ijk} = R^h_{ijk} - g_{h[j}B_{k]i} + g_{i[j}B_{k]h} + e(g_{h\alpha}B_{i\beta} - g_{i\alpha}B_{h\beta})F^\alpha_{[j}F^\alpha_{k]} -$$

$$2eg_{h\alpha}F^\alpha_i B_{j\beta}F^\beta_k - 2eB_{h\alpha}F^\alpha_i g_{j\beta}F^\beta_k,$$

  where $B_{ij} = \frac{1}{n+4}(R_{ij} - \frac{R}{2(n+2)}g_{ij})$.

  The Bochner tensors B^h_{ijk} and $B^h_{\cdot\, ij}{}^{\cdot}_{k}$ are F-traceless.

- We directly get the F-traceless decomposition of $R^h_{\cdot\, ij}{}^{\cdot}_{k}$ in the form

  $R^h_{\cdot\, ij}{}^{\cdot}_{k} = K^h_{\cdot\, ij}{}^{\cdot}_{k} - \frac{4}{n^2-16}(\delta^h_i E^k_j + \delta^k_j E^h_i + eF^h_i \overline{E}^k_j + F^k_j \overline{E}^h_i) + \frac{n}{n^2-16}(\delta^h_j E^k_i + \delta^k_i E^h_j + eF^h_j \overline{E}^k_i + eF^h_i \overline{E}^h_j) - \frac{2R}{n(n^2-4)}(\delta^h_i \delta^k_j + eF^h_i F^k_j) - \frac{R}{n^2-4}(\delta^h_j \delta^k_i + eF^h_j F^k_i)$,

  where $\overline{E}^h_i = E^h_\alpha F^\alpha_i$, $K^h_{\cdot\, ij}{}^{\cdot}_{k}$ is an F-traceless tensor, which may be expressed from the formula above.

3.8.6 Traceless decompositon and recurrency

M. Crasmareanu has expressed the connection between traceless decompositions and recurency by the following way (see [360]):

A tensor field $A \in T^p_q(M)$ defined on a manifold M with affine connection ∇ is called *k-recurrent with respect to* ∇ if there exists a k-form ω, $1 \leq k \leq n$, such that

$$\nabla_{X_k} \cdots \nabla_{X_1} A = \omega(X_1, \ldots, X_k) \cdot A \tag{3.183}$$

for all vector fields $X_1, \ldots, X_k \in T^0_1(M)$ on M. In local coordinates, it may be written as follows:

$$A^{i_1 \cdots i_p}_{j_1 \cdots j_q, l_1 \cdots l_k} = \omega_{l_1 \cdots l_k} A^{i_1 \cdots i_p}_{j_1 \cdots j_q}, \tag{3.184}$$

where " , " denotes the covariant derivative with respect to ∇. A form ω is called the *coefficient of recurrency for A*.

The following theorem holds:

Theorem 3.31 *Let $\mathbb{A}_n$ be an n-dimensional manifold with affine connection, let $A \in T^1_q(M)$, where $q \leq n$. If A is a k-recurrent tensor field, the all tensors B from its traceless decomposition are also k-recurrent with the same form of recurrency.*

4 MAPPINGS AND TRANSFORMATIONS OF MANIFOLDS

4.1 Theory of Mappings

4.1.1 Introduction to mappings and transformation theory

By examining affine, geodesic, conformal and other types of mappings, some useful formalisms are introduced. They have been developed for the general case of diffeomorphisms of manifolds endowed with a linear connection. Let us mention the following most frequent conventions used in this text.

Let $f\colon M_n \to \overline{M}_n$ be a diffeomorphism (possibly a bijection of "sufficiently high" differentiability class).

4.1.2 Formalism of a "common coordinate system"

Let us restrict ourselves to a coordinate neighborhood U of some point $m \in M$. Then its image $f(U)$ is a neighborhood around $f(m)$. We will suppose that in our neighborhoods, a particular choice of local maps is made. Namely, if (U, φ), $\varphi = (x^1, \dots, x^n)$ is a map around $m \in U$ then around $f(m)$, we can choose just the map $(f(U), \varphi \circ f^{-1})$. In other words, for points which correspond to each other in the map, the preimage $m \in U$ and its image $f(m) \in f(U)$ have the same local coordinates. If this is the case we say that on our neighborhoods, the manifolds are related to a "common coordinate system" with respect to the mapping f, or that they are "correspondingly coordinatized". We may apply this view-point to a coordinate neighborhood of any point of M.

4.1.3 Formalism of a "common manifold"

Let us start with a diffeomorphism $f\colon M \to \overline{M}$ of manifolds endowed with affine connections ∇ and $\overline{\nabla}$, respectively. If $\{(U_\alpha, \varphi_\alpha), \alpha \in \Lambda\}$ is an atlas on M then due to the properties of the diffeomorphism f, $\{(f(U_\alpha), \varphi_\alpha \circ f^{-1}); \alpha \in \Lambda\}$ can serve as an atlas on $\overline{M}$. The above considerations allow us to suppose that the manifolds in fact coincide, $\overline{M} \equiv M$ (the topology as well as the differentiable structure are the same).

4.1.4 Deformation tensor of a mapping

Hence $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}, \overline{\nabla}) \equiv (M, \overline{\nabla})$, and we can suppose that both connections ∇ and $\overline{\nabla}$ determining the spaces are defined on the same underlying manifold. The technical advantage is that in further considerations,

we can employ a difference tensor of two connections on the same manifold,

i.e. the type $(1,2)$ tensor field

$$P = \overline{\nabla} - \nabla. \tag{4.1}$$

In this situation, P is sometimes called the *deformation tensor* of the connections $\overline{\nabla}$ and ∇ with respect to f.

Similarly, also other objects, such as e.g. metric, Riemannian or Ricci tensor etc., are supposed to live on M, and we distinguish by "bar" those which belong to the image space $\bar{\mathbb{A}}_n$. This approach has a global character.

So let us choose a local map (U,φ) around $x \in M$, $\varphi = (x^1, x^2, \dots, x^n)$. Since the components of connections are transformed according to (2.13) we get

$$\overline{\Gamma'}{}^{\gamma}_{\alpha\beta} - \Gamma'^{\gamma}_{\alpha\beta} = \left(\overline{\Gamma}^k_{ij} - \Gamma^k_{ij}\right) \frac{\partial x^i}{\partial x'^{\alpha}} \cdot \frac{\partial x^j}{\partial x'^{\beta}} \cdot \frac{\partial x'^{\gamma}}{\partial x^k}$$

which means that

$$P^k_{ij}(x) = \overline{\Gamma}^k_{ij}(x) - \Gamma^k_{ij}(x), \tag{4.2}$$

are components of a tensor field P of type $(1,2)$, see (4.1).

For $P = 0$, the mapping $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is connection-preserving; in this case, the mapping is also called *affine*. From the view-point of manifolds with affine connection, under affine mappings the underlying manifolds really coincide.

The formulae (4.1)[61] imply the following relations between the Riemannian and the Ricci tensors, respectively, of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \nabla_X P(Z,Y) - \nabla_Y P(Z,X) \\ &+ P(P(Z,Y),X) - P(P(Z,X),Y),\end{aligned} \tag{4.3}$$

and alternatively in a common coordinate system,

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^{\alpha}_{ik}P^h_{\alpha j} - P^{\alpha}_{ij}P^h_{\alpha k}, \tag{4.4}$$

where the comma "," is the covariant derivative corresponding to ∇. By contraction with respect to the indices h and k we obtain for the Ricci tensors

$$\begin{aligned}\overline{Ric}(X,Y) = Ric(X,Y) &- Tr(Z \to \nabla_X P(Z,Y) - \nabla_Y P(Z,X) \\ &- P(P(Z,Y),X) + P(P(Z,X),Y)),\end{aligned} \tag{4.5}$$

and alternatively in a common coordinate system

$$\overline{R}_{ij} = R_{ij} - P^{\beta}_{i\beta,j} + P^{\beta}_{ij,\beta} - P^{\alpha}_{i\beta}P^{\beta}_{\alpha j} + P^{\alpha}_{ij}P^{\beta}_{\alpha\beta}. \tag{4.6}$$

[61] In local coordinates this formula reads (4.2). Here and in what follows, we will write only one of the pair of equivalent formulas.

4.1.5 On equations of mappings onto Riemannian manifolds

Let a mapping $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ be characterized by the deformation tensor P. Then the formulas (4.1), and in local transcription (4.2), can be taken as fundamental equations of this mapping. Formally, they can be rewritten as

$$\overline{\nabla} = \nabla + P \tag{4.7}$$

or in local transcription,

$$\overline{\Gamma}^k_{ij}(x) = \Gamma^k_{ij}(x) + P^k_{ij}(x). \tag{4.8}$$

The following holds:

Theorem 4.1 *Let $\mathbb{A}_n$ be a torsion-free manifold with affine connection and $\bar{\mathbb{A}}_n$ be a Riemannian space $\overline{\mathbb{V}}_n$ with metric tensor $\overline{g}$. Then the equation (4.7) is equivalent to the equation*

$$\nabla_Z \overline{g}(X,Y) = \overline{g}(X, P(Y,Z)) + \overline{g}(Y, P(X,Z)) \tag{4.9}$$

or in local transcription, to

$$\overline{g}_{ij,k} = \overline{g}_{i\alpha} P^\alpha_{jk} + \overline{g}_{j\alpha} P^\alpha_{ik}, \tag{4.10}$$

where the comma " , " is a covariant derivative of ∇ in $\mathbb{A}_n$.

Proof. Obviously, in this case $\nabla(X,Y) = \nabla(Y,X) + [X,Y]$ and $\overline{\nabla}(X,Y) = \overline{\nabla}(Y,X) + [X,Y]$ hold. Consequently $P(X,Y) = P(Y,X)$. To verify the equivalence completely, it remains to prove that (4.8) implies (4.10). So let $\mathbb{A}_n$ and $\overline{\mathbb{V}}_n$ be given, let P be the deformation tensor, and let (4.8) be satisfied. Using (4.8) we get

$$\begin{aligned} \overline{g}_{ij,k} &= \partial_k \overline{g}_{ij} - \overline{g}_{i\alpha}\Gamma^\alpha_{jk} - \overline{g}_{\alpha j}\Gamma^\alpha_{ik} \\ &= \partial_k \overline{g}_{ij} - \overline{g}_{i\alpha}\overline{\Gamma}^\alpha_{jk} - \overline{g}_{\alpha j}\overline{\Gamma}^\alpha_{ik} + \overline{g}_{i\alpha}P^\alpha_{jk} + \overline{g}_{\alpha j}P^\alpha_{ik} \\ &= \overline{g}_{i\alpha}P^\alpha_{jk} + \overline{g}_{\alpha j}P^\alpha_{ik}. \end{aligned} \tag{4.11}$$

Let us introduce an auxiliary tensor field Q of type $(1,3)$,

$$Q(X,Y,Z) = \overline{g}(Z, \overline{\nabla}(X,Y) - \nabla(X,Y) - P(X,Y)), \tag{4.12}$$

with components $Q_{ijk} = \overline{g}_{k\alpha}(\overline{\Gamma}^\alpha_{ij} - \Gamma^\alpha_{ij} - P^\alpha_{ij})$. From the first two rows of (4.11) we get the equality $Q_{ikj} + Q_{jki} = 0$, eventually $Q(X,Y,Z) + Q(Z,Y,X) = 0$.

Since from (4.12) it follows $Q(X,Y,Z) = Q(X,Z,Y)$, we get from the last formula by elementary calculation[62)] that $Q(X,Y,Z) = 0$ holds for any $X,Y,Z \in \mathcal{X}(M)$. Hence the left hand side in (4.7) is zero, and $\overline{g}$ is regular, hence it follows by (4.12) that $\overline{\nabla}(X,Y) - \nabla(X,Y) - P(X,Y) = 0$ which finishes the proof. □

[62)] We used the fact that if a tensor Q_{hij} is symmetric in a pair of indices and skew-symmetric in another pair of indices then it is zero.

4.2 Transformation Lie Groups

4.2.1 Introduction

The concept of a group was founded by E. Galois during his work on algebraic problems. C. Jordan has found further applications of group theory. The theory of continuous groups was founded by S. Lie. Investigating a possibility to use extended methods of Galois for the solution of problems concerning integration of differential equations, Lie discovered a new type of groups he called himself *continuous* transformation groups (they are called *Lie groups* nowadays).

F. Klein pointed out the role of such groups in geometry: consider a set M consisting of arbitrary elements, and assume that G is a group consisting of transformations of M, i.e. bijections of M onto M (permutations of M). Elements of M are called *points*, the set M itself is a *space*. A family of points is a *configuration*, or a *figure*. Two configurations A and B are called *equivalent* when there is a mapping $g\colon M \to M$ such that $g(A) = B$. The properties which are invariant under the action of the group G, that is, invariant under all transformations $g \in G$, form the geometry of the group, according to Klein. These ideas are well-known as Klein's Erlangen program.

The theory of continuous transformation groups have been applied also to manifolds, and was developed by many famous mathematicians. Let us mention W. Killing, É. Cartan, Schur, H. Weyl, J.A. Schouten, P.L. Eisenhart, L.S. Pontryagin, N.G. Chebotarev, K. Yano, I.P. Egorov etc. [30, 33, 45–47, 52, 142, 160].

The theory of Lie groups has many applications in theoretical physics, mechanics, differential geometry etc.

4.2.2 Transformation Groups

Let M be a differentiable manifold of the class C^r. As well known, the family of transformations of M (i.e. bijections $f\colon M \to M$) endowed with map composition forms a group that is non-commutative in general.

Recall that a set G with binary operation $\circ\colon G \times G \to G$ is a group $(G, \circ)$ if the following conditions (axioms) hold true:

G1. $\forall a, b, c \in G$: $a \circ (b \circ c) = (a \circ b) \circ c$,
G2. $\exists e \in G$: $\forall a \in G$: $a \circ e = e \circ a = a$,
G3. $\forall a \in G$: $\exists a^{-1} \in G$: $a \circ a^{-1} = a^{-1} \circ a = e$.

A group is commutative (abelian) if it satisfies

G4. $\forall a \in G$: $a \circ b = b \circ a$.

The element e is the *unit*, and a^{-1} is the *inverse*.

For transformation groups, the unit e is always the identity map $\mathrm{Id}\colon M \to M$, $\mathrm{Id}(x) = x$ for all $x \in M$. Moreover, the following holds:

Theorem 4.2 *A set G of transformations of the set M is a group if and only if the following conditions are satisfied:*

$$\text{if } f_1, f_2 \in G \text{ then } f_1 \circ f_2 \in G \quad \text{and} \quad \text{if } f \in G \text{ then } f^{-1} \in G.$$

4.2.3 Continuous transformation groups. Lie groups.

We restrict ourselves to the formulation of those properties of Lie groups which are valid in some coordinate neighborhood, in the spirit of Eisenhart [52].

Let M be an n-dimensional elementary manifold and $(x) = (x^1, \dots, x^n)$ its coordinates. Similarly, let A be an r-dimensional elementary manifold and $(a) = (a^1, \dots, a^n)$ its coordinates. The formula $y = f(x; a)$, in a more detailed form,

$$y^h = f^h(x^1, \dots, x^n; a^1, \dots, a^r), \qquad h = 1, 2, \dots n, \tag{4.13}$$

defines, for any value $a \in A$, a transformation $M \to M$; a point $p(x) \in M$ is mapped onto a point $p'(y) \in M$. The functions f^h are supposed to be continuous, and usually sufficiently differentiable. Hence if $y = f(x; a)$ and a' is close to a then $f(x; a')$ is in a neighborhood of the point y. Another speaking, the transformation (4.13) is continuous.

Let us formulate the fundamental theorems on continuous transformation groups (Lie groups).

Theorem 4.3 (First Lie Theorem) *The functions (4.13), determining the r-parametric Lie group of M, are solutions of the totally integrable system of PDEs of Cauchy type*

$$\frac{\partial y^h}{\partial a^\sigma} = \xi^h_\mu(y)\, A^\mu_\sigma(a), \qquad \begin{matrix} h = 1, 2, \dots, n; \\ \sigma, \mu = 1, 2, \dots, r, \end{matrix} \tag{4.14}$$

where the coefficients $\xi^h_\mu(y)$ are linearly independent, with constant coefficients, and

$$\det \|A^\mu_\sigma(a) \neq 0\|. \tag{4.15}$$

The functions $\xi^h_\mu(x)$ determine the *infinitesimal operator of the Lie transformation group*:

$$X_\sigma = \xi^\alpha_\sigma(x)\ \partial_\alpha\ . \tag{4.16}$$

The Poisson bracket of X_σ and X_μ is defined by

$$[X_\sigma, X_\mu] = (\xi^\beta_\sigma\, \partial_\beta \xi^\alpha_\mu - \xi^\beta_\mu\, \partial_\beta \xi^\alpha_\sigma)\ \partial_\alpha \tag{4.17}$$
$$\alpha, \beta = 1, 2, \dots, n; \quad \sigma, \mu = 1, 2, \dots, r.$$

Theorem 4.4 (Second Lie Theorem) *For the fundamental operators the relations*

$$[X_\sigma, X_\mu] = C^\nu_{\sigma\mu}\, X_\nu \tag{4.18}$$

hold, where $C^\nu_{\sigma\mu}$ are constants.

Theorem 4.5 (Third Lie Theorem) *The structure constants $C^\nu_{\sigma\mu}$ are antisymmetric and satisfy the Jacobi identity, i.e.*

$$C^\nu_{\sigma\mu} = -C^\nu_{\mu\sigma}; \tag{4.19}$$

$$C^\nu_{\lambda\mu} C^\sigma_{\nu\tau} + C^\nu_{\mu\tau} C^\sigma_{\nu\lambda} + C^\nu_{\tau\lambda} C^\sigma_{\nu\mu} = 0. \tag{4.20}$$

4.2.4 One-parameter groups of continuous transformations

Let us give one-parameter groups of continuous transformations $f_t\colon M\times\mathbb{R}\to M$, i.e. $f_t(x)=f(x;t)$, and in coordinates $y^h=f^h(x^1,\dots,x^n;t)$. The system of equations (4.14) has the form $\frac{dy^h}{dt}=\xi^h(y)A(t)$.

After change of the parameter $t=t(\tau)$ we can get this one-parameter group $f_\tau:=M\times R\to M$ in the form

$$y^h=f^h(x^1,x^2,\dots,x^n,\tau) \tag{4.21}$$

provided $Id(x)=f(x,0)$, and the system is

$$\frac{dy^h}{d\tau}=\xi^h(y^1,\dots,y^n,\tau). \tag{4.22}$$

The parameter τ is said to be natural.

Recall that a local one-parameter group of continuous transformations defined on $U\times I_\varepsilon$, where $I_\varepsilon=(-\varepsilon,\varepsilon)$, $\varepsilon>0$, and U is a open subset of M, is a continuous map $(x,\tau)\in U\times I_\varepsilon\to f_\tau(x)\in f_\tau(U)$ such that $f_\tau: x\to f_\tau(x)$ is a diffeomorphism of U onto an open set $f_\tau(U)$ of M and

$$f_{\tau_1+\tau_2}(x)=f_{\tau_1}\circ f_{\tau_2}(x)$$

whenever $\tau_1,\tau_2,\tau_1+\tau_2\in I_\varepsilon$ and $f_{\tau_2}(x)\in U$.

The solution of the equations (4.22) can be written in some domain in the form

$$y^h(\tau)=x^h+\xi^h(x)\tau+\xi^\alpha(x)\frac{\partial\xi^h(x)}{\partial x^\alpha}\frac{\tau^2}{2!}+\ \cdots\ . \tag{4.23}$$

From this formula we can deduce the concept of an infinitesimal transformation of M:

$$y^h=x^h+\xi^h(x)\,\delta\tau, \tag{4.24}$$

where $\delta\tau$ is an infinitesimal parameter. Under infinitesimal transformations, the coordinates of x are changed infinitesimally

$$\delta x^h=\xi^h(x)\,\delta\tau. \tag{4.25}$$

The above equations are uniquely determined by the operator

$$X=\xi^\alpha(x)\,\partial_\alpha \tag{4.26}$$

which Lie himself called a *symbol of infinitesimal group transformation.*

The operator X is also called a *generator* or *operator* of the one-parameter Lie transformation group. A one-parameter Lie group is fully determined by the operator X.

The solution (4.23) can be written in the form

$$y^h(\tau)=\Big(1+\tau\xi^\alpha\partial_\alpha+\frac{\tau}{2!}(\xi^\alpha\partial_\alpha)^2+\ \cdots\ \Big)\,x^h,$$

or in the symbolic shape, well-known form the theory of differential equations, as

$$y^h(\tau) = \mathrm{e}^{\tau X} x^h.$$

The solution generates a transformation $f_\tau : (x^h) \to (y^h(\tau))$. From this it follows

$$f_{\tau_1} \circ f_{\tau_2} = f_{\tau_1+\tau_2} \quad \text{and} \quad f_\tau^{-1} = f_{-\tau}.$$

Let X_σ, $\sigma = 1, 2, \ldots, r$, be r linearly independent infinitesimal transformations that satisfy (4.18) and $C^\sigma_{\mu\nu}$ satisfy (4.19) and (4.20). Then these operators form a group G_r consisting of all one-parameter groups, with generators

$$a^\sigma X_\sigma, \tag{4.27}$$

where a_σ are constants.

On the other hand, each group G_r can be generated by r linearly independent operators X_σ which satisfy (4.18), (4.19) and (4.20).

Recall that on M, we can choose a coordinate system such that a vector field $\xi^h(x)$ has the following coordinate form:

$$\xi^h(x) = \partial_1^h. \tag{4.28}$$

In this case, the equations (4.22) can be integrated:

$$y^h = x^h + \tau\, \delta_1^h \tag{4.29}$$

4.2.5 Lie derivative

An important tool for the investigation of Lie groups is the so-called Lie derivative which in general can be introduced for geometric objects introduced in Section 1.3.3, p. 80. It means that the Lie derivative is introduced not only for tensors, but also for affine connections. Let us restrict ourselves to some well known facts, e.g. [8, 56] etc. Let us given a one-parameter group of continuous transformations f_τ on M with natural parameter τ. Further, let $\mathcal{A}$ be a field of a geometric object on M.

Under a *Lie derivative of the geometric object $\mathcal{A}$ with respect to an operator* $X(\equiv \xi^\alpha \partial_\alpha)$ we mean a geometric object

$$(\mathcal{L}_X \mathcal{A})_x = (\mathcal{L}_\xi \mathcal{A})_x = \lim_{\tau\to 0} \frac{1}{\tau}(\mathcal{A}_x - \tilde{\mathcal{A}}_x(\tau)) = \lim_{\tau\to 0} \frac{1}{\tau}(\mathcal{A}_x - (\tilde{f}_\tau \mathcal{A})_x). \tag{4.30}$$

Here $\tilde{\mathcal{A}}_x(\tau) = (\tilde{f}_\tau \mathcal{A})_x$ is the value of a geometrical object $\mathcal{A}$ which is called a *pull-back* of the geometrical object $\mathcal{A}$. This value is evaluated in the point $f_\tau^{-1}(x)$, i.e. $(\tilde{f}_\tau \mathcal{A})_x = \tilde{f}(\mathcal{A}_{\varphi_\tau^{-1}(x)})$. The Lie derivative has the following properties

$$\begin{aligned} \mathcal{L}_X(\mathcal{A} \pm \mathcal{B}) &= \mathcal{L}_X \mathcal{A} \pm \mathcal{B}, \\ \mathcal{L}_X(\mathcal{A} \cdot \mathcal{B}) &= \mathcal{A} \cdot \mathcal{L}_X \mathcal{B} + \mathcal{L}_X \mathcal{A} \cdot \mathcal{B}. \end{aligned}$$

For a function $f(x)$ on M,

$$\mathcal{L}_X f(x) = Xf(x).$$

For vector fields X and Y, the Lie derivative satisfies

$$\mathcal{L}_X Y = [X, Y],$$

$$\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y] \equiv \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X.$$

For a tensor field T of type (p, q), the Lie derivative $\mathcal{L}_X T$ is a vector field of the same type; in components,

$$\mathcal{L}_X T^{h_1 \dots h_p}_{i_1 \dots i_q} = \xi^\alpha \partial_\alpha T^{h_1 \dots h_p}_{i_1 \dots i_q} + \sum_{s=1}^{q} T^{h_1 \dots h_p}_{i_1 \dots \alpha \dots i_q} \partial_{i_s} \xi^\alpha - \sum_{s=1}^{p} T^{h_1 \dots \alpha \dots h_p}_{i_1 \dots i_q} \partial_\alpha \xi^{h_s}. \quad (4.31)$$

Among others, it follows $\mathcal{L}_X \delta^h_i = 0$.

If M is equipped with an affine connection ∇ then the equations (4.31) can be written in the more "invariant" form

$$\mathcal{L}_X T^{h_1 \dots h_p}_{i_1 \dots i_q} = \xi^\alpha \, T^{h_1 \dots h_p}_{i_1 \dots i_q, \alpha} + \sum_{s=1}^{q} T^{h_1 \dots h_p}_{i_1 \dots \alpha \dots i_q} \, \xi^\alpha_{, i_s} - \sum_{s=1}^{p} T^{h_1 \dots \alpha \dots h_p}_{i_1 \dots i_q} \, \xi^{h_s}_{, \alpha} \quad (4.32)$$

where "," denotes the covariant derivative on ∇.

E.g. for the metric tensor g, the following holds:

$$\mathcal{L}_\xi g_{ij} = \xi^\alpha \partial_\alpha g_{ij} + g_{\alpha j} \partial_i \xi^\alpha + g_{i\alpha} \partial_j \xi^\alpha = \xi_{i,j} + \xi_{j,i}. \quad (4.33)$$

The Lie derivative of an affine connection ∇ with components Γ^h_{ij} is a type $(1, 2)$ tensor field

$$\mathcal{L}_X \Gamma^h_{ij} = \xi^h_{, ij} - R^h_{ij\alpha} \xi^\alpha, \quad (4.34)$$

where R^h_{ijk} are components of the curvature tensor.

In a special coordinate system in which the operator X is of the form

$$X = \partial_1 , \quad \text{i.e.} \quad \xi^h(x) = \delta^h_1, \quad (4.35)$$

for many objects $\mathcal{A}$ (such as tensors, affine connections, projective connections) the Lie derivative has the following simple form:

$$\mathcal{L}_X \mathcal{A}(x) = \partial_1 \mathcal{A}(x). \quad (4.36)$$

In other words, if $\mathcal{L}_X \mathcal{A} = 0$ then the components of the geometric object $\mathcal{A}$ are constant along the x^1-coordinate curves.

4.3 Affine mappings and transformations

Let us demonstrate the concepts of the just introduced mappings and transformations in special cases.

4.3.1 Affine mappings of manifolds with affine connection

Let $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}, \overline{\nabla})$ be two manifolds with affine connection ∇ and $\overline{\nabla}$, respectively.

Definition 4.1 A diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is called an *affine mapping* if it maps any parallel vector field along $\ell \subset M$ into a parallel vector field along $f(\ell) \subset \overline{M}$.

Two manifolds $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ are *affine* (*locally affine*, respectively) if there exists (there exists locally) an affine mapping between them.

The following holds:

Theorem 4.6 *A diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is an affine map if and only if $f_*(\overline{\nabla}) = \nabla$, i.e. locally in a common coordinate system, the components of the affine connections (∇ and $\overline{\nabla}$) are related by*

$$\overline{\Gamma}{}^h_{ij}(x) = \Gamma^h_{ij}(x). \tag{4.37}$$

That is, the deformation tensor P, see (4.1), is zero.

It follows that if there is an affine mapping between $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ then the manifolds are "the same" from the view-point of affine geometry.

Proof. Let there be an affine mapping $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$. Let us choose a coordinate neighborhood U with a common coordinate system (x^i) under f. Let $\lambda = \lambda^h(t)$ be a parallel vector field along a curve $\ell\colon\ x = x^h(t)$, $t \in I$, that is, $\nabla_t \lambda = 0$ holds, which in local coordinates reads

$$\frac{\lambda^h(t)}{dt} + \Gamma^h_{ij}(x(t))\,\lambda^j(t)\,\frac{dx^i(t)}{dt} = 0. \tag{4.38}$$

According to the assumptions, the vector field $\lambda(t)$ is parallel also in $\bar{\mathbb{A}}_n$, that is, $\overline{\nabla}_t \lambda = 0$, i.e.

$$\frac{\lambda^h(t)}{dt} + \overline{\Gamma}{}^h_{ij}(x(t))\,\lambda^j(t)\,\frac{dx^i(t)}{dt} = 0. \tag{4.39}$$

Subtracting (4.38) from (4.39) we obtain

$$\left(\overline{\Gamma}{}^h_{ij}(x(t)) - \Gamma^h_{ij}(x(t))\right)\,\lambda^j(t)\,\frac{dx^i(t)}{dt} = 0. \tag{4.40}$$

In the bracket, we have in fact components of the deformation tensor $P = \tilde{\nabla} - \nabla$, see (4.2).

The formula (4.40) is satisfied for an arbitrary curve ℓ and any parallel vector field λ. That is why the bracket in (4.40) must be zero, hence the formula (4.37) holds. On the other hand, if (4.37) is satisfied then from (4.38) one obtains (4.39). □

Let us now show how to determine whether the given $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ are locally affine or not. Let us consider manifolds $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}, \overline{\nabla})$ with affine connections and an affine mapping $f\colon M \to \overline{M}$. Let $\overline{x} = f(x)$ and let (U, φ), $(\overline{U}, \overline{\varphi})$ be charts containing x or $\overline{x}$, respectively. The diffeomorphism is expressed by the formulas

$$\overline{x}^h = \overline{x}^h(x^1, \dots, x^n) \equiv \overline{x}^h(x). \tag{4.41}$$

From (4.37) and the "transformation formulas" for affine connection (2.13) it follows

$$\frac{\partial^2 \overline{x}^h(x)}{\partial x^i \partial x^j} = \Gamma_{ij}^{\alpha}(x) \frac{\partial \overline{x}^h(x)}{\partial x^\alpha} - \overline{\Gamma}_{\alpha\beta}^{h}(\overline{x}(x)) \frac{\partial \overline{x}^\alpha(x)}{\partial x^i} \frac{\partial \overline{x}^\beta(x)}{\partial x^j}. \tag{4.42}$$

If we put $P_i^h(x) \equiv \dfrac{\partial \overline{x}^h(x)}{\partial x^i}$ then we can write the equations (4.42) as

$$\begin{aligned} \frac{\partial \overline{x}^h(x)}{\partial x^i} &= P_i^h(x), \\ \frac{\partial P_i^h(x)}{\partial x^j} &= \Gamma_{ij}^{\alpha}(x) P_\alpha^h(x) - \overline{\Gamma}_{\alpha\beta}^{h}(\overline{x}(x)) P_i^\alpha(x) P_j^\beta(x). \end{aligned} \tag{4.43}$$

The system of PDE's (4.43) forms a system of PDE's of Cauchy type, described in section 1.5.2, in $n^2 + n$ unknown functions $\overline{x}^h(x)$ and $P_i^h(x)$.

The integrability conditions for the first equation from (4.43) are satisfied according to the second equations (4.43). The integrability conditions for the second equation from (4.43) read

$$P_\alpha^h R_{ijk}^{\alpha}(x) = \overline{R}_{\alpha\beta\gamma}^{h}(\overline{x}(x)) P_i^\alpha P_j^\beta P_k^\gamma, \tag{4.44}$$

where R and $\overline{R}$ are the curvature tensors of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

By subsequent covariant derivation we find differential prolongations

$$\begin{aligned} P_\alpha^h R_{ijk,l}^{\alpha} &= \overline{R}_{\alpha\beta\gamma|\delta}^{h} P_i^\alpha P_j^\beta P_k^\gamma P_l^\delta, \\ &\dots \\ P_\alpha^h R_{ijk,l_1 \cdots l_m}^{\alpha} &= \overline{R}_{\alpha\beta\gamma|\delta_1 \cdots \delta_m}^{h} P_i^\alpha P_j^\beta P_k^\gamma P_{l_1}^{\delta_1} \cdots P_{l_m}^{\delta_m} \end{aligned} \tag{4.45}$$

where "," and "|" denote the covariant derivative on $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

The conditions (4.44) and (4.45) are algebraic with respect to their unknowns, but not linear.

Remark 4.1 Let us consider a common coordinate system with respect to an affine mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$, that is, let $f\colon \overline{x}^h = x^h$. Then $P_i^h(x) = \delta_i^h$ holds. As a consequence, the necessary and sufficient condition (4.42) takes the form (4.37). The conditions (4.44) read:

$$R_{ijk}^{h}(x) = \overline{R}_{ijk}^{h}(x). \tag{4.46}$$

Therefore also the differential prolongations of (4.45) take the form of an equality of covariant derivatives of the Riemannian tensor.

4.3.2 Affine mappings onto Riemannian manifolds

Let a manifold $\mathbb{A}_n = (M, \nabla)$, equipped with affine connection ∇, admit a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n = (M, \overline{g})$, where $\overline{g}$ is the metric tensor of $\overline{\mathbb{V}}_n$. Since under an affine mapping $f\colon \mathbb{A}_n \to \overline{\mathbb{V}}_n$, the deformation tensor is zero, we obtain as a consequence of Theorem 4.1:

Theorem 4.7 *A diffeomorphism $f\colon \mathbb{A}_n \to \overline{\mathbb{V}}_n$ is an affine mapping if and only if the metric tensor $\overline{g}$ of $\overline{\mathbb{V}}_n$ is totally parallel in $\mathbb{A}_n$, i.e.*

$$\nabla \overline{g} = 0. \tag{4.47}$$

The problem of the existence of affine mappings onto Riemannian manifolds is equivalent with the problem of metrizability of a manifold endowed with an affine connection.

As already stated, the equations $\nabla \overline{g} = 0$ can be locally written as

$$\overline{g}_{ij,k} = 0, \tag{4.48}$$

which form a linear system of PDEs of Cauchy type in $\frac{1}{2}\, n(n+1)$ unknown functions $\overline{g}_{ij}(x)$.

For given initial values $\overline{g}_{ij}(x_o) = \overset{o}{\overline{g}}_{ij}$, this sytem has at most one solution. The number of parameters on which the general solution depends will be denoted by

$$r_{af} \leq \frac{1}{2}\, n(n+1). \tag{4.49}$$

The maximal value $r_{af} = \frac{1}{2} n(n+1)$ is reached in (pseudo-) Euclidean manifolds. It follows from the fact that in these spaces, and only in these spaces, the integrability conditions of (4.48)

$$\overline{g}_{h\alpha} R^{\alpha}_{ijk} + \overline{g}_{i\alpha} R^{\alpha}_{hjk} = 0 \tag{4.50}$$

are satisfied indentically. Here R^h_{ijk} are components of the curvature tensor.

4.3.3 Product manifolds and affine mappings

The problem of solvability of equations (4.48) on Riemannian manifold $\mathbb{V}_n$ and manifolds $\mathbb{A}_n$ with affine connection have been studied in various aspects. In this paragraph, let us mention breafly those results that are related to product spaces.

Definition 4.2 A Riemannian manifold $\mathbb{V}_n$ is called a *product manifold* of Riemannian manifolds $\overset{1}{\mathbb{V}}_{n_1}, \overset{2}{\mathbb{V}}_{n_2}, \ldots, \overset{m}{\mathbb{V}}_{n_m}$ $(n_1 + n_2 + \cdots + n_m = n)$:

$$\mathbb{V}_n = \overset{1}{\mathbb{V}}_{n_1} \otimes \overset{2}{\mathbb{V}}_{n_2} \otimes \cdots \otimes \overset{m}{\mathbb{V}}_{n_m} \tag{4.51}$$

if the metrics are related by

$$g = g_1 \otimes g_2 \otimes \cdots \otimes g_m. \tag{4.52}$$

Locally this means that there exists a coordinate system (x^i) such that the metric forms of these Riemannian manifolds satisfy

$$ds^2 = ds_1^2 + ds_2^2 + \cdots + ds_m^2 \tag{4.53}$$

where

$$ds^2 = g_{ij}(x^k)\,dx^i dx^j \quad \text{and} \quad ds_\sigma^2 = \overset{\sigma}{g}{}_{i_\sigma j_\sigma}(x^{k_\sigma})\,dx^{i_\sigma}dx^{j_\sigma} \tag{4.54}$$

$$i,j,k = \langle 1,n\rangle; \quad i_\sigma, j_\sigma, k_\sigma = \langle p_\sigma, r_\sigma\rangle;$$

$$1 = p_1 \le r_1 < p_2 \le r_2 < \cdots < p_m \le r_m = n.$$

A symmetric tensor a which has the form

$$a = (\alpha_1\, g_1) \otimes (\alpha_2\, g_2) \otimes \cdots \otimes (\alpha_m\, g_m) \tag{4.55}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are constants, is a solution of the following equation

$$\nabla a = 0. \tag{4.56}$$

Therefore on a product manifod $\mathbb{V}_n$, solutions

$$\overline{g} = (\alpha_1\, g_1) \otimes (\alpha_2\, g_2) \otimes \cdots \otimes (\alpha_m\, g_m), \quad \alpha_\sigma \neq 0, \tag{4.57}$$

generate affine mappings $f\colon \mathbb{V}_n(M,g) \to \overline{\mathbb{V}}_n(M,\overline{g})$.

Let us recall that $\mathbb{V}_n$ is called *irreducible*, if no decomposition (4.51) of the metric is possible.

For proper Riemannian spaces $\mathbb{V}_n$, if the decomposition (4.51) is maximal, i.e. all $\overset{\sigma}{\mathbb{V}}_{n_\sigma}$ are irreducible, general solutions of the equation (4.56) have the form (4.55).

In pseudo-Riemannian manifolds $\mathbb{V}_n$, metric forms for which there are solutions of the equation (4.56) are very complicated. The general solution of this problem appears in the paper by V.N. Abdulin [244].

4.3.4 Affine motions

An affine mapping of $\mathbb{A}_n = (M, \nabla)$ onto itself is called an *affine transformation* of $\mathbb{A}_n$. The set of all affine transformations of $\mathbb{A}_n$ form an affine group on $\mathbb{A}_n$.

A vector field X on $\mathbb{A}_n$ is called an *infinitesimal affine* or *affine (Killing) vector field*, also an *affine motion*, if this field generates a local one-parameter group f_t in a neighborhood U of any point $p \in M$, which preserves the affine connection, i.e. the mapping $f_t : M \to M$ is an affine transformation.

Obviously, in a special coordinate system (x^i) in which $X = \partial_1$, an affine transformation is characterized by

$$\partial_1 \Gamma_{ij}^h(x) = 0. \tag{4.58}$$

By (4.34) and (4.36), these conditions can be written as

$$\mathcal{L}_X \Gamma_{ij}^h \equiv \xi_{,ij}^h - R_{ij\alpha}^h \xi^\alpha = 0. \tag{4.59}$$

The solutions of this equation form a Lie group. It follows from the fact that if $\mathcal{L}_X\Gamma = 0$ and $\mathcal{L}_Y\Gamma = 0$ then we can easily prove that

$$\mathcal{L}_{[X,Y]}\Gamma = \mathcal{L}_X\mathcal{L}_Y\Gamma - \mathcal{L}_Y\mathcal{L}_X\Gamma = 0.$$

Denoting $\xi^h_i \equiv \xi^h_{,i}$ the conditions from (4.59) can be written in a form of a system of equations of Cauchy type in covariant derivatives with respect to $n + n^2$ unknowns ξ^h and ξ^h_i:

$$\begin{aligned} \xi^h_{,i} &= \xi^h_i, \\ \xi^h_{i,j} &= R^h_{ij\alpha}\xi^\alpha. \end{aligned} \tag{4.60}$$

Obviously, the general solution of (4.60) depends on $r_{af} \leq n+n^2$ real parameters. The number r_{af} is the degree of the affine transformation.

The integrability conditions of (4.60) have the form $\mathcal{L}_\xi R^h_{ijk} = 0$, i.e.

$$\xi^\alpha R^h_{ijk,\alpha} + \xi^h_{,\alpha}R^\alpha_{ijk} - \xi^\alpha_{,i}R^h_{\alpha jk} - \xi^\alpha_{,j}R^h_{i\alpha k} - \xi^\alpha_{,k}R^h_{ij\alpha} = 0. \tag{4.61}$$

Analysing these conditions we check easily that the system (4.60) has a solution for any initial values in the case that $\mathbb{A}_n$ is flat, i.e. $R \equiv 0$. In the affine coordinates (x) in which $\Gamma^h_{ij}(x) \equiv 0$, the equations (4.59) have the solution

$$\xi^h = a^h + b^h_i x^i,$$

where a^h and b^h_i are constants. In these coordinates, the affine transformation takes the classical form

$$y^h = A^h_i x^i + B^h$$

where A^h_i, B^h are constants and $\det(A^h_i) \neq 0$.

A detailed analysis of the equations for affine motion is included in the works of I.P. Egorov [45–47] who proved, among others, that a non-flat manifold $\mathbb{A}_n$ admits a group of affine motions of order

$$r_{af} \leq n(n-2) + 5. \tag{4.62}$$

4.4 Isometric mappings and transformations

Isometric mappings were defined in Section 2.1.3, p. 69, as the mappings that preserve the lenght of all arcs. Further we show their following properties.

4.4.1 Fundamental equations of isometric mappings

Let us consider Riemannian manifolds $\mathbb{V}_n = (M, g)$, $\overline{\mathbb{V}}_n = (\overline{M}, \overline{g})$ and an isometric diffeomorphism $f\colon M \to \overline{M}$.

Let $\overline{x} = f(x)$ and let (U, φ), $(\overline{U}, \overline{\varphi})$ be charts containing x or $\overline{x}$, respectively. The diffeomorphism is expressed by the formulae

$$\overline{x}^h = \overline{x}^h(x^1, \ldots, x^n) \equiv \overline{x}^h(x). \tag{4.63}$$

As already mentioned (Definition 2.2, p. 69), the diffeomorphism f is an isometry if the lengths of all arcs are preserved under f. The equations (3.5) read

$$g_{ij}(x) = \overline{g}_{\alpha\beta}(\overline{x}(x))\frac{\partial \overline{x}^{\alpha}(x)}{\partial x^{i}}\frac{\partial \overline{x}^{\beta}(x)}{\partial x^{j}}, \tag{4.64}$$

where g_{ij} and $\overline{g}_{\alpha\beta}$ are components of the metric tensors g and $\overline{g}$, respectively. The equations (4.64) form a system of PDE's in the unknown functions $\overline{x}^{h}(x)$.

From these equations it follows

$$\frac{\partial^{2}\overline{x}^{h}(x)}{\partial x^{i}\partial x^{j}} = \Gamma_{ij}^{\alpha}(x)\frac{\partial \overline{x}^{h}(x)}{\partial x^{\alpha}} - \overline{\Gamma}_{\alpha\beta}^{h}(\overline{x}(x))\frac{\partial \overline{x}^{\alpha}(x)}{\partial x^{i}}\frac{\partial \overline{x}^{\beta}(x)}{\partial x^{j}}, \tag{4.65}$$

in fact, the "transformation formulae" for Christoffels: (2.13).

If we put $P_{i}^{h}(x) \equiv \dfrac{\partial \overline{x}^{h}(x)}{\partial x^{i}}$ then we can write the equation (4.65) as

$$\begin{aligned} \frac{\partial \overline{x}^{h}(x)}{\partial x^{i}} &= P_{i}^{h}(x), \\ \frac{\partial P_{i}^{h}(x)}{\partial x^{j}} &= \Gamma_{ij}^{\alpha}(x)P_{\alpha}^{h}(x) - \overline{\Gamma}_{\alpha\beta}^{h}(\overline{x}(x))P_{i}^{\alpha}(x)P_{j}^{\beta}(x). \end{aligned} \tag{4.66}$$

The conditions (4.64) read

$$g_{ij}(x) = \overline{g}_{\alpha\beta}(\overline{x}(x))P_{i}^{\alpha}(x)P_{j}^{\beta}(x). \tag{4.67}$$

The system of PDE's (4.66) together with the algebraic conditions (4.67) form a mixed systems of PDE's of Cauchy type, described in Section 1.5.2, in the unknown functions $\overline{x}^{h}(x)$ and $P_{i}^{h}(x)$. The integrability conditions for the first equations (4.66) are satisfied according to the second equations (4.66).

The integrability conditions for the second equations (4.66) read

$$P_{\alpha}^{h}R_{ijk}^{\alpha}(x) = \overline{R}_{\alpha\beta\gamma}^{h}(\overline{x}(x))P_{i}^{\alpha}P_{j}^{\beta}P_{k}^{\gamma}, \tag{4.68}$$

where R and $\overline{R}$ are the Riemannian tensors of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$, respectively. By subsequent covariant derivation we find the differential prolongations (4.39). The conditions (4.68) and (4.39) are algebraic with respect to their unknowns, but not linear.

Remark 4.2 Let us consider a common coordinate system with respect to an isometric mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$, that is, let $f\colon \overline{x}^{h} = x^{h}$. Then $P_{i}^{h}(x) = \delta_{i}^{h}$ hold. As a consequence, the necessary and sufficient condition (4.64) takes the form

$$\overline{g}_{ij}(x) = g_{ij}(x). \tag{4.69}$$

The conditions (4.65) and (4.68) read:

$$\overline{\Gamma}_{ij}^{h}(x) = \Gamma_{ij}^{h}(x) \quad \text{and} \quad \overline{R}_{ijk}^{h}(x) = R_{ijk}^{h}(x). \tag{4.70}$$

Therefore also the differential prolongations of (4.45) take the form of an equality of covariant derivatives of the Riemannian tensor.

4.4.2 On local isometry of spaces of constant curvature

By means of the above theory we can prove the following well-known property of spaces of constant curvature.

Theorem 4.8 *Two spaces with constant curvature are locally isometric if and only if they have the same curvature and at the same time, their metrics are of the same signature.*

Proof. We must find an isometric mapping (4.63) between two spaces of constant curvature K and $\overline{K}$, respectively, that have Riemannian tensors of the form

$$R^h_{ijk} = K(\delta^h_j g_{ik} - \delta^h_k g_{ij}) \text{ and } \overline{R}^h_{ijk} = \overline{K}(\delta^h_j \overline{g}_{ik} - \delta^h_k \overline{g}_{ij}). \tag{4.71}$$

We will seek for solutions of the system (4.66) and (4.67). The integrability conditions (4.68), after substituting (4.71) and accounting (4.67), take the form:

$$(K - \overline{K})(P^h_j g_{ik} - P^h_k g_{ij}) = 0.$$

Contracting with g^{ik} we find $(K - \overline{K})P^h_j = 0$. It follows immediately that in the case $\overline{K} \neq K$, the spaces cannot be isometric, neither locally nor globally. If $\overline{K} = K$ the corresponding integrability conditions hold identically. The solution of (4.66) must satisfy (4.67). Their differential prolongations are identically satisfied. The initial conditions $P^h_i(x_0)$ satisfying $|P^h_i(x_0)| \neq 0$ and (4.67), can be found only in case that the matrices $\|g_{ij}\|$ and $\|\overline{g}_{ij}\|$ have the same signature. For these initial conditions $P^h_i(x_0)$ and $\overline{x}^h(x_0)$, in some point x_0, there is always a solution $\overline{x}^h(x)$, and this solution realizes the isometry. □

4.4.3 On local isometry of spaces of constant holomorphic curvature

A similar property to those being formulated for spaces of constant curvature in the Theorem 5.1 holds for Kähler spaces $\mathbb{K}_n$ of constant holomorphical curvature.

Theorem 4.9 *Two Kähler spaces of constant holomorphical curvature are locally isometric if and only if they have the same holomorphical curvature and at the same time their metrics are of the same signature and their structure is of the same type, i.e. either both are elliptic or both are hyperbolic.*

Proof. We must find an isometric mapping (4.63) between two spaces of constant holomorphical curvature k and $\overline{k}$, respectively, that have Riemannian tensors of the form

$$R^h_{ijk} = \frac{k}{4}\, G^h_{ikj} \text{ and } \overline{R}^h_{ijk} = \frac{\overline{k}}{4}\, \overline{G}^h_{ikj} \tag{4.72}$$

where $G^h_{ikj} = g^{h\alpha} G_{\alpha ijk}$ and G_{hijk} are components of the tensor from (3.99), p. 138.

We shall seek solutions of the system (4.66) and (4.67). From the integrability conditions (4.68) we obtain for the Ricci tensor

$$R_{ij}(x) = \overline{R}_{\alpha\beta}(\overline{x}(x)) P^\alpha_i P^\beta_j.$$

Since the space of constant holomorphical curvature is an Einstein space, it follows from the previous formula that an isometry can exist only in the case $k = \overline{k}$.

In the case $k = \overline{k} = 0$, the assertion is trivial. So in what follows, let us suppose $k = \overline{k} \neq 0$. If this is the case, accounting (4.72) the integrability conditions take the form

$$\mathcal{F}^h_j g_{i\alpha} \mathcal{F}^\alpha_k - \mathcal{F}^h_k g_{i\alpha} \mathcal{F}^\alpha_j - 2\mathcal{F}^h_i g_{j\alpha} \mathcal{F}^\alpha_k = 0$$

where $\mathcal{F} = F^\alpha_j P^h_\alpha - \overline{F}^h_\alpha P^\alpha_j$. It follows that $\mathcal{F} = 0$, i.e.

$$F^\alpha_j P^h_\alpha = \overline{F}^h_\alpha P^\alpha_j. \tag{4.73}$$

Moreover, the metrics g and $\overline{g}$ must satisfy the condition (4.68). The differential prolongations of (4.68) and (4.73) are automatically satisfied since the metrics and structures are covariantly constant.

From the formulas (4.68) and (4.73) in a fixed point x_0, we are able to find initial values $P^h_i(x_0)$ such that $\det(P^h_i(x_0)) \neq 0$ if and only if the structures F and $\overline{F}$ are of the same type, i.e. both are either elliptic or hyperbolic, and the metrics are of the same signature. Then for such initial values $P^h_i(x_0)$, together with initial values $\overline{x}^h(x_0)$ in some point x_0, there exists always a solution $\overline{x}^h(x)$, and this solution realizes the isometry. □

4.4.4 Groups of motions

Isometric mappings of a Riemannian space $\mathbb{V}_n = (M, g)$ onto itself are called *isometric transformations* on $\mathbb{V}_n$ or *motions* of $\mathbb{V}_n$.

A vector field X on $\mathbb{V}_n$ is called an *infinitesimal isometry* or a *Killing vector field* if for each point $p \in M$ there is a neighborhood U of p such that the local one-parameter group f_t determined by the vector field preserves the metric, that is, the mapping $f_t \colon M \to M$ is an isometric transformation. In a special coordinate system (x^i) in which $X = \partial_1$, the isometric transformation is characterized by

$$\partial_1 g_{ij}(x) = 0. \tag{4.74}$$

By (4.33) and (4.36) these conditions take the form

$$\mathcal{L}_X g_{ij} \equiv \xi_{i,j} + \xi_{j,i} = 0 \tag{4.75}$$

where $\xi_i = g_{i\alpha}\xi^\alpha$, $X = \xi^\alpha \partial_\alpha$. Let us recall that ξ^h is called a Killing vector.

We can easily check that the set of all Killing vectors depends on

$$r_{mot} \leq \frac{1}{2}\,(n+1)(n+2) \tag{4.76}$$

parameters and isometric transformations form a Lie group of order r_{mot}. The number r_{mot} is also called the *degree of mobility* of $\mathbb{V}_n$.

The group of motions is a subgroup of the affine group. From the equations (4.75), the equations (4.59) follow as a consequence. Hence we can write the

equations of motion in the form of a mixed system of equations of Cauchy type in the unknowns $\xi_i(x)$ and $\xi_{ij}(x)$:

$$\begin{aligned} \xi_{i,j} &= \xi_{ij}, \\ \xi_{ij,k} &= \xi_\alpha R^\alpha_{kji} \end{aligned} \tag{4.77}$$

and

$$\xi_{ij} + \xi_{ji} = 0. \tag{4.78}$$

A detailed investigation of groups of motions was done in papers by I.P. Egorov, and can be found in his monograph [46], see also [45, 47].

It is known since the times of Fubini that the maximum

$$r_{mot} = \frac{1}{2}\, n(n+1) \tag{4.79}$$

is realized only in spaces of constant curvature and globally, only in n-dimensional Euclidean spaces and on spheres. It follows from the fact that the integrability conditions for (4.77), which take the form $\mathcal{L}_X R^h_{ijk} = 0$, are satisfied due to the shape of the Riemannian tensor (3.17) and (4.75).

Fubini proved that there are no spaces $\mathbb{V}_n$ for which $r_{mot} = \frac{1}{2}n(n+1) - 1$. These results were later specified by K. Yano and particularly by I.P. Egorov, see [45–47].

Theorem 4.10 *There are no spaces $\mathbb{V}_n$ for which the degree r_{mot} satisfies the inequalities*

$$\begin{aligned} \frac{n(n-1)}{2} + 1 < r_{mot} < \frac{n(n+1)}{2}, \\ \frac{(n-1)(n-2)}{2} + 5 < r_{mot} < \frac{n(n-1)}{2}. \end{aligned} \tag{4.80}$$

The group of motions with

$$r_{mot} = \frac{n(n-1)}{2} \tag{4.81}$$

is admitted only by subprojective manifolds in the sence of Kagan [45–47, 79].

The group of motions with

$$r_{mot} = \frac{1}{2}\,(n-1)(n-2) + 5 \tag{4.82}$$

is admitted by the symmetric spaces for which the Riemannian tensor has the following form:

$$R_{hijk} = \varepsilon(a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \;\; \varepsilon = \pm 1, \tag{4.83}$$

where a_i and b_i are orthogonal isotropic absolutely parallel vector fields.

4.5 Homothetic mappings and transformations

Homothetic mappings and transformations are generalization of isometric mappings and transfomations.

4.5.1 Homothetic mappings

Let us consider the Riemannian manifolds $\mathbb{V}_n = (M, g)$ and $\overline{\mathbb{V}}_n = (\overline{M}, \overline{g})$.

Definition 4.3 A diffeomorphism f of $\mathbb{V}_n = (M, g)$ onto $\overline{\mathbb{V}}_n = (\overline{M}, \overline{g})$ is called *homothetic* if the length of any arc $\ell \subset \mathbb{V}_n$ and the length of its image $f(\ell) \subset \overline{\mathbb{V}}_n$ are proportional with a constant coefficient, i.e. $|f(\ell)| = k \cdot |\ell|$.

A necessary and sufficient condition for a map $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ to be homothetic is $\overline{g} = k\, f^* g$, $k = \text{ const}$. In a common coordinate system (x^i)

$$\overline{g} = k\, g, \quad k = \text{ const}\,; \tag{4.84}$$

in coordinate form, $\overline{g}_{ij}(x) = k \cdot g_{ij}(x)$.

As $\nabla \overline{g} = 0$ holds, any homothetic mapping is a special affine mapping, and for $k = 1$ it is isometric.

An example of homothetic mappings are central projections of two concetric spheres.

We can prove easily that two spaces of nonzero constant curvature and the same signature of the metric are locally homothetic. A similar property holds for Kähler spaces of constant holomorphic curvature.

4.5.2 Groups of homothetic motions

Homothetic mappings of a Riemannian space $\mathbb{V}_n = (M, g)$ onto itself are called *homothetic transformations* on $\mathbb{V}_n$ or *homothetic motions* of $\mathbb{V}_n$.

A vector field X on $\mathbb{V}_n$ is called an *infinitesimal homothety* or a *Killing homothetic vector field* if for each point $p \in M$ there is a neighborhood U of p such that the local one-parameter group f_t determined by the vector field preserves the metric up to a constant factor k, that is, the mapping $f_t\colon M \to M$ is an homothetic transformation. In a special coordinate system (x^i) in which $X = \partial_1$, a homothetic transformation is characterized by

$$\partial_1 g_{ij}(x) = k\, g_{ij}(x), \quad k = \text{ const}\,. \tag{4.85}$$

By (4.33) and (4.36) these conditions take the form

$$\mathcal{L}_X g_{ij} \equiv \xi_{i,j} + \xi_{j,i} = k\, g_{ij} \tag{4.86}$$

where $\xi_i = g_{i\alpha}\xi^\alpha$. Let us recall that ξ^h is called the Killing homothetic vector.

We can easily check that the set of all Killing homothetic vectors depends on

$$r_{hom} \le \frac{1}{2}\,(n+1)(n+2) + 1 \tag{4.87}$$

parameters, and homothetic transformations form a Lie group of order r_{hom}.

The group of homothetic motions is a subgroup of the affine group, and it contains a subsgroup of motions. From the equations (4.86), the equations (4.59) follow as a consequence. Hence we can write the equations of motion in the form of a mixed system of equations of Cauchy type in the unknowns $\xi_i(x)$, $\xi_{ij}(x)$ and the constant k:

$$\begin{aligned} \xi_{i,j} &= \xi_{ij}, \\ \xi_{i,jk} &= \xi_\alpha R^\alpha_{kji} \end{aligned} \tag{4.88}$$

and

$$\xi_{ij} + \xi_{ji} = k\, g_{ij}. \tag{4.89}$$

The maximum $r_{hom} = \frac{1}{2}\, n(n+1) + 1$ is reached only for the (pseudo-) Euclidean manifolds. It follows from the fact that the integrability conditions for (4.88), which take the form $\mathcal{L}_X R^h_{ijk} = 0$, are satisfied due to the shape of the Riemannian tensor (3.17) and (4.86).

For spaces $\mathbb{V}_n$ different from (pseudo-) Euclidean manifolds, the inequality

$$r_{hom} \le \frac{n(n-1)}{2} \tag{4.90}$$

holds, see [45–47].

4.5.3 Transformation groups and special mappings

Let us investigate problems and questions that have a general character and connect various classes (groups) of mappings with corresponding transformations and deformations.

Many particular transformation groups of Riemannian spaces are defined by equations of the form

$$\mathcal{L}_\xi \Gamma^h_{ij} = P^h_{ij}, \tag{4.91}$$

where Γ^h_{ij} are Christoffel symbols of second type, $\mathcal{L}_\xi$ is the Lie derivative in the direction of the vector $\xi = (\xi^h)$, and P^h_{ij} are components of the tensor P of the given structure. E.g. provided $P^h_{ij} = 0$, the equations (4.91) define affine motions, and when $P^h_{ij} = \delta^h_i \psi_j + \delta^h_j \psi_i$, the same equations (4.91) characterize projective motions (see (6.52), p. 273).

The equations (4.91) can be written as follows:

$$\xi^h_{,ij} - \xi^\alpha R^h_{ij\alpha} = P^h_{ij}. \tag{4.92}$$

Similarly as above, "," denotes the covariant derivative in $\mathbb{V}_n$, and R^h_{ijk} are components of the Riemann tensor of $\mathbb{V}_n$.

Lowering the index h, and then symmetrizing in the indices h and i, we get

$$(\mathcal{L}_\xi g_{ij})_{,k} \equiv \xi_{(i,j)k} = P_{(ij)k}, \tag{4.93}$$

where $\xi_i = g_{i\alpha}\xi^\alpha$, $P_{ijk} = g_{i\alpha} P^\alpha_{jk}$. Alternating the equations (4.93) in j and k and accounting the Ricci identity we obtain

$$\xi_{j,ik} - \xi_{k,ij} + \xi_\alpha R^\alpha_{ijk} = P_{jik} - P_{kij}.$$

Alternating the last expression in the indices i and k, accounting (4.93), the Ricci identity and properties of the Riemann tensor we get, after rising indices, the equations (4.92). Hence the following holds [635]:

Theorem 4.11 *The equations (4.91) and (4.93) are equivalent.*

L.P. Eisenhart [50] proved the above theorem for projective mappings, T. Ōtsuki and Y. Tashiro [737] for holomorphically-projective mappings.

Instead of the equations (4.93) let us consider

$$a_{ij,k} = P_{(ij)k}, \tag{4.94}$$

where a_{ij} is a symmetric tensor.

Let the general solution of the equations (4.91) and (4.94) with respect to the vector ξ^h and tensor a_{ij}, respectively, depend on a finite number of essential parameters r and r_a, respectively. The parameter r is in fact the total order of the corresponding special transformation group.

The following holds [635]:

Theorem 4.12 *The inequality*

$$r \leq r_a + r_{mot} \tag{4.95}$$

is satisfied where r_{mot} is the order of the full group of motions on $\mathbb{V}_n$.

Proof. In fact, let a_{ij} be a particular solution of the equations (4.94), which depends on r_a essential parameters. Consider the equation

$$\xi_{(i,j)} = a_{ij}. \tag{4.96}$$

Any vector ξ_i satisfying the equations (4.96) satisfies also the equations (4.93), and consequently, due to the Theorem 4.11 it also satisfies the equations (4.91). Let us denote by ξ_i^* a particular solution of (4.96). Then, obviously, the general solution of the equations (4.96) takes the form $\xi_i = \xi_i^* + \overline{\xi}_i$, where $\overline{\xi}_i$ is a solution of the homogeneous equation $\overline{\xi}_{(i,j)} = 0$, where $\overline{\xi}_i$ is a Killing vector. As well known, such vectors determine full groups of motions of Riemannian spaces. Consequently, the general solution of the equations (4.91) depends on at most $r_a + r_{mot}$ parameters, which finishes the proof. □

In what follows let us suppose that $a_{ij} = \text{const}\ g_{ij}$ is a particular solution of the equations (4.94). (in other words, the tensor P_{ij}^h can vanish.) If this is the case we can prove [635]:

Theorem 4.13 *The following inequality is satisfied*

$$r \leq r_a + r_{hom} - 1, \tag{4.97}$$

where r_{hom} is the order of the full group of homothetic transformations of the space $\mathbb{V}_n$.

Proof. The proof is analogous to the proof of the Theorem 4.12. But if to any solution a_{ij} ($\neq \text{const}\ g_{ij}$) of the equation (4.94) there corresponds a solution of the equation (4.96), then the inequality (4.97) becomes equality. □

4.6 Metric connections, Metrization problem

As well known, the Fundamental Principle of Riemannian Geometry tells that a non-degenerate metric g (positive definite or not) on a manifold M determines a unique torsion-free affine connection ∇ for which the metric is parallel, $\nabla g = 0$, namely the Riemannian (or Levi-Civita) connection of (M, g).

From other point of view we can see this Metrization problem as Problem of existence of affine mappings from space with affine connection $\mathbb{A}_n = (M, \nabla)$ onto Riemannian space $\overline{\mathbb{V}}_n = (M, \overline{g})$, see Theorem 4.7, p. 191.

If $\nabla g = 0$ we say that a connection ∇ and a metric tensor g on a manifold M are *compatible*. The geometric meaning of the condition $\nabla g = 0$ is that whatever (at least piecewise differentiable) curve we choose the scalar product is preserved under parallel transport along the curve.

Let us contribute here to the reverse, namely to the *Metrization Problem*, MP, for affine connections which means: given a manifold M with a symmetric affine connection ∇, decide whether the connection arises from some metric tensor g as the Levi-Civita connection of the metric. It is in fact a kind of inverse problem.

MP was discussed – in various spaces (in manifolds endowed with a connection, in vector bundles), eventually under various constraint conditions – by various authors, both by mathematicians and mathematical physicists (L.P. Eisenhart and O. Veblen [394], S. Golab [438], A. Jakubowicz [506], B.G. Schmidt [780], S.B. Edgar, O. Kowalski [554, 555], L. Tamássy, M. Anastasiei [265], G. Thompson [266, 895], K.S. Cheng and W.T. Ni [328], M. Cocos [354] etc.). In [780], a possibility to use holonomy groups and holonomy algebras is pointed out, and difficulties arising in C^∞-class are discussed; in [554], among others, positive definite metrics for a symmetric connection with regular curvature are constructed in the favourale case; in [555], positive definite metrics for analytic connections on analytic manifolds are determined by means of an algorithm based on the de Rham decomposition and holonomy algebras; cf. [901] (the case of indefinite metrics, particularly Lorentzian, is different).

The metrization problem for connections can be formulated as follows: given an affine manifold (M, ∇), consisting of an n-dimenisonal manifold M endowed with a torsion-free affine connection ∇, under what conditions is ∇ metrizable; that is, when is there a (pseudo-) Riemannian (=non-generate) metric g such that ∇ is just the Levi-Civita connection of (M, g)? Although the problem is also of intrinsic mathematical importance, a quite natural motivation comes from theoretical physics, see explanations in [328] etc.

As well known, the Levi-Civita connection is uniquely determined by zero torsion and the requirement that g should be covariant constant, $\nabla g = 0$ (as mentioned, the scalar product of tangent vectors with respect to g should be preserved under the parallel transport with respect to ∇ along all curves). Components (Christoffel's symbols of the second type) Γ^i_{jk} of the Levi-Civita connection are related to components of the metric by the well-known formula, see p. 110:

$$\Gamma^h_{ij} = g^{hk}\Gamma_{ijk} \quad \text{and} \quad \Gamma_{ijk} = 1/2\ (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \tag{4.98}$$

4.6.1 Metrization according to Eisenhart and Veblen

In 1920', L.P. Eisenhart and O. Veblen [394] started to study geometries which arise on an (analytic) n-manifold when a system of curves called *paths* is given as the family of solutions of a system of differential equations (2.26):

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \quad i,j,k = 1, \ldots, n, \tag{4.99}$$

where $\Gamma^i_{jk}(x^1, \ldots, x^n)$ are analytic functions of the coordinates in the manifold. They also mentioned earlier attempts of H. Weyl [193] and A.S. Eddington [390], who prefered *generalization of Levi-Civita's concept of infinitesimal parallelism* to the idea of paths.

To give the system (4.99) is (locally) the same as to give a symmetric affine connection ∇ with the functions $\Gamma^k_{ij}(x)$ as Chistoffels in (some coordinate neighbourhood of) M. The solutions of (4.99) were sometimes called *paths* in the very classical terminology, and the *geometry of paths* was the geometry of (M, ∇). As a motivation coming from gravitation theory, let us mentione free-fall trajectories as example of a "preferred family" of curves. The natural idea, to introduce geometries directly by their paths, was developed in 1920s ([394], where also previous attempts made by H. Weyl and A.S. Eddington were discussed, both preferring the Levi-Civita's concept of parallelism; cf. [51]).

In [394], the problem of determining under what conditions the geometry of paths given as solutions of the differential equations (4.99) in an analytic (C^ω) n-manifold is (pseudo-) Riemannian, was established and solved by analytical methods.

(M, ∇) is (pseudo-)Riemannian if and only if there are functions g_{ij} such that the curves for which the integral $\int \sqrt{g_{ij}\dot{x}^i\dot{x}^j}ds$ is stationary satisfy (4.99); this condition can be equivalently written as

$$g_{\alpha k}\Gamma^\alpha_{ij} = 1/2\ (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \tag{4.100}$$

Solving for the derivatives of the g's we get an equivalent formulation in terms of the connection: $\nabla g = 0$, or, in an expanded form

$$g_{ij,k} = \partial_k g_{ij} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk} = 0. \tag{4.101}$$

At first we solve a bit more general problem: *find a symmetric type* $(0,2)$ *tensor field* $g(g_{ij})$ *such that the system of partial differential equations (4.101) holds with* g_{ij} *as unknowns.* If there is a solution of the system then $0 \leq \operatorname{rank}(g_{ij}) \leq n$ holds in general; even the case $q = 0$ might come. If the maximum q of ranks of all possible solutions of (4.101) is less than n then there is no metric tensor compatible with the system of curves (4.99), and (M, ∇) is not metrizable. If a solution exists and also the condition $q = n$ concerning its rank is satisfied, i.e. $\det(g_{ij}) \neq 0$, then the solution is a metric tensor.

We can apply higher order covariant derivatives and, employing the curvature tensor, write the necessary integrability conditions in the form of an infinite linear sytem of (algebraic) equations in $\frac{1}{2}n(n+1)$ unknown functions g_{ij} with

coefficients that are functions of Γ's (and their partial derivatives) alone. In coordinate-free form, the equations read

$$g(R(X,Y)Z,W) + g(Z,R(X,Y)W) = 0, \tag{4.102}$$

$$\begin{aligned} g(\nabla^r R(X,Y,Z_1,\dots,Z_r)(Z),W)+ \\ g(Z,\nabla^r R(X,Y,Z_1,\dots,Z_r)(W)) = 0 \end{aligned} \tag{4.103}$$

for all $X,Y,Z,W,Z_1,\dots,Z_r \in \mathcal{X}(M)$, $1 \leq r < \infty$, or locally,

$$g_{sj}R^s_{ik\ell} + g_{is}R^s_{jk\ell} = 0, \tag{4.104}$$

$$g_{sj}R^s_{ik\ell,m_1,\dots,m_r} + g_{is}R^s_{jk\ell,m_1,\dots,m_r} = 0, \quad 1 \leq r, \tag{4.105}$$

where R^i_{hjk} are components of the curvature tensor R. In the "matrix form", [895], we can write

$$g \circ \nabla^r R + (\nabla^r R)^T \circ g^T = 0, \quad r = 0,1,\dots < \infty. \tag{4.106}$$

If a solution should exist, the above linear conditions must stabilize for some positive integer N in the sense that from the $(N+1)^{\text{th}}$ stage, the conditions are algebraic consequences of the previous ones. Obviously on flat parts, where the Riemann curvature vanishes, $R = 0$, we get no conditions and the connection is (at least locally) metrizable.

The existence problem for compatible metrics has been considered by various authors, [438, 506] ($n = 2$), [507, 508] ($n = 4$), [592, p. 75] ($n = 2$), [895] ($n = 2$), [900, 901] ($n = 2$) and the references therein. For any $n \geq 2$, there are non-metrizable n-dimensional manifolds with connection - it happens whenever the maximum q of ranks of all possible solutions of (4.102) is less than n.

If the connection is a metric one the above system up to some N^{th} stage of differentiation should possess at least a 1-dimensional solution space, i.e. an obvious condition for existence of a (non-degenerate) metric is that the system up to some N^{th} stage should possess at least a one-dimensional solution space (over the ring of smooth functions). The following result, useful in examples, tells us that in some cases, it suffices to take $N = 1$:

Theorem 4.14 *Let M be an n-dimensional manifold with a symmetric affine connection ∇ on M. If there is a non-trivial solution g_{ij} of the system (4.104) in a local chart (U,x) and each element of the solution space satisfies also the system*

$$g_{sj}R^s_{ik\ell,m} + g_{is}R^s_{jk\ell,m} = 0 \tag{4.107}$$

then g_{ij} are componets of a covariantly constant symmetric tensor field of type $(0,2)$ on U.

Recall crutial some steps from the proof that provide a method for finding compatible metrics in examples. Suppose that the system (4.104) is solvable

and any solution to (4.104) satisfies (4.107). Choose a basis $\langle G^{(1)}, \dots, G^{(p)} \rangle$ of the solution space, any solution g takes the form

$$g = \sum_{\alpha=1}^{p} \varphi^{(\alpha)} G^{(\alpha)} \tag{4.108}$$

where $\varphi^{(\alpha)}$ are some functions on M. The covariant derivatives $G^{\alpha}_{sj,m}$ satisfy (4.104), since $G^{(\alpha)}_{sj,m} R^s_{ik\ell} + G^{(\alpha)}_{is,m} R^s_{jk\ell} = 0$ for $\alpha = 1, \dots, p$, according to (4.107). It means that we can express them through generators

$$G^{(\alpha)}_{ij,k} = \sum_{\beta=1}^{p} \mu^{(\alpha\beta)}_k G^{(\beta)}_{ij}, \quad \alpha = 1, \dots, p, \tag{4.109}$$

where $\mu^{(\alpha\beta)}_k(x)$ are functions. The coefficients can be calculated as follows. Since second covariant derivatives satisfy the so-called Ricci indentity $G^{(\alpha)}_{ij,k\ell} - G^{(\alpha)}_{ij,\ell k} = G^{(\alpha)}_{sj} R^s_{ik\ell} + G^{(\alpha)}_{is} R^s_{jk\ell}$, and the right hand sides vanish for our $G^{(\alpha)}_{ij}$, we get $G^{(\alpha)}_{ij,k\ell} - G^{(\alpha)}_{ij,\ell k} = 0$, and further (after some evaluations) we obtain

$$\frac{\partial \mu^{(\alpha\beta)}_k}{\partial x^\ell} - \frac{\partial \mu^{(\alpha\beta)}_\ell}{\partial x^k} + \sum_{\gamma=1}^{p} \left(\mu^{(\alpha\gamma)}_k \mu^{(\gamma\beta)}_\ell - \mu^{(\alpha\gamma)}_\ell \mu^{(\gamma\beta)}_k \right) = 0. \tag{4.110}$$

If g of the form (4.108) shall satisfy $\nabla g = 0$ then the φ's must satisfy the equations

$$\frac{\partial \varphi^{(\alpha)}}{\partial x^k} + \sum_{\beta=1}^{p} \varphi^{(\beta)} \mu^{(\alpha\beta)}_k = 0, \quad \alpha = 1, \dots, p, \quad k = 1, \dots n. \tag{4.111}$$

But according to (4.110), the system (4.111) is completely integrable, hence there exist functions $\varphi^{(1)}, \dots, \varphi^{(p)}$ which, by means of (4.108), determine a compatible (pseudo-)Riemannian metric.

The proof of Theorem 4.14 provides us by an algorithm useful in examples. Moreover, if the general solution of (4.104) forms a 1-dimensional space, over smooth functions, that is, it is determined uniquely up to a multiple by function of local coordinates, i.e. $g = \varphi(x^1, \dots, x^n) \cdot G$ is the general solution where G is a fixed non-trivial solution, then the solution procedure is simplified considerably. Instead of (4.109), we have the conditions $G_{ij,k} = \mu_k G_{ij}$ for suitable functions $\mu_k(x^i)$, $k = 1, \dots, n$, it means that G should be recurrent, that is, there must exist a one-form μ with μ_k as components such that $\mu \otimes G = \nabla G$. Instead of (4.110), we have the integrability conditions ($k, \ell = 1, \dots, n$)

$$\frac{\partial \mu_k}{\partial x^\ell} - \frac{\partial \mu_\ell}{\partial x^k} + \mu_k \mu_\ell - \mu_\ell \mu_k = 0. \tag{4.112}$$

The system (4.111) is reduced to

$$\frac{\partial \varphi}{\partial x^k} + \varphi \mu_k = 0, \quad k = 1, \dots, n. \tag{4.113}$$

If the conditions (4.112) hold then there exists a function $f(x^1, \dots, x^n)$ such that for $k = 1, \dots, n$, $\frac{\partial f}{\partial x^k} = \mu_k$ holds, that is, μ is exact (=gradient), $\mu = df$, and (4.113) reads

$$\frac{\partial \varphi}{\partial x^k} + \varphi(x^1, \dots, x^n)\frac{\partial f}{\partial x^k} = 0, \quad k = 1, \dots, n. \tag{4.114}$$

Then (4.114) is completely integrable, and all solutions are just $\varphi = e^{-f}$ with $\mu = df$. Finally, each $g = e^{-f} \cdot G$ is covariantly constant, $\nabla g = 0$.

As a consequence of the above considerations, we get

Corollary 4.1 *A manifold M with a symmetric connection ∇ is locally metrizable if and only if the system (4.104) for components R^i_{hjk} of the curvature tensor has non-degenerate solution, and any solution of (4.104) satisfies (4.107).*

As we have seen, we can give an algorithm (known already to classics of differential geometry, [51,394]) that provides a solution, and may be implemented to a simple computer program.

4.6.2 Application to the calculus of variations

Let us mention the relationship of the problem MP to the Calculus of Variations. The so-called Inverse Problem (IP) of the calculus of variations is: if a system $\ddot{x}^i = f^i(t, x^k, \dot{x}^k)$, $i, k = 1, \dots, n$ of second order differential equations (SODEs) is given, find (sufficiently differentiable) Lagrangian functions $L(t, x^k, \dot{x}^k)$ and a multiplier matrix $g_{ij}(t, x^k, \dot{x}^k)$ such that

$$g_{ij}(\ddot{x}^i - f^i) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}.$$

Given a system of second order ODEs of a particular type

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \qquad k = 1, \dots, n, \tag{4.115}$$

that is, second derivatives can be expressed as quadratic forms in first derivatives, we can use the above theory for deciding whether the system (4.115) is derivable from a Lagrangian. In fact, provided $\det(g_{ij}) \neq 0$, the system (4.115) is equivalent to the system

$$g_{mi}(\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k) = 0, \quad i, m = 1, \dots, n. \tag{4.116}$$

Another speaking, MP can be viewed as a particular case of IP, where $f^i = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k$ (that is, f^i are quadratic forms in components of velocities, with coefficients depending only on components of positions) in the particular case when the multipliers are time- and velocities-independent. We can assume that the coefficients in (4.115), the functions $\Gamma^k_{rs}(x)$, are components of a symmetric linear connection ∇ on some neighbourhood $U \subset \mathbb{R}^n$. If ∇ is (locally) metrizable, and $g_{ij}(x)$ (with $\det(g_{ij}(x)) \neq 0$ at any $x \in U$) are components of some non-degenerate metric g compatible with ∇ on U, then (4.115) and

(4.116) are equivalent, hence the functions $g_{ik}(x)$ can be taken as the desired variational multipliers. One of particular Lagrangians comming from MP (and solving IP) is

$$L = T = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j, \tag{4.117}$$

the kinetic energy. There might exist multipliers of a more general form $g_{ik}(t, x, \dot{x})$, depending on "time, positions and velocities", which might bring more complicated Lagrangians.

4.6.3 Metrization in dimension two

Dimension $n = 2$ appears to be rather exceptional, and especially in non-flat parts of the manifold, a very simple answer can be given (while for $n > 3$, the problem is more difficult). We show two different approaches in what follows.

Metrization of affine 2-manifolds by Cheng and Ni (revisited). Essentially the same "very classical" approach as above was used by Kuo-Shung Cheng and Wei-Tou Ni in [328] (without any refererence to Veblen's and Eisenhart's results) in the particular case $n = 2$. Necessary and sufficient conditions for existence of a metric on a two-dimensional affine manifold were formulated as conditions on components of the curvature tensor and its covariant derivatives (which can be easily checked), but without any geometric interpretation. Note that some kind of answer, dealing with various tensors related to the curvature tensor, can be found also in [592].

Let us briefly sketch a free paraphrase of the approach from [328][63]. We employ components of the Ricci tensor rather than components of the curvature. For a symmetric connection, the conditions (4.98) and $\nabla g = 0$ are equivalent. In local coordinates,

$$g_{ij,k} = \partial_k g_{ij} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk}. \tag{4.118}$$

Hence the formula $\nabla g = 0$ reads

$$\partial_k g_{ij} = g_{sj}\Gamma^s_{ik} + g_{is}\Gamma^s_{jk}. \tag{4.119}$$

If $n = 2$, the condition $\nabla g = 0$ takes the form of six PDEs in three variables g_{11}, g_{12}, g_{22}

$$\begin{aligned}
\partial_1 g_{11} &= 2\left(\Gamma^1_{11}g_{11} + \Gamma^2_{11}g_{12}\right), \quad \partial_1 g_{22} = 2\left(\Gamma^1_{12}g_{12} + \Gamma^2_{12}g_{22}\right),\\
\partial_2 g_{11} &= 2\left(\Gamma^1_{12}g_{11} + \Gamma^2_{12}g_{12}\right), \quad \partial_2 g_{22} = 2\left(\Gamma^1_{22}g_{12} + \Gamma^2_{22}g_{22}\right),\\
\partial_1 g_{12} &= \Gamma^1_{12}g_{11} + \left(\Gamma^1_{11} + \Gamma^2_{12}\right)g_{12} + \Gamma^2_{11}g_{22},\\
\partial_2 g_{12} &= \Gamma^1_{22}g_{11} + \left(\Gamma^1_{12} + \Gamma^2_{22}\right)g_{12} + \Gamma^2_{12}g_{22}.
\end{aligned} \tag{4.120}$$

Partial differentiating of (4.120) followed by substituting for partial derivatives $\partial_k g_{ij}$ from (4.118) gives

$$\frac{\partial^2 g_{sk}}{\partial x^r \partial x^j} = g_{s\ell}\left(\frac{\partial \Gamma^\ell_{rk}}{\partial x^j} + \Gamma^\ell_{jt}\Gamma^t_{rk}\right) + g_{k\ell}\left(\frac{\partial \Gamma^\ell_{rs}}{\partial x^j} + \Gamma^\ell_{jt}\Gamma^t_{rs}\right) + g_{\ell t}\left(\Gamma^t_{jk}\Gamma^\ell_{rs} + \Gamma^\ell_{rk}\Gamma^t_{js}\right).$$

[63] We hope to succeed in improving of some gaps in the proof given in [328], as well as some missprints, e.g. in formulas (28), (35) from [328]

Substituting for components of the Riemannian tensor, and accounting interchangeability of second partial derivatives, we get, as a necessary condition, that g_{ij} must satisfy a homogeneous system of linear algebraic equations $g_{sj}R^s_{ik\ell}+g_{is}R^s_{jk\ell}=0$. Employing the Ricci tensor we get a homogeneous system of three linear algebraic equations in three unknowns g_{11}, g_{12}, g_{22},

$$\begin{aligned} R_{12}\,g_{11} \;-\; R_{11}\,g_{12} &= 0,\\ R_{22}\,g_{11}+(R_{12}-R_{21})\,g_{12}-R_{11}\,g_{22} &= 0,\\ R_{22}\,g_{12}-R_{21}\,g_{22} &= 0. \end{aligned} \tag{4.121}$$

The system (4.121) has a non-trivial solution iff its matrix has vanishing determinant on M, i.e. if and only if

$$0=\begin{vmatrix} R_{12} & -R_{11} & 0\\ R_{22} & R_{12}-R_{21} & -R_{11}\\ 0 & R_{22} & -R_{21}\end{vmatrix} = (R_{12}-R_{21})\cdot\det(R_{ij}) \tag{4.122}$$

holds. If $R\equiv 0$ (or $R(x)=0$) then (4.122) holds identically, and we obtain no new condition for g_{ij} from (4.121). If $R(x)\neq 0$ in a point $x\in M$ then (from continuity) $R\neq 0$ on some neighbourhood U of x ("R is regular"). Points with non-vanishing curvature form an open subset.

Suppose $R\neq 0$ on M (the manifold is "no-where flat"). Then also the Ricci tensor is non-vanishing. Recall that on a 2-dimensional pseudo-Riemannian manifold, the Ricci tensor Ric is proportional to the metric tensor, $\mathrm{Ric}=K\cdot g$ where K denotes the sectional curvature (or Gaussian curvature, respectively, for embedded submanifolds in the positive definite case), [900]. Hence regularity of the Ricci tensor is a necessary condition for metrizability of nowhere flat 2-manifolds[64] Under the regularity assumption, $\det(R_{ij})\neq 0$, we obtain $R_{12}=R_{21}$, symmetry of Ric, as another necessary condition. Accounting this condition, the system (4.121) reads

$$\begin{pmatrix} R_{12} & -R_{11} & 0\\ R_{22} & 0 & -R_{11}\\ 0 & R_{22} & -R_{12}\end{pmatrix}\begin{pmatrix} g_{11}\\ g_{12}\\ g_{22}\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\end{pmatrix}. \tag{4.123}$$

Under the conditions $R(x)\neq 0$, $\det(R_{ij}(x))\neq 0$, the following cases might occur in $x\in M_2$:

(a) $R_{11}(x)=0$ (or $R_{22}(x)=0$, or even both are equal zero), but $R_{12}(x)\neq 0$,
(b) $R_{12}(x)=0$, but $R_{11}(x)\cdot R_{22}(x)\neq 0$,
(c) $R_{ij}(x)\neq 0$ for $i,j\in\{1,2\}$.

In each of these three cases, there exists a non-vanishing subderterminant of order two, hence the coefficient matrix is of rank two. So under the assumptions $R\neq 0$, Ric symmetric, the coefficient matrix has rank two, and the general solution is one-dimensional, $g_{ij}(x)=k(x)\cdot R_{ij}(x)$, $k(x):M\to\mathbb{R}-\{0\}$, or shortly,

$$g_{11}:g_{12}:g_{22}=R_{11}:R_{12}:R_{22}. \tag{4.124}$$

[64] Such geometric arguments seem to be missing in [328, 900].

This solution satisfies the starting system of PDE's. If we prescribe initial data $x_0 \in M$, $K(x_0) \neq 0$, and set $k(x) = \frac{1}{K(x_0)}$ then the solution is unique, $g_{ij} = \frac{1}{K(x_0)} R_{ij}$ (the choice of a constant has a geometric meaning of a "constant scale change" of the metric). In the case (c), and only in this case, we can proceed as in [328]. We can calculate the ratios

$$\frac{\partial_k g_{ij}}{g_{ij}} = \frac{\partial \ln |g_{ij}|}{\partial x^k} = \Gamma^s_{ik} \frac{R_{sj}}{R_{ij}} + \Gamma^s_{jk} \frac{R_{is}}{R_{ij}} \tag{4.125}$$

from (4.120), derive (necessary and sufficient) integrability conditions for the first four of these equations (namely with $i = j = 1$ and $i = j = 2$),

$$\begin{aligned} \partial_2(\Gamma^1_{11} + \frac{R_{12}}{R_{11}}\Gamma^2_{11}) - \partial_1(\Gamma^1_{12} + \frac{R_{12}}{R_{11}}\Gamma^2_{12}) &= 0, \\ \partial_1(\Gamma^2_{12} + \frac{R_{12}}{R_{22}}\Gamma^1_{12}) - \partial_2(\Gamma^2_{22} + \frac{R_{12}}{R_{22}}\Gamma^1_{22}) &= 0. \end{aligned} \tag{4.126}$$

In [328], after some evaluation, s symmetry of the Ricci tensor, $R_{12} = R_{21}$, further necessary conditions are obtained, originally formulated in components of the curvature tensor, which in our denotation read

$$\begin{aligned} R_{12}R_{11,1} = R_{11}R_{12,1}, \quad R_{12}R_{11,2} = R_{11}R_{12,2}, \\ R_{12}R_{22,1} = R_{22}R_{12,1}, \quad R_{12}R_{22,2} = R_{22}R_{12,2}. \end{aligned} \tag{4.127}$$

This family of conditions can be rewritten as

$$\begin{aligned} \varrho_1 = R_{11,1} : R_{11} = R_{12,1} : R_{12} = R_{22,1} : R_{22}, \\ \varrho_2 = R_{11,2} : R_{11} = R_{12,2} : R_{12} = R_{22,2} : R_{22}. \end{aligned} \tag{4.128}$$

Obviously, (4.128) is equivalent to recurrency: $\nabla \mathrm{Ric} = \varrho \otimes \mathrm{Ric}$, $\varrho = \varrho_i dx^i$, $i = 1, 2$ (it was realized in [895, 900]). If the integrability conditions $R_{12} = R_{21}$ and (4.127) are satisfied, that is, if the (non-degenerate) Ricci tensor is symmetric and recurrent, then (4.127) hold, (4.126) can be integrated[65)], and the solution of (4.120) can be given in the following form, [328]:

$$g_{11} = \exp\left(2 \int\limits_{(x^1_0, x^2_0)}^{(x^1, x^2)} (\Gamma^1_{11} + \frac{R_{12}}{R_{11}}\Gamma^2_{11})dx^1 + (\Gamma^1_{12} + \frac{R_{12}}{R_{11}}\Gamma^2_{12})dx^2 + c_1\right), \tag{4.129}$$

$$\begin{aligned} g_{12} = \exp(2 \int\limits_{(x^1_0, x^2_0)}^{(x^1, x^2)} (\Gamma^1_{11} + \Gamma^2_{12} + \tfrac{R_{11}}{R_{12}}\Gamma^1_{12} - \tfrac{R_{22}}{R_{12}}\Gamma^2_{11})dx^1 \\ + (\Gamma^1_{12} + \Gamma^2_{22} + \tfrac{R_{11}}{R_{12}}\Gamma^1_{22} - \tfrac{R_{22}}{R_{12}}\Gamma^2_{12})dx^2 + c_2), \end{aligned} \tag{4.130}$$

$$g_{22} = \exp\left(2 \int\limits_{(x^1_0, x^2_0)}^{(x^1, x^2)} (\Gamma^2_{12} + \frac{R_{12}}{R_{22}}\Gamma^1_{12})dx^1 + (\Gamma^2_{22} + \frac{R_{12}}{R_{22}}\Gamma^1_{22})dx^2 + c_3\right), \tag{4.131}$$

[65)] Note that sufficiency of the reached system of conditions $R_{12} = R_{21}$, (4.127) is only asserted in [328], without any detailed proof; the possibilities (a) and (b) are not discussed at all.

where the constants c_1, c_2, c_3 must be properly selected in order to satisfy (4.121); in fact the choice of c_2 is free (and corresponds to the constant scale change of the compatible metric), c_1 and c_3 depend on c_2.

In practise, we can try to find all compatible metrics on (open) nowhere flat parts, and then try to glue the metrics together, using suitable metrics on flat parts; we might succeed or not (obstructions might arise on the boundary, [780]).

Ricci tensor as a tool for metrization of 2-manifolds

Let (x^1, x^2) denote local coordinates on a coordinate neighbourhood U of a manifold M_2. In dimension two, the curvature is simply given by $R_{hijk} = K(x)(g_{hj}g_{ik} - g_{hi}g_{jk})$ and the function $K(x)$ is the so-called *Gauss curvature*. The Riemann curvature R in type $(1,3)$ and the Ricci tensor Ric are related by $R^i_{hjk} = \delta^i_j R_{kh} - \delta^i_k R_{jh}$. As far as $R^i_{hjj} = 0$ and $R^i_{hij} = R_{jh}$ holds for $j \neq i$, the curvature tensor of a linear connection ∇ on M_2 is completely determined by its Ricci tensor; explicitely,

$$\begin{aligned} R_{11} = -R^2_{112} = R^2_{121},\ R_{21} = -R^1_{121} = R^1_{112}, \\ R_{12} = -R^2_{212} = R^2_{221},\ R_{22} = -R^1_{221} = R^1_{212}. \end{aligned} \tag{4.132}$$

Particularly, $R = 0$ if and only if $\mathrm{Ric} = 0$, and recurrency is also inherited:

Lemma 4.1 *For (M_2, ∇),* Ric *is recurrent if and only if R is recurrent.*

Indeed, let Ric be recurrent, $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$. In local coordinates, if $\omega = \omega_j dx^j$ then $\nabla_\ell R^i_{hjk} = \delta^i_j \nabla_\ell R_{kh} - \delta^i_k \nabla_\ell R_{jh} = \delta^i_j \omega_\ell R_{kh} - \delta^i_k \omega_\ell R_{jh} = \omega_\ell R^i_{hjk}$, hence $\nabla R = \omega \otimes R$. Vice versa, if $\nabla R = \omega \otimes R$ holds then $\nabla_\ell R_{jk} = \omega_\ell R^i_{kij} = \omega_\ell R_{jk}$, and $\nabla \mathrm{Ric} = \omega \otimes \mathrm{Ric}$.

On (M_2, g), non-zero components of type $(0,4)$ curvature $\tilde{R}$ are (up to a sign) equal just R_{1212}, and we have $R_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij})$ where the sectional curvature $K = K(x)$ is the Gauss curvature, $K = \dfrac{R_{1212}}{\det(g_{ij})}$.

Lemma 4.2 *The curvature tensor of a two-dimensional pseudo-Riemannian manifold (M_2, g) satisfies*

$$R^i_{hjk} = K(\delta^i_k g_{hj} - \delta^i_j g_{hk}), \tag{4.133}$$

and the Ricci tensor is proportional to the metric tensor,

$$\mathrm{Ric} = K \cdot g = \frac{1}{2}\varrho \cdot g \tag{4.134}$$

where ϱ is the scalar curvature.

Indeed, $R^t_{hij} = R^s_{hij}\delta^t_s = R^s_{hij} g_{sk} g^{kt} = R_{hijk} g^{kt} = K(g_{hj}g_{ik} - g_{hk}g_{ij})g^{kt} = K(\delta^t_i g_{hj} - \delta^t_h g_{ij})$. It follows immediately for the Ricci tensor that $R_{hj} = \Sigma_i R^i_{hij} = K \cdot \Sigma_i (\delta^i_i g_{hj} - \delta^i_h g_{ij}) = K \cdot g_{hj}$, hence $\mathrm{Ric} = Kg$, and $\varrho = R_{hj} g^{hj} = 2K$.

Corollary 4.2 *(M_2, g) is always an Einstein space. For a nowhere flat (M_2, g), the Ricci tensor is symmetric and non-degenerate.*

Note that according to [90, I, p. 280], any non-flat Riemannian 2-manifold has a recurrent curvature provided its sectional curvature does not vanish. We can check:

Lemma 4.3 *The Ricci tensor of a nowhere flat pseudo-Riemannian manifold* (M_2, g) *is recurrent, and the corresponding* 1*-form is exact* [66].

Indeed, $R \neq 0$ is equivalent with $K(x) \neq 0$ on M (from continuity, K is either positive, or negative). Since by (4.134), $g = \alpha(x)\cdot\mathrm{Ric}$ with $\alpha(x) = \frac{1}{K(x)} \neq 0$, and $\nabla g = 0$, we get easily that $\alpha(x) \cdot \mathrm{Ric}$ is parallel. Now $\nabla\mathrm{Ric} = d(-\ln|\alpha|) \otimes \mathrm{Ric}$ holds.

It follows from the above discussion on pseudo-Riemannian manifolds that two conditions are necessary for local metrizability of a (symmetric) connection on a 2-manifold: the Ricci tensor must be symmetric, and must be also recurrent, with the corresponding 1-form being closed; Ric may be degenerate only in the case $R = 0$, and then $\mathrm{Ric} = 0$ holds. Furthermore, for global metrizability, the 1-form from the recurrency condition must be even exact. A flat connection is always (globally) metrizable, with $\frac{1}{2}n(n+1)$-parameter solution space; even the signature can be prescribed. So let us pay attention to the situation when the curvature tensor (or equivalently, the Ricci tensor) is non-zero in one point $x_0 \in M$, and due to continuity, in some neighbourhood of x_0 [67].

Theorem 4.15 (Existence of local metrics on two-manifolds) *Let a* 2*-dimensional manifold* (M_2, ∇) *with a symmetric linear connection be given such that the Ricci tensor is regular,* $|R_{ij}| \neq 0$*, symmetric,* $R_{ij} = R_{ji}$*, and recurrent,* $\nabla Ric = \varrho \otimes Ric$ *for some* 1*-form* ϱ*. Then locally, there is a metric compatible with the connection.*

Let us prove it. Let $x_0 \in M$. $|R_{ij}| \neq 0$ implies existence of a pair (i, j) of indices such that $R_{ij} \neq 0$ about[68] x_0. Recurrency together with regularity guarantee that $d\varrho = 0$. Hence about x_0, there is a function f such that $\varrho = df$. Consequently, $e^{-f} \cdot \mathrm{Ric}$ is parallel about x_0. Therefore $g = e^{-f} \cdot \mathrm{Ric}$ is a local metric on a nbd of x_0 compatible with ∇. Of course, the function f from the proof is not unique. Any function $\tilde{f}$ with the same differential, $d\tilde{f} = df$, also gives a metric; such a function differs up to a constant, $\tilde{f} = f + a$, $a \in \mathbb{R}$.

If R is nowhere zero, a similar proof quarantees existence of global metrizability of a nowhere flat affine manifold:

Proposition 4.1 *Let* (M_2, ∇) *be a two-dimensional manifold with a symmetric linear connection. If the Ricci tensor of* ∇ *is regular, symmetric, and recurrent,* $\nabla Ric = \varrho \otimes Ric$*, and the* 1*-form* ϱ *is exact, i.e.* $\varrho = df$ *for some function* $f \in \mathcal{F}(M)$*, then* $g = e^{-f} \cdot \mathrm{Ric}$ *is a (global) metric tensor compatible with* ∇*.*

[66] and consequently closed

[67] The subset of non-flat points is open.

[68] Under "about x" we mean on some neighbourhood of x.

Theorem 4.16 (Global metrizability of no-where flat connections on 2-manifolds) *A nowhere flat symmetric linear connection on M_2 is metrizable if and only if its Ricci tensor is regular, symmetric, recurrent, and the corresponding* 1*-form is exact. If this is the case, and $\nabla Ric = df \otimes Ric$ holds for some smooth function $f \in \mathcal{F}(M)$, then all global metrics compatible with ∇ form a* 1*-parameter family described by the formula*

$$g_b = \exp(-f + b) \cdot \mathrm{Ric}, \qquad b \in \mathbb{R}, \tag{4.135}$$

that is, any of them arises from the Ricci tensor as a multiple by a smooth function. Moreover, any two compatible metrics differ up to a scalar multiple.

The main statement has been already proved – the "if" part in Theorem 4.15 and Proposition 4.1, and the "only if" part in Corollary 4.2 and Lemma 4.3. As to the rest, let $g = e^{-f} \cdot \mathrm{Ric}$, $\tilde{g} = e^{-\tilde{f}} \cdot \mathrm{Ric}$ be two compatible metrics, then $\tilde{f} - f = a$, $\mathrm{Ric} = e^{\tilde{f}} g$, and $g = e^a \tilde{g}$. Choosing $b = -a$ we get $\tilde{g} = e^{-f-a} \cdot \mathrm{Ric}$, i.e. (4.135) holds.

As an immediate consequence of Theorem 4.16 we obtain:

Corollary 4.3 *Two pseudo-Riemannian metrics g_1, g_2 compatible with the same nowhere flat (symmetric) linear connection on M_2 are homothetic.*

Unicity of g declared in [895, p. 532] must be understood in this way.

For positive-definite metrics, this result is a special case of the Theorem 1 of O. Kowalski from [554, p.131] (recall that two metrics g_1, g_2 on a manifold are called *conformally equivalent* if there is a function κ on M such that $g_2 = \kappa g_1$):

Let g, g' be two Riemann metrics on a smooth manifold M with the same Riemann curvature tensor R. Then g, g' are conformally equivalent on the closure of the set of all regular points of R.

Connections with constant Christoffels in plane domains

To demonstrate application of the above statments consider the class of symmetric connections with constant Christoffels in plane domains. A torsion-free affine connection ∇ in a domain U of $\mathbb{R}^2[u, v]$ is given uniquely by a family of components (Christoffel symbols) which are eight functions $a, b, c, d, , e, f$ in two variables u, v (local coordinates in U) such that

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= a\partial_u + b\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v, \\ \nabla_{\partial_v}\partial_u &= \tilde{c}\partial_u + \tilde{d}\partial_v, \quad \nabla_{\partial_v}\partial_v = e\partial_u + f\partial_v \end{aligned} \tag{4.136}$$

where $\partial_u = \frac{\partial}{\partial u}$, $\partial_v = \frac{\partial}{\partial y}$. In the usual notation $\Gamma^1_{11} = a(u, v)$, $\Gamma^2_{11} = b(u, v)$, $\Gamma^1_{12} = \Gamma^1_{21} = c(u, v)$, $\Gamma^2_{12} = \Gamma^2_{21} = d(u, v)$, $\Gamma^1_{22} = e(u, v)$, $\Gamma^2_{22} = f(u, v)$. A connection with constant Christoffels defined in $U \subset \mathbb{R}^2[u, v]$ has the curvature tensor R as well as the Ricci tensor constant, moreover Ric is always symmetric, with components

$$R_{11} = b(f - c) + d(a - d), \quad R_{12} = R_{21} = cd - be, \quad R_{22} = e(a - d) + c(f - c),$$

a torsion-free connection with constant Christoffels on M_2 is recurrent if and only if there exist constants χ_1, χ_2 such that the following system holds:

$$\begin{aligned}
&(\chi_1+2a)R_{11}+2bR_{12}=0,\quad bR_{11}+(\chi_1+a+d)R_{12}+bR_{22}=0,\\
&\qquad 2cR_{12}+(\chi_1+2d)R_{22}=0,\quad (\chi_2+2c)R_{11}+2dR_{12}=0,\\
&eR_{11}+(\chi_2+c+f)R_{12}+dR_{22}=0,\ 2eR_{12}+(\chi_2+2f)R_{22}=0.
\end{aligned}\tag{4.137}$$

If we account also non-singularity and symmetry of Ric we find [904, 905]:

Lemma 4.4 *Exactly for the following choices of constants we get a metrizable connection ∇ of type A in M_2*

(i) $a, c, e, f \in \mathbb{R}$ *are constants satisfying* $ae - c^2 + cf = 0$, $b = d = 0$;

(ii) $a, b, e, f \in \mathbb{R}$ *are constants,* $d \neq 0$, $c = be/d$ *and the equality* $d^2a - eb^2 + dbf - d^3 = 0$ *is satisfied.*

In all cases, Ric $= 0$. Hence s locally flat connections, there are no other metrizable torsion-free type A connections (with constant Christoffels) on 2-manifolds.

4.6.4 Metrization via holonomy groups and holonomy algebras

In the real analytic case, an algorithmic decision procedure can be given, and in the affirmative case, all metrics compatible with the given connection can be described. The method is based also on geometrical considerations, on the de Rham decomposition of the tangent space and its consequences for the holonomy algebra, but the decision process itself is more or less an algebraic one, and, what is a great adavntage, it can be presented as an algorithm for a computer search.

The method of Eisenhart and Veblen gives a very little insight into the geometric meaning of the integrability conditions and the restrictions imposed by them on the linear connection. Another approach, a more geometric one, is those using parallel displacement determined by the connection and holonomy groups, [265, 780, 902], or, in the real analytic case, holonomy algebras, [555, 780, 901, 903]; even here, examining the "classical" equations is an important and fruitful starting point.

Recall that the *holonomy* of (M, ∇) at $x \in M$ along the piecewise C^1-loop c is a linear transformation (automorphism) $h_c: T_xM \to T_xM$, $X_0 \mapsto X(1)$ of the tangent space (given by parallel transport of vectors; alternatively, parallel propagation of frames [90, I, p. 85, Th. 7.2.]) can be used. If $c \simeq \tilde{c}$ then $h_c = h_{\tilde{c}}$, [354], hence we can pass to equivalence classes. Due to the properties of parallel transport, $h_{c_2} \circ h_{c_1} = h_{c_1 * c_2}$ and $h_c^{-1} = h_{-c}$ hold. The set of all holonomies along loops based at x with composition (of endomorphisms) is a subgroup in the automorphism group $Aut(T_xM) \simeq GL(T_xM)$ of the fiber (hence a Lie transformation group), the so-called full linear *holonomy group* Hol_x^{∇} of (M, ∇) at x. If M is connected the holonomy groups at different points are isomorphic, and a simplified notation *Hol* can be used. In what follows we suppose connectedness. Restricting ourselves to loops homotopic

zero (contractible to a point) we arrive at the *restricted holonomy group* Hol^0, [90, 901].

If ∇ is a Riemannian connection of some (pseudo-)Riemannian manifold (M, g) then the scalar product defined by g in particular tangent spaces is preserved by parallel transport, particularly by holonomies. Elements of the holonomy group are isometries of the tangent space, and Hol_x^∇ can be identified with a subgroup of $O(n)$, or of $O(k, \ell)$, $k + \ell = n$, respectively, according to the signature of the metric g. The restricted holonomy group Hol^0 identifies with a subgroup of $SO(n)$, or of $SO(k, \ell)$, respectively. Hence the holonomy group "decides" whether a connection is metrizable or not: obviously, a connection can only be a Riemannian connection of a metric g, if the holonomy group (restricted holonomy group) is a subgroup of the generalized orthogonal (special orthogonal) group corresponding to the signature of g.

On the other hand, if M is connected the condition is also sufficient. The basic idea is to construct the pull-back of the scalar product given at one fixed point by means of parallel traslations along paths connecting the fixed point with other points in M.

Theorem 4.17 *Let ∇ be a linear connection on a smooth connected manifold M. If there is a point $x \in M$ such that the holonomy (restricted holonomy) group is contained in the generalized orthogonal (special orthogonal) group of T_xM then ∇ is metrizable.*

Proof. The tangent space T_xM may be considered isomorphic to $(\mathbb{R}^n, \langle\,,\,\rangle)$ via a fixed chart around x where $\langle\,,\,\rangle$ is the standard scalar product of the corresponding signature. Any other point $y \in M$ can be joint with x by a path in M (since M is connected). The holonomy groups Hol_y^∇ and Hol_x^∇ are isomorphic (via parallel transport, but not canonically). Hence we can use parallel propagation to pull the scalar product back; it can be checked that the resulting scalar product g_y on T_yM is independent of the particular path [265]. □

Yet another formulation: (M, ∇) *is metrizable if and only if the bundle of all frames is reducible to the orthogonal group, or generalized orthogonal group, respectively.* The way how to employ holomomy groups in order to solve the metrization problem for linear connections was discussed already in [265, 780].

Theorem 4.18 *Let M be a connected manifold endowed with a torsion free connection ∇. Let there exist a point $x \in M$ and a Hol_x^∇-invariant non-degenerate quadratic form G_x on T_xM. Then ∇ is the Riemannian connection of a metric g on M which has the same signature as G_x. Moreover, if G_x is positive definite then ∇ is Riemannian.*

Proof. The symmetric polar bilinear form g_x of the given quadratic form G_x defines a Hol_x^∇-invariant scalar product on T_xM; as above, assume the fibre T_xM endowed with g_x as isomorphic to $\mathbb{R}^n$ with the standard scalar product of the corresponding signature. Now pull the scalar product back, i.e. g_y is determined by the condition that the parallel transport is an isometry. The

constructed metric tensor g is preserved by the connection ∇, hence the (unique) Riemannian connection of g must coincide with ∇. □

Given (M, ∇) where M is a *connected* and *simply connected* manifold then *Hol* is a connected Lie subgroup of the automorphism group of the fibre, and hence is uniquely determined by its Lie algebra.

If we restrict ourselves to loops homotopic zero (contractible to a point), the construction as above gives rise to the linear *restricted holonomy group* Hol^0 of ∇ with the reference point x, a connected Lie transformation group which is a component of unit in *Hol*. Denote by $\underline{h}(x) := \underline{Hol}$ the corresponding Lie algebra of *Hol*. Particular Lie subgroups in *Hol*, namely the linear *local holonomy group* Hol^* and the *infinitesimal holonomy group* Hol' can be introduced, and the inclusions $Hol' \subseteq Hol^* \subseteq Hol^0$ can be checked. The corresponding holonomy Lie algebras satisfy [90, I, Ch. II, p. 94, 95] $\underline{h}'(x) \subseteq \underline{h}^*(x) \subseteq \underline{h}(x)$. For a smooth connection, the infinitesimal holonomy algebra $\underline{h}'(x)$ can be calculated from the curvature tensor and its covariant derivatives:

Proposition 4.2 *For smooth connections, the Lie algebra* $\underline{h}'(x)$*, as a vector space, is a span of the linear maps*

$$\nabla^k R(X, Y, Z_1, \dots, Z_k), \qquad X, Y, Z_1, \dots, Z_k \in T_x M, \quad 0 \leq k < \infty, \tag{4.138}$$

which can be (in local coordinates) respresented by matrices with components

$$R^i_{k\ell m, s_1, \dots, s_k} \xi^\ell \eta^m \nu^{s_1} \dots \nu^{s_k}. \tag{4.139}$$

([90, Ch. III, p. 152, Lemma 1, Th. 9.2.])

Proposition 4.3 *At a fixed point, holonomy groups of a real analytic connection on a real analytic manifold satisfy* $Hol' = Hol^* = Hol^0$*.* ([90, Ch. II, p. 101, Th. 10.8.])

As a consequence, in the real analytic case the holonomy algebra coincides with the infinitesimal holonomy algebra, $\underline{h}(x) = \underline{h}'(x)$, hence the component Hol^0 of unit can be retrieved from its Lie algebra $\underline{h}'(x)$. Moreover, the restricted holonomy group of a connected real analytic manifold (M, ∇) with an analytic connection is fully determined by the curvature tensor R and its iterated covariant derivatives $\nabla^\ell R$, $\ell \in \mathbb{N}$.

Now let us pay more attention to the Riemannian case. Note the following.

Lemma 4.5 *Let* g*,* $g(x, y) = \sum x^i y^i$*, be the standard positive definite scalar product on* $\mathbb{R}^n$*. The Lie algebra* $\underline{o}(n) = T_e SO(n)$ *of the orthogonal group* $O(n)$ *can be identified with the algebra of all endomorphisms* $A \in End(\mathbb{R}^n)$ *of* $\mathbb{R}^n$ *which are skew-symmetric with respect to* g*, i.e. satisfy*

$$g(Au, v) + g(u, Av) = 0 \quad \textit{for all} \quad u, v \in \mathbb{R}^n. \tag{4.140}$$

Proof. For any $A \in \underline{o}(n)$ consider the corresponding one-parameter subgroup $R^A\colon \mathbb{R} \to SO(n)$, $t \mapsto R^A_t$ given by the initial data

$$R^A_0 = e, \qquad (R^A_0)' := \left(\frac{d}{dt}\right)_{t=0} R_t = A. \tag{4.141}$$

Since the scalar product g is $O(n)$-invariant, $g(Ru, Rv) = g(u, v)$ for R from $O(n)$, we get

$$g(R^A_t u, R^A_t w) = g(u, w), \qquad u, w \in \mathbb{R}^n.$$

Let us differentiate with respect to t making use of $(f \cdot g)' = f' \cdot g + f \cdot g'$, consider $t \to 0$ and pass to limits:

$$g((R^A_0)'u, R^A_0 w) + g(R^A_0 u, (R^A_0)'w) = 0.$$

By (4.141) we get (4.140). Hence we have constructed a mapping of $\underline{o}(n)$ to the set $\{A \mid g(Au, v) + g(u, Av) = 0\}$ which is obviously bijective since the dimensions coincide; namely, are equal $\frac{1}{2}n(n-1)$. □

In general, we can not calculate the holonomy group from the curvature tensor and its covariant derivatives. It might be even difficult to find the holonomy group itself, and a quadratic form invariant under it. The situation is easier in the real analytic case: the assumptions for *Hol* can be reformulated as assumptions for $\underline{Hol}$. The following shows how to characterize quadratic forms invariant under the holonomy group in terms of the holonomy algebra.

Theorem 4.19 *Let ∇ be a torsion free connection on a smooth connected manifold M, $x \in M$ a fixed point. A symmetric bilinear form G on the tangent space T_xM is Hol^{∇}_x-invariant if and only if for all elements $A \in \underline{Hol}$ of the Lie algebra,*

$$G(AX, Y) + G(X, AY) = 0 \qquad \text{for } X, Y \in T_xM. \tag{4.142}$$

Proof. Here we check only that for any *Hol*-invariant symmetric bilinear form G, elements of the holonomy algebra satisfy (4.142). We use $\exp\colon \underline{h}(x) \to Hol$. So let $A \in \underline{h}(x)$. Consider the corresponding one-parameter subgroup $s^A\colon \mathbb{R} \to Hol$, $t \mapsto s^A(t)$ uniquely determined by the initial data $s^A(0) = 1$, $(s^A)'(0) := (\frac{d}{dt})_{t=0}\, s^A(t) = A$. Let G be invariant under the holonomy group, that is, $G(\tau X, \tau Y) = G(X, Y)$ for all $\tau \in Hol^{\nabla}_x$. Then we get $G(s^A(t)X, s^A(t)Y) = G(X, Y)$ for all $X, Y \in T_xM$. Let us differentiate with respect to t, making use of the formula for scalar product, and pass to limits for $t \to 0$:

$$G((s^A)'(0)(X), s^A(0)(Y)) + G(s^A(0)(X), (s^A)'(0)(Y)) = 0.$$

That is, (4.142) is satisfied. □

The above gives us a quite natural motivation for introducing the vector subspace $H(x)$, $x \in M$, of all symmetric bilinear forms satisfying (4.142),

$$\begin{aligned} H(x) := \{G_x \in S^2(T^*_xM) \,|\, & G_x(AX, Y) + G_x(X, AY) = 0, \\ & A \in \underline{h}(x),\ X, Y \in T_xM\}. \end{aligned} \tag{4.143}$$

If ∇ is Riemannian then for every $x \in M$, $H(x)$ includes a positive definite form. Under additional assumptions, the converse also holds. Denote

$$\begin{aligned} \underline{h}_r(x) = \; & span\,\{\nabla^k R(X,Y,Z_1,\ldots,Z_k), \\ & X,Y,Z_1,\ldots,Z_k \in T_xM, \quad 0 \le k \le r\}. \end{aligned} \tag{4.144}$$

Recall that a point $x \in M$ is called *Hol-regular* (regular) or *generic* if for each $r \in \mathbb{N}$ the dimension of the subspace $\underline{h}_r(x)$ attains its maximum in a neighbourhood U_x of the point x. At a *Hol*-regular point, the sequence of subspaces stabilizes for some $\nu \in \mathbb{N}$, $\underline{h}_\nu(x) = \underline{h}_{\nu+1}(x)$, and (using covariant differentiation) we get the equality $\underline{h}(y) = \underline{h}_\nu(y)$ at any point $y \in U_x$. Note that the set of all *Hol*-regular points is a dense open subset in M. Due to parallel transport, ∇ is Riemannian if and only if there is a Hol_x^∇-invariant positive definite scalar product at one generic point x. If Hol^0 is a subgroup of the special orthogonal group of the fibre at one "nice", generic point, then we can define a scalar product on this particular fibre T_xM and create a compatible metric using parallel transport, [265, 555, 780]. In simple examples, it works directly.

Now use the above Theorems 4.17, 4.18. If ∇ is Riemannian, i.e. comes from a positive definite metric, then for every $x \in M$, $H(x)$ includes a positive definite form; under additional assumptions, the converse also holds. If M is a connected simply connected manifold with symmetric linear connection, $x \in M$ a fixed point, and if there exists a positive definite form G_x in $H(x)$, then the connection ∇ is Riemannian. Moreover, if M is connected and there is a non-degenerate quadratic form $G_x \in H(x)$ at one point $x \in M$, then ∇ is the Riemannian connection of some (pseudo-Riemannian) metric on M with the same signature.

Theorem 4.20 [555, Prop. 1] *Given a connected simply connected (M, ∇) and $x \in M$, let there be a positive definite form $G_x \in H(x)$. Then ∇ is Riemannian.*

It might be a problem to find out whether there is a positive definite form in $H(x)$. As far as we know, no direct decision algorithm is available based on Theorem 4.20 and using exclusively linear algebra. That is why we propose an algorithm using more geometrical tools based especially on properties of the (canonical) de Rham decomposition of the tangent space T_xM of a Riemannian manifold (M, g) with respect to the holonomy group.

For a symmetric real analytic connection on an analytic manifold (satisfying some additional conditions), we can construct an algorithm that decides effectively whether the given connection is positive definite Riemannian or not, as explained in [555, 901].

We start with calculation of the Lie algebra of the restricted holonomy group in a generic point. The method, in principle based on the de Rham decomposition of Riemannian manifolds, makes use of linear algebra, deals with subspaces invariant under the action of the holonomy group, anables us to find all relevant geometric objects *unless* the metric tensor is known, and allows to construct all compatible Riemannian metrics in the affirmative case.

Let (M, g) be a given Riemannian manifold and ∇ the corresponding Levi-Civita connection. Fix a point $x \in M$. If $W \subseteq T_xM$ is a subspace of the tangent space at x, denote $W^\perp$ the g-orthogonal complement.

The tangent space T_xM has a natural decomposition with respect to the holonomy group $Hol = Hol_x^\nabla$. Indeed, denote by T_0 the maximal linear subspace of T_xM on which Hol acts trivially, $Hol|T_xM = id$. Consider the g-orthogonal complement ${T_0}^\perp$ in T_xM. Both T_0 and ${T_0}^\perp$ are Hol-invariant, and the complement can be decomposed into a direct sum of mutually orthogonal, Hol-invariant and Hol-irreducible subspaces: ${T_0}^\perp = \oplus_{i=1}^k T_i$, [90, I, p. 185]. Hence we get the so-called *de Rham decomposition* of the tangent space in x of the Riemannian manifold (M, g):

$$T_xM = T_0 \oplus {T_0}^\perp = \oplus_{i=0}^k T_i.$$

If M is simply connected then the decomposition is unique up to order. The holonomy group itself decomposes as a direct product of normal subgroups:

$$Hol = \Phi_0 \oplus \Phi_1 \oplus \ldots \oplus \Phi_k;$$

here $\Phi_0 = (id)$, Φ_i is irreducible on T_i, and Φ_i is trivial on T_j for any $j \neq i$.

Our first goal is to analyze in details and find the canonical decomposition of T_xM, unless g is used. In what follows, we need:

Lemma 4.6 [90, I, Appendix 5, p. 277] *If a subgroup $H \subset O(n)$ of the orthogonal group acts irreducibly on the vector space $\mathbb{R}^n$ then any H-invariant symmetric bilinear form G is a multiple of the standard inner product*

$$\langle x, y\rangle = \sum_{i=1}^n x^i \cdot y^i, \qquad x = (x^1, \ldots, x^n), \quad y = (y^1, \ldots, y^n).$$

Theorem 4.21 *Let (M, g) be a connected and simply connected smooth Riemannian manifold with the corresponding Riemannian connection ∇. Let Hol be the holonomy group of ∇ framed at the fixed point $x \in M$, and let*

$$T_xM = T_0 \oplus T_1 \oplus \ldots \oplus T_s \tag{4.145}$$

be the corresponding canonical Hol-invariant g-orthogonal de Rham decomposition of the tangent space where T_0 is the maximal subspace on which the holonomy group acts trivially, and $T_1, \ldots, T_s$ complete the list of all irreducible subspaces under the action. Let $T_i^\perp$ be the g-orthogonal complement of T_i in T_xM, $i \in \{0, 1, \ldots, s\}$. Then the subspace $H(x)$ introduced by (4.143) consists just of all linear combinations

$$G_0 + \lambda_1 G_1 + \ldots + \lambda_s G_s \in S^2(T_x^*M), \qquad \lambda_1, \ldots, \lambda_s \in \mathbb{R} \tag{4.146}$$

where G_0 runs over all forms such that $T_0^\perp \subseteq Ker\, G_0$, and $G_1, \ldots, G_s$ are fixed semi-definite forms such that G_i is positive definite on T_i, and $Ker\, G_i = T_i^\perp$, $i = 1, \ldots, s$.

Proof. To verify that all linear combinations of the shape (4.146) belong to $H(x)$ it is sufficient to verify that the forms $G_0, G_1, \dots, G_s$ with the properties declared above belong to $H(x)$. So let $G_0 \in S^2(T_x^*M)$ satisfies $T_0^\perp \subseteq Ker\, G_0$. Let $\varphi \in Hol$, and consider tangent vectors splitted with respect to the decomposition $T_0 \oplus T_0^\perp$, $X = X_0 + X_0'$, $Y = Y_0 + Y_0' \in \mathbb{T}_x M$. Then $G_0(\varphi(X), \varphi(Y)) = G_0(\varphi(X_0), \varphi(Y_0)) = G_0(X_0, Y_0) = G_0(X, Y)$, and G_0 is Hol-invariant. Let us introduce the form G_i by the following conditions: $Ker\, G_i = T_i^\perp$, $G_i|T_i = g|T_i$, $i \in 1, \dots, s$. Since the holonomy group preserves the scalar product g, it preserves also its restriction, i.e. $G_i|T_i$ is Hol-invariant, and G_i takes zero value whenever (at least) one of the arguments belongs to the complement $T_i^\perp$. Hence the forms G_i belong to $H(x)$ for $i = 1, \dots, s$, and their linear combinations as well.

On the other hand, let $G \in H(x)$ hold. To find the desired decomposition (4.146) let us show at first that whenever $X \in T_i$ and $Y \in T_j$, $i \neq j$, $0 \leq i, j \leq s$ we get $G(X, Y) = 0$. Take i, j and $Y \in T_j$ fixed. Then $X \mapsto G(X, Y) := L(X)$, $L : T_i \to \mathbb{R}$ defines a linear form $L \in T_i^*$ that is Φ_i-invariant (since G is invariant). But it means that $L = 0$ (due to irreducibility of the decomposition, under the dual representation of Hol in T_i^*, no fixed directions of forms are admitted). Hence the decomposition is G-orthogonal. Now let us define the form $G^{(i)}$ by setting $G^{(i)} = G|T_i$, $G^{(i)}(X, Y) = 0$ in other cases, $i \in \{0, 1, \dots, s\}$. Our construction guarantees that $G^{(i)} \in H(x)$, and $G = \sum_{i=0}^{s} G^{(i)}$. To finish the proof, let us note that the forms G_i introduced above have some privileged position. According to Lemma 4.6, any Hol-invariant symmetric bilinear form on T_i must be a scalar multiple of $g|T_i$, therefore a scalar multiple of $G_i|T_i$. Hence $G^{(i)}|T_i = G|T_i = \lambda_i G_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \dots, s$, $G^{(0)} = G_0$, and $G = G_0 + \sum_i \lambda_i G_i$ as required. □

Now let us examine regular forms from $H(x)$. Let $\hat{h}$ be a fixed regular form belonging to $H(x)$ expanded as in (4.146),

$$\hat{h} = \hat{h}_0 + \hat{\alpha}_1 G_1 + \dots + \hat{\alpha}_s G_s.$$

The restriction $\hat{h}_0|T_0$ is a symmetric, regular form (since $\hat{h}$ is regular, and $G_j|T_0$ are zero forms), and $\hat{\alpha}_j \neq 0$ for all $j = 1, \dots, s$. According to Lemma 4.6, the restriction $\hat{h}|T_j = \hat{\alpha}_j G_j|T_j$ is a non-zero multiple of the standard scalar product, hence it is either positive, or negative definite; due to Hol-invariance, the complements of T_0 with respect to $\hat{h}$ and g coincide, $\hat{T}_0 = T_0{}^\perp$. Moreover, $\hat{h}|\hat{T}_0$ is regular, i.e. $Ker(\hat{h}|\hat{T}_0) = \{0\}$. Further, let us fix a basis

$$\{G^1, \dots, G^p\} \tag{4.147}$$

of $H(x)$. Each base form can be again written as in (4.146),

$$G^\ell = G_0^\ell + \sum_j \lambda_j^\ell G_j, \qquad j = 1, \dots, s, \quad \ell = 1, \dots, p.$$

On the vector space $(T_x M, \hat{h})$ there are endomorphisms $S^1, \dots, S^p$ satisfying

$$\hat{h}(S^\ell X, Y) = G^\ell(X, Y), \qquad X, Y \in T_x M, \quad \ell = 1, \dots, p. \tag{4.148}$$

It can be easily checked when we use the matrix representation of the linear and bilinear maps occuring in the formula. The mappings S^ℓ are $\hat{h}$-symmetric. Let us compare the expressions

$$\begin{aligned}\hat{h}(S^\ell X, Y) &= G^\ell(X,Y) = G_0^\ell(X,Y) + \textstyle\sum_{j=1}^s \lambda_j^\ell G_j(X,Y),\\ \hat{h}(S^\ell X, Y) &= \hat{h}_0(S^\ell X, Y) + \textstyle\sum_{i=1}^s \hat{\alpha}_j^\ell G_j(S^\ell X, Y)\end{aligned}$$

calculated on pairs of decomposed vectors $X = \sum_{i=1}^s X_i$, $Y = \sum_{i=1}^s Y_i$, accounting properties of the forms $\hat{h}, G^\ell \in H(x)$ and the decomposition

$$S^\ell X = \sum_{i=0}^{s} S^\ell_{(i)} X \quad \text{with } S^\ell_{(i)} X \in T_i.$$

Let us introduce auxiliar linear endomorhisms $Q_i, E_0^\ell \in End(T_xM)$, $\ell = 1, \dots, p$

$$Q_i(X) = \frac{1}{\hat{\alpha}_i} X \ \text{ for } X \in T_i, \quad Q_i(X) = 0 \ \text{ for } X \in T_i^\perp, \ \ i = 1, \dots, s, \quad (4.149)$$

$$\hat{h}(E_0^\ell(X), Y) = G^i(X,Y) \ \text{ for } X, Y \in T_0, \quad E_0^\ell = 0 \ \text{ for } X \in T_0^\perp. \quad (4.150)$$

The operators satisfy $S^\ell_{(0)}(X) = E_0^\ell(X)$, $S^\ell_{(j)}(X) = Q_j(X)$, and are obviously $\hat{h}$-symmetric. We get finally

$$S^\ell = E_0^\ell + \sum_{i=1}^{s} \lambda_i^\ell Q_i, \qquad \ell = 1, \dots, p. \quad (4.151)$$

Note that the null-space of each E_0^ℓ contains $T_0^\perp$; this fact will be useful in what follows. Let us introduce the span in $End(T_xM)$ of all commutators of the above endomorphisms,

$$C_x = span\{[S^\ell, S^m] \mid \ell, m = 1, \dots, p\} \subset End(T_xM).$$

C_x will be called a *commutant* of the set $\{S^1, \dots, S^p\}$. Denote by N_x the common null-space in T_xM of all endomorphisms from C_x,

$$N_x = \{X \in T_xM \mid L(X) = 0 \ \text{ for all } L \in C_x\}.$$

Proposition 4.4 *In the above notation, if the commutator is non-trivial, $C_x \neq (0)$, then $T_0 = \hat{N}_x$ where $\hat{N}_x$ is the $\hat{h}$-orthogonal complement of N_x. In the trivial case, $C_x = (0)$, we get $\dim T_0 \in \{0, 1\}$. The restrictions $G^\ell|T_0$, $\ell = 1, \dots, p$ of elements of the distinguished basis $\{G^1, \dots, G^p\}$ of $H(x)$ are generators of the space $S^2T_0^*$ of all symmetric bilinear forms on T_0, and the restriction $\hat{h}|N_x$ is regular. The subspace $H(x)|N_x$ in $S^2N_x^*$ is a span of positive semi-definite forms $h_1, \dots, h_s$ with null-spaces $N_1, \dots, N_s$ such that the corresponding $\hat{h}|N_x$-orthogonal complements $\hat{N}_j$, $j = 1, \dots, s$ in N_x satisfy the $\hat{h}|N_x$-orthogonal decomposition*

$$\hat{N}_1 \oplus \dots \oplus \hat{N}_s = N_x. \quad (4.152)$$

Proof. Due to the definitions (4.149) and (4.150), $[Q_i, Q_j] = [E_0^\ell, Q_j] = 0$. Using the decompositions (4.151) we find

$$[S^\ell, S^m] = [E_0^\ell, E_0^m], \qquad \ell, m = 1, \dots, p. \tag{4.153}$$

Since the null-space of $[E_0^\ell, E_0^m]$ contains $T_0^\perp$, it follows that the null-space of any form belonging to C_x must contain the subspace $T_0^\perp$, hence

$$\hat{T}_0 = T_0^\perp \subseteq N_x, \qquad T_0 \supseteq \hat{N}_x. \tag{4.154}$$

Let us prove the other inclusion. First of all, note that by Lemma 4.6, the space of all $\hat{h}|T_0$-symmetric endomorphisms of the subspace T_0 is generated by the restrictions $S^\ell|T_0$, $\ell = 1, \dots, p$. Let us fix a $\hat{h}$-orthonormal basis $\langle e_1, \dots, e_r \rangle$ of the space $(T_0, \hat{h}|T_0)$. That is, $\hat{h}(e_i, e_i) = \varepsilon_i = \pm 1$ holds, and $\hat{h}(e_i, e_j) = 0$ for $i \neq j$, $i, j = 1, \dots, r$. Introduce endomorphisms E_{ij}, $1 \leq i \leq j \leq r$ by

$$E_{ij}(e_i) = \varepsilon_j e_j, \quad E_{ij}(e_j) = \varepsilon_i e_i, \quad E_{ij}(e_k) = 0 \ \text{ otherwise.} \tag{4.155}$$

It can be checked that $\{E_{ij}, 1 \leq i \leq j \leq r\}$ is a basis of the space of all $\hat{h}|T_0$-symmetric endomorphisms of T_0. Since each E_{ij} must be a linear combination of generators $S^\ell|T_0$ ($\ell = 1, \dots, p$) the brackets[69] $[E_{ij}, E_{kq}]$ are expressed through $[E_0^\ell, E_0^m]$. Further observe that the commutant C_x is in fact a span of all $[E_0^\ell, E_0^m]$. Consequently $[E_{ij}, E_{kq}]$ belong to the restriction $C_x|T_0$.

Let us examine the space $C_x|T_0$ and the brackets in detail. Consider the space of all skew-symmetric endomorphisms of the vector space $(T_0, \hat{h}|T_0)$ and introduce the operators ($i \neq j$)

$$J_i^j(e_i) = \varepsilon_j e_j, \quad J_i^j(e_j) = -\varepsilon_i e_i, \quad J_i^j(e_k) = 0 \ \text{ otherwise;} \tag{4.156}$$

$\{J_i^j \mid i, j = 1, \dots, r\}$ is a basis of the space, and $J_i^j = -\varepsilon_i [E_{ij}, E_{ii}]$ holds. Therefore the brackets $[E_{ij}, E_{ii}]$ span the space of all skew-symmetric endomorphisms of $(T_0, \hat{h}|T_0)$. Together with the above arguments, $C_x|T_0$ must contain all skew-symmetric endomorphisms of $(T_0, \hat{h}|T_0)$, particularly it contains all J_i^j. We can check that if $\dim T_0 \geq 2$ then the null-space N_x of the commutant C_x must be a subspace in $\hat{T}_0$ (in fact, if $X_0 = \sum_k a_k e_k \in T_0$ and $J_i^j(X_0) = 0$ for all i, j then $a_k = 0$ for all k, and $X_0 = 0$). Hence $\hat{T}_0 \supseteq N_x$, and finally, together with (4.154), $T_0 = \hat{N}_x$ whenever $C_x \neq \{0\}$. As far as the forms h_i are concerned, it suffices to choose

$$h_i = G_i|N_x, \qquad i = 1, \dots, s \tag{4.157}$$

where $G_i = g|T_i$ are the forms from Theorem 4.21, and the proof is finished if $\dim T_0 \neq 1$.

If $\dim T_0 = 1$ the situation is a bit different: the only skew-symmetric form on a 1-dimensional space is a trivial form, $C_x = \{0\}$, hence $N_x = T_x M$, and we take semi-definite forms $h_1, \dots, h_s, h_{s+1}$ where the attached form $h_{s+1} \in H(x)$

[69] Note that if Q, S are two endomorphisms symmetric w.r.t. the regular symmetric bilinear form $\hat{h}$ then the commutator $[Q, S]$ is skew-symmetric.

is a new form that has null-space $\hat{T}_0$ and is positive-definite on T_0, and the Proposition 4.4 holds with s replaced by $s+1$. □

As alreay mentioned, the restrictions $G^\ell|T_0$, $\ell \in \{1,\dots,p\}$, are generators of the space of all symmetric bilinear forms on T_0, $\hat{h}|N_x$ is regular, the restrictions $h_i = G_i|N_x$ are positive semi-definite forms with null-spaces N_i in N_x for all $i = 1,\dots,s$, N_x admits the orthogonal decomposition (w.r.t. $\hat{h}|N_x$) $N_x = \oplus_{i=1}^s \hat{N}_i$, and $H(x)|N_x \subset S^2N_x$ is a span of $\{h_1,\dots,h_s\}$.

Proposition 4.5 *Let $N_x = \hat{N}_1 \oplus \dots \oplus \hat{N}_s$ be the $\hat{h}$-orthogonal decomposition from (4.154). Let $h^{(1)},\dots,h^{(r)}$ be a basis of $H(x)|N_x$, and let $S^{(1)},\dots,S^{(r)}$ be the corresponding endomorphisms introduced by*

$$\hat{h}(S^{(\ell)}X,Y) = h^{(\ell)}(X,Y), \qquad \ell = 1,\dots,r. \tag{4.158}$$

For each endomorphism $S^{(\ell)}$, let us consider the set of all eigenspaces labelled e.g. by distinct eigenvalues $\lambda^\ell_1,\dots,\lambda^\ell_{p_\ell}$ of $S^{(\ell)}$ (written in some fixed order)

$$\{Z^{(\ell)}_{\lambda^\ell_1},\dots,Z^{(\ell)}_{\lambda^\ell_{p_\ell}}\}.$$

Hence we have a collection of eigenspaces (in some order)

$$\{Z^{(1)}_{\lambda^1_1},\dots,Z^{(1)}_{\lambda^1_{p_1}},\dots,Z^{(r)}_{\lambda^r_1},\dots,Z^{(r)}_{\lambda^r_{p_r}}\}.$$

Each of the eigenspaces $Z^{(\ell)}_\lambda$ can be expressed as a direct sum of some subspaces $\hat{N}_j$, and each of the subspaces $\hat{N}_j$ is an intersection of the form $Z^{(1)}_{\lambda^1_{j_1}} \cap \dots \cap Z^{(r)}_{\lambda^r_{j_r}}$ where $j_i \in \{1,\dots,p_i\}$, $i = 1,\dots,r$.

Moreover, $s = r$, and if we write $h^{(\ell)} = \sum_{j=0}^s \lambda^{(\ell)}_j h_j$ where h_j are forms from (4.157) then any restriction $h^{(\ell)}|\hat{N}_j$ is a multiple of the corresponding form $h_j|\hat{N}_j$.

4.6.5 Decision Algorithm

Let (M,∇) be connected, simply connected and analytic.

If $x \in M$ is a regular point the sequence of subspaces stabilizes for some natural number ν, $\underline{h}_\nu(x) = \underline{h}_{\nu+1}(x)$, and the same must hold in a neighbourhood. Using covariant differentiation we get $\underline{h}(y) = \underline{h}_\nu(y)$ at all points $y \in U_x$. So if we find the subspaces $\underline{h}_\nu(y)$ in a coordinate neighbourhood of a point x we are able to decide whether x is regular or not, and in a coordinate neighbourhood of the regular point, we calculate the algebra $\underline{h}(y)$ in the given coordinate chart. Let us establish the decision algorithm, according to which also a computer program can be performed.

Decision algorithm.
Step (1) Choose local coordinates in an open subset $U \subset M$ about x. Calculate the curvature tensor and, step by step, the covariant derivatives of the curvature

at $x \in U$. Determine the subspaces $\underline{h}_o(x) \subset \underline{h}_1(x) \subset \ldots$ at $x \in U$. If $\underline{h}_\nu(x) = \underline{h}_{\nu+1}(x)$ for some $\nu \in N \cup \{0\}$ the point x is *Hol*-regular.

Step (2) If the point $x \in U$ is *Hol*-regular continue and calculate, by means of the Frobenius Theorem, the space $H(x)$ defined in (4.143) as a solution space of the homogeneous system of algebraic equations obtained from (4.142) if we put $A = \nabla^r R$, $r = 0, \ldots, \nu$; that is, the equations from (4.102), (4.103) corresponding to $r \leq \nu$, or (4.104), (4.105), respectively. If $p = \dim H(x) = 0$, ∇ is neither Riemannian nor metrizable. If $p = \dim H(x) \geq 1$ choose a basis $G^1, \ldots, G^p$ of $H(x)$. Any element of the subspace $H(x)$ can be uniquely expressed as a linear combination of basic forms for some constants, $G = \lambda_1 G^1 + \ldots + \lambda_p G^p$. The local components G^ℓ_{ij} of the basic forms are rational functions of the components of covariant derivatives $R^i_{k\ell m, s_1, \ldots, s_k}$. We continue by

Step (3) As far as to find out whether there is a regular form in $H(x)$ compute the determinant $\det(\sum_m \lambda_m G^m_{k\ell})$, $k, \ell \in \{1, \ldots, n\}$ which is in fact a polynomial of independent variables λ_ℓ. If we get the zero polynomial, the connection ∇ is neither metrizable nor Riemannian. If the obtained polynomial is non-zero then there is a regular form in $H(x)$. Step by step, we choose suitable *integers* $\hat{\lambda}_1, \ldots, \hat{\lambda}_p$ and construct a particular regular form $\hat{h}$ from $H(x)$.

Step (4) From the formula (4.148), calculate the linear operators $S^1, \ldots, S^p$ corresponding to respective $G^1, \ldots, G^p$ via the regular form $\hat{h}$, and the commutant C_x of the set $\{S^1, \ldots, S^p\}$. In our local coordinates, find the null-space N_x of C_x. If N_x is not invariant under all of $S^1, \ldots, S^p$ or if $\hat{h}|N_x$ is not regular then ∇ is not Riemannian. Otherwise we continue.

Step (5) If $C_x = (0)$ go to Step (6). If $C_x \neq (0)$ first calculate the restrictions $G^\ell|\hat{N}_x = G^\ell|T^0_x$, $\ell = 1, \ldots, p$. If they do not generate $S^2(T^0_x)$ then ∇ is not Riemannian. If they generate $S^2(T^0_x)$ continue.

Step (6) Among the restrictions $S^1|N_x, \ldots, S^p|N_x$, find a set of independent generators $S^{(1)}, \ldots, S^{(s)}$ of the space $H(x)|N_x$. Calculate all eigenspaces of $S^{(1)}, \ldots, S^{(s)}$ and all possible intersections $Z^{(1)\alpha} \cap \ldots \cap Z^{(s)\gamma}$ of various eigenspaces of $S^{(1)}, \ldots, S^{(s)}$. Let the set of all intersections is $(0), L_1, \ldots, L_r$. Then the necessary conditions for ∇ to be Riemannian are:

- $r = s$, and $N_x = L_1 \oplus \ldots \oplus L_r$ (the orthogonal decomposition w.r.t. $\hat{h}$).

If the above conditions are satisfied go to Step (7).

Step (7) ∇ is Riemannian if and only if each of the restrictions $\hat{h}|L_j$ is either positive or negative definite, Theorem 4.20.

Note that if $n = 2$ and the manifold (M_2, ∇) is real analytic connected and simply connected then the decision procedure is much simpler, [908]: ∇ is Riemannian only in two cases, namely, at the given regular point x, either $p = \dim H(x) = 1$ and the space $H(x)$ is generated by a positive definite form, or $p = 3$, which happens just if and only if $R = 0$, and then the (flat) connection ∇ is Euclidean. A simplification can be also made when the curvature tensor od the given connection is regular, [554, 903].

The following results enable us to construct the metric explicitly.

Theorem 4.22 [555, p. 8] *Let (M, ∇) be a connected simply connected analytic manifold endowed with an analytic connection, let U_x be an open neighbourhood of $x \in M$ formed exclusively by regular points. Let the connection ∇ be metrizable and positive definite (Riemannian), and let $\hat{h} \in H(x)$ be regular on U. Let $H^{(1)}, \dots, H^{(t)}$ be analytic tensor fields on U such that for each $y \in U$, $H^{(1)}_y, \dots, H^{(t)}_y$ are linearly independent symmetric bilinear forms on the tangent space T_yM, with the common null-space equal to N_y. Let the restrictions $H^{(1)}|\hat{N}_y, \dots, H^{(t)}|\hat{N}_y$ to the complement $\hat{N}_y$ of N_y span the space $S^2(\hat{N}^*_y)$. Then there are 1-forms ω^i_j on U such that*

$$H^{(i)} = \sum \omega^i_j \otimes H^{(j)}, \qquad 1 \le i, j \le t. \tag{4.159}$$

The homogeneous system of linear PDE's

$$\mathrm{d}\lambda_i + \lambda_k \omega^k_i = 0, \qquad 1 \le i \le t \tag{4.160}$$

is always completely integrable.

Theorem 4.23 [555, p. 9] *Under the same assumptions and notation as above, suppose that $N_y = L_{1,y} \oplus \cdots \oplus L_{s,y}$ is the orthogonal decomposition w.r.t. $\hat{h}$ from Step (6) for any $y \in U_x$. Let h_i, $1 \le i \le s$, denote the tensor field on U such that its null-space at $y \in U$ coincides with the orthogonal complement of $L_{i,y}$ in T_yM w.r.t. $\hat{h}$, and which coincides with $\hat{h}$ on $L_{i,y}$ for any $y \in U$. Then there exist exact 1-forms ω_i, first integrals $\omega_i = \mathrm{d}f_i$, such that $\nabla h_i = \omega_i \otimes h_i$, $1 \le i \le s$, i.e. h_i are recurrent.*

Theorem 4.24 *Under the same assumptions as above, with $H^{(i)}$ and h_i analytic tensor fields on U, all admissible Riemannian metrics are of the form*

$$g = \sum_{i,k=1}^{t} b_i \lambda^i_k H^{(k)} + \sum_{k=1}^{s} c_k e^{-f_k} h_k, \tag{4.161}$$

where f_j is some primitive function of the exact differential form ω_j, $1 \le j \le s$, the functions $(\lambda^i_1, \dots, \lambda^i_t)$, $1 \le i \le t$ form a basis of the solution space of the completely integrable system from Theorem 4.22, and the real parameters b_i, c_k are chosen in such a way that g is positive definite.

4.6.6 Metrization of connections with regular curvature

We will keep the following notation. If M is a smooth n-dimensional manifold, $p\colon TM \to M$ denotes its tangent bundle, $\mathcal{X}(M)$ is the $\mathcal{F}(M)$-module of smooth vector fields on M where $\mathcal{F}(M)$ denotes the ring of smooth functions on M. Consider the vector bundles $\Lambda^2(TM)$, $\Lambda^2(T^*M)$, $\mathrm{Hom}(TM, TM)$, the vector space $\mathcal{L}(T_xM)$ of all homomorphisms $\Lambda^2(T_xM) \to End(T_xM)$, and the space $\mathcal{L}(TM)$ of all smooth bundle morphisms $\Lambda^2(TM) \to \mathrm{Hom}(TM, TM)$.

If (M, g) is a (pseudo-)Riemannian manifold (i.e. g is a metric on M of arbitrary signature) then its curvature tensor[70] R of type $(1,3)$ gives rise to the $(0,4)$ curvature tensor R_g, $R_g(X,Y,Z,W) = g(R(X,Y)Z,W)$, which is usually denoted by the same symbol R. It is a well known fact that among others, $R = R_g$ satisfies $R(X,Y,Z,W) = -R(Y,X,Z,W)$, $R(X,Y,Z,W) = -R(X,Y,W,Z)$, and $R(X,Y,Z,W) = R(Z,W,X,Y)$. Moreover it can be verified that at any point $x \in M$, R induces a homomorphism[71] $\hat{R}_x\colon \Lambda^2(T_xM) \to End(T_xM)$, $\sigma \mapsto \hat{R}_x(\sigma)$, such that if $\sigma = \sum_i c_i X_i \wedge Y_i \in \Lambda^2(T_xM)$ then

$$\hat{R}_x(\sigma)(Z) = \sum_i c_i R(X_i, Y_i)Z \quad \text{for any } Z \in T_xM. \tag{4.162}$$

Consequently, a bundle morphism $\hat{R}\colon \Lambda^2(TM) \to \mathrm{Hom}(TM, TM)$ is induced.

We pay now attention to some related algebraic structures with similar characteristic algebraic features or behaviour.

Curvature structures for inner product

Let us keep the following notation: if V is an n-dimensional real vector space, V^* denotes its dual, $End(V) = \mathrm{Hom}\,(V, V)$ is the vector space of all endomorphisms of V. The second exterior power[72] of V, $\Lambda^2(V)$, consists of antisymmetric type $(0,2)$ tensors on V. The space $\Lambda^2(V^*)$ of antisymmetric $(0,2)$ tensors on the dual V^* will be identified with the dual of $\Lambda^2(V)$, i.e. we use the identification $(\Lambda^2(V))^* \approx \Lambda^2(V^*)$. $S^2(V^*)$ denotes the space of all symmetric bilinear forms on V. $\mathcal{L}(V)$ denotes the space of all homomorphisms $\varrho\colon \Lambda^2(V) \to End(V)$.

A linear map $\varrho \in \mathcal{L}(V)$ will be called *regular* if any non-vanishing[73] decomposable bivector is mapped onto a non-zero endomorphism[74],

$$X, Y \in V, \quad X \wedge Y \neq 0 \implies \varrho(X \wedge Y) \neq 0.$$

Let $G \in S^2(V^*)$ be a fixed positive definite symmetric bilinear form on V.

Definition 4.4 Under a *curvature structure with respect to* G we mean a linear map $\varrho \in \mathcal{L}(V)$ such that the following two conditions hold $(X_1, X_2, Y_1, Y_2 \in V)$:

(i) the map $G(\varrho(X_1 \wedge X_2)-, -)\colon V^2 \to \mathbb{R}$ is antisymmetric, i.e. it satisfies

$$G(\varrho(X_1 \wedge X_2)(Y_1), Y_2) = -G(\varrho(X_1 \wedge X_2)(Y_2), Y_1);$$

(ii) the pairs (X_1, X_2), (Y_1, Y_2) are interchangeable,

$$G(\varrho(Y_1 \wedge Y_2)(X_1), X_2) = G(\varrho(X_1 \wedge X_2)(Y_1), Y_2).$$

[70] In terms of the Riemannian (Levi-Civita) connection ∇ of (M, g), the curvature (Riemannian) tensor is defined by $R(X,Y)(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ for $X, Y, Z \in \mathcal{X}(M)$.

[71] Given $X_1, X_2, Z \in T_xM$, $X_i' = a_i^j X_j$ then $X_1' \wedge X_2' = \det(a_i^j)\, X_1 \wedge X_2$, and we find easily that $R(X_1', X_2')Z = \det(a_i^j)\, R(X_1, X_2)Z$.

[72] Its elements, called bi-vectors, are of the form $\sum_{i,j} c^{ij} Z_i \wedge Z_j$, $Z_i \in V$, $c^{ij} \in \mathbb{R}$; elements of the form $X \wedge Y$ are called *decomposable*.

[73] Recall that $X \wedge Y = 0$ if and only if either $Y = 0$ or $X = kY$ for some $k \in \mathbb{R}$.

[74] Of course, we can characterize regularity by the equivalent condition that any non-zero bi-vector is mapped onto a non-zero endomorphism but the above condition is easier to check.

All curvature structures belonging to a fixed $G \in S^2(V^*)$ form a linear subspace $\mathcal{L}(V,G) \subset \mathcal{L}(V)$. The property (i) can be equivalently written as

$$\text{(i')}: G(\varrho(X_1 \wedge X_2)(Y_1), Y_2) + G(Y_1, \varrho(X_1 \wedge X_2)(Y_2)) = 0\,.$$

Remark that for any $\varrho \in \mathcal{L}(V)$ and $G \in S^2(V^*)$, the assignment

$$(\varrho, G) \mapsto \varrho_G, \quad \varrho_G(w, X \otimes Y) = G(\varrho(w)(X), Y), \quad w \in \Lambda^2(V),\ X, Y \in V$$

gives rise to a map $S^2(V^*) \times \mathcal{L}(V) \to \Lambda^2(V^*) \otimes (V \otimes V)^*$. There is a canonical injection $\iota\colon \Lambda^2(V^*) \otimes \Lambda^2(V^*) \to \Lambda^2(V^*) \otimes (V \otimes V)^*$. If we denote by $\mathcal{C}(V)$ the linear subspace of all symmetric tensors from $\Lambda^2(V^*) \otimes \Lambda^2(V^*)$, we can check:

ϱ is a curvature structure w.r.t. G if and only if $\varrho_G \in \mathcal{C}(V)$.

Lemma 4.7 *Let $\varrho \in \mathcal{L}(V,G)$ be a regular curvature structure with respect to a positive definite symmetric bilinear form $G \in S^2(V^*)$. Then for any vectors $X \in V\backslash\{0\}$, $Y \in V$ satisfying $G(X,Y) = 0$ (i.e. forming a G-orthogonal pair) there exists a bivector $w \in \Lambda^2(V)$ such that $\varrho(w)(X) = Y$.*

Proof. For arbitrary $X \in V$, $X \neq 0$, the subset of images of the above shape forms a linear subspace $W_X = \{\varrho(w)(X) \,|\, w \in \Lambda^2(V)\}$ in V. Since G is positive definite, $G(X,X) \neq 0$ holds, and $V = W_X \oplus W_X^{\perp}$. Let us check that W_X is just the orthogonal complement of $span\{X\}$, or equivalently, $span\{X\}^{\perp} = W_X$. Due to symmetry and (i'), $G(\varrho(w)(X), X) = 0$, therefore $\varrho(w)(X) \perp X$. Assume $Y \neq 0$ with $Y \perp X$. Consider the orthogonal decomposition $Y = Y_1 + Y_2$, $Y_1 \in W_X$, $Y_2 \in {W_X}^{\perp}$. Obviously, $Y_2 \perp \varrho(w)(X)$ for any $w \in \Lambda^2(V)$. Consequently, for any $Z_1, Z_2 \in V$, $G(\varrho(X \wedge Y_2)(Z_1), Z_2) = G(\varrho(Z_1 \wedge Z_2)(X), Y_2) = 0$. Hence $\varrho(X \wedge Y_2) = 0$. Due to $X \neq 0$ and regularity, the zero morphism can arise only if $Y_2 = kX$ for certain $k \in \mathbb{R}$. But $0 = G(X, Y_1 + Y_2) = G(X, Y_2) = kG(X,X)$, that is, $k = 0$, and $Y = Y_1 \in W_X$. Hence ${W_X}^{\perp} = span\{X\}$, and $Y = \varrho(w)(X)$ for some w whenever Y and X are G-orthogonal. □

For any $\varrho \in \mathcal{L}(V)$, let us introduce a linear subspace H_ϱ in $S^2(V^*)$ by

$$H_\varrho = \{F \in S^2(V^*) \,|\, F(\varrho(X_1 \wedge X_2)(Y_1), Y_2) + F(Y_1, \varrho(X_1 \wedge X_2)(Y_2) = 0\}. \quad (4.163)$$

That is, endomorphisms $\varrho(w)$, $w \in \Lambda^2(V)$ are skew-adjoint relative to any symmetric form $F \in H_\varrho \subset S^2(V^*)$. Obviously, $G \in H_\varrho$ whenever ϱ is a curvature structure relative G.

Theorem 4.25 *Let $G \in S^2(V^*)$ be positive definite. If ϱ is a regular curvature structure w.r.t. G then the space H_ϱ is 1-dimensional, $H_\varrho = span\{G\}$.*

Proof. Let $F \in H_\varrho$. We find $k \in \mathbb{R}$ such that $F = kG$. In (V,G), choose a G-orthonormal basis $\langle e_1, \dots e_n\rangle$ of V. For any pair $X \perp Y$ (orthogonal w.r.t. G), $X \neq 0$, we get orthogonality w.r.t. F. Indeed, by Lemma 4.7, $Y = \varrho(w)(X)$ for some $w \in \Lambda^2(V)$. Due to symmetry and (4.163), $F(X,Y) = F(X, \varrho(w)(X)) = -F(\varrho(w)(X), X) = -F(Y,X) = 0$. Consequently, $F(e_i, e_j) = 0$ for $i \neq j$, $1 \leq i, j \leq n$, and, since $e_i + e_j \perp e_i - e_j$, we get $0 = F(e_i + e_j, e_i - e_j) = F(e_i, e_i) - F(e_j, e_j)$. That is, $k = F(e_i, e_i) = F(e_j, e_j)$ must be a fixed constant. Hence $F(X,Y) = \sum_{i,j} X^i Y^j F(e_i, e_j) = \sum_{i=1}^n X^i Y^i F(e_i, e_i) = kG(X,Y)$, and $F = kG$ with $k = F(e_1, e_1)$. □

Riemannian metrics

Let (M, ∇) be an n-dimensional manifold endowed with a linear connection, and let R be its curvature. Let us use the above algebraic results on any fibre T_xM of TM, $x \in M$. We say that $x \in M$ is a *regular point* of $\varrho \in \mathcal{L}(TM)$ if ϱ_x is regular on T_xM, and that ϱ is *regular* on M if all points of M are regular.

If $G_x \in S^2(T_x^*M)$ is a positive definite scalar product on the tangent space T_xM, $x \in M$, then $\hat{R}_x \in \mathcal{L}(T_xM)$, derived from R by the formula (4.162), is surely a curvature structure[75] for G_x. If g is a Riemannian metric on M we define a curvature structure with respect to g pointwise, and introduce the subspace $\mathcal{L}(M, g) \subset \mathcal{L}(M)$; the curvature tensor R of (M, g) satisfies $R \in \mathcal{L}(M, g)$. Similarly as in (4.163), for every $x \in M$ we introduce a subspace $H_{\hat{R}_x} =: H^0(x)$ consisting of all $G_x \in S^2(T_x^*M)$ relative to which all elements $\hat{R}_x(X_1 \wedge X_2)$ are skew-adjoint, i.e. the following holds for any $X_1, X_2, Y_1, Y_2 \in T_xM$:

$$G_x(\hat{R}_x(X_1 \wedge X_2)Y_1, Y_2) + G_x(Y_1, \hat{R}_x(X_1 \wedge X_2)Y_2 = 0.$$

Their collection forms the bundle $H^0(M) \to M$, $\quad H^0(M) = \bigcup_{x \in M} H_{\hat{R}_x}$.

As a consequence of Lemma 4.7 and Theorem 4.25 we get

Corollary 4.4 *Let (M, g) be a Riemannian manifold such that each point of M is regular w.r.t. the curvature tensor R. Then at each point $x \in M$, the space $H^0(x) = H_{\hat{R}_x}$ is 1-dimensional, that is, $H^0(M)$ is a line-bundle.*

On a connected manifold M with $\dim M \geq 3$, a Riemannian metric is determined by its curvature R, provided the subset of R-regular points is dense, uniquely up to a scalar multiple, [554, p. 133].

Riemannian metrizability in regular case

Let us formulate necessary and sufficient metrizability conditions for linear connection with regular curvature tensor.

Recall that a one-form $\omega\colon M \to T^*M$ on M is *exact* (= gradient) if $\omega = df$ for a certain function f on M.

Theorem 4.26 *Let (M, ∇) be a manifold with torsion-free linear connection ∇, let the curvature R be regular on M, and let $H^0(M) = \bigcup_{x \in M} H_{\hat{R}_x}$ be the bundle corresponding to the curvature tensor. Then ∇ is a Riemannian connection of a positive-definite metric g if and only if the following conditions hold:*

(1) $H^0(M)$ is the line bundle,

(2) the bundle $H^0(M)$ is metric,

(3) any Riemannian metric $\tilde{g}\colon M \to H^0(M)$ is recurrent, $\nabla\tilde{g} = \omega \otimes \tilde{g}$, and the 1-form ω is exact on M.

[75] As above, we can introduce $R_{G_x,x}$ by $R_{G_x,x}(\sigma, Y \otimes Z) = G_x(\hat{R}_x(\sigma)(Y), Z)$ for $Y, Z \in T_xM$, $\sigma \in \Lambda^2(T_xM)$. Then we have a map $(G_x, \hat{R}_x) \mapsto R_{G_x,x}$ of $S^2(T_x^*M) \times \mathcal{L}(T_xM)$ to a particular subspace of $\Lambda^2(T_x^*M) \otimes (T_xM \times T_xM)^*$.

Proof. To verify that the conditions are sufficient, let $\tilde{g}\colon M \to H^0(M)$ be a Riemannian metric, and let $\nabla\tilde{g} = \delta f \otimes \tilde{g}$ for some function f. Then the tensor field $g = \exp(-f)\cdot\tilde{g}$ is parallel; $\nabla g = 0$. Therefore ∇ is the Levi-Civita connection of (M, g). The conditions are necessary according to [554]. □

Since the condition (1) means that $H^0(x)$ is one-dimensional at any point x, it is sufficient to suppose that the third condition (3) is satisfied for an arbitrary fixed metric. The second condition tells that $H^0(x)$ involves a positive definite symmetric bilinear form on each fibre T_xM, $x \in M$.

Real analytic case with regular curvature

We have seen an algorithm for positive definite metrics on an analytic, connected and simply connected manifold with an analytic affine connection. Briefly, the procedure was based on the philosophy that a manifold carries a structure invariant under parallel transport if and only if this stucture is invariant at a single point under the holonomy group (which can be expressed in terms of the corresponding Lie algebra). The Lie algebra of the holonomy group was generated by the curvature endomorphisms, arising from the curvature tensor and its covariant derivatives. All compatible positive metrics were described explicitely.

Of course, the algorithm discussed above allows us to answer the metrization problem even without regularity assumption, but if the curvature tensor is regular, the process is simplified considerably.

So let M be a connected simply connected analytic n-manifold endowed with an analytic symmetric affine connection ∇ whose curvature R is regular. Recall that in the analytic case, the holonomy group $\mathrm{Hol}(x)$ is a connected Lie subgroup of the automorphism (transformation) group $GL(T_xM)$ of the fibre, coincides with the restricted holonomy group (component of unit), $\mathrm{Hol}(x) = \mathrm{Hol}_0(x)$, and is therefore uniquely determined by its Lie algebra $\underline{hol}(x)$, i.e. its tangent space at unit. Holonomy groups in different points are isomorphic, hence we can define the abstract holonomy group of the connection, Hol^∇, [90, I], with the Lie *holonomy algebra* $\underline{hol}$. Recall that Hol_0^∇ is trivial if and only if the connnection is flat.

Furthermore, in the analytic case $\mathrm{Hol}_0(x) = \mathrm{Hol}'(x)$ (the infinitesimal holonomy group), the same for Lie algebras. But for smooth connections, $\underline{hol}'(x)$ is, as a vector space, a span of endomorphisms $\nabla^k R(X, Y, Z_1, \ldots, Z_k)$, $0 \le k < \infty$, $X, Y, Z_1, \ldots, Z_k \in T_xM$, [90, I]. Hence the restricted holonomy group of a real analytic connection is fully determined by values of all $\nabla^k R$, $0 < k$, in a point x.

The restricted holonomy group of any Riemannian manifold (M, g) is a closed connected subgroup of the orthogonal group, and in particular it is compact; $\mathrm{Hol}(x)$ identifies with a subgroup of $O(T_xM)$, g is $\mathrm{Hol}(x)$-invariant. For connected, simply connected M, it is sufficient to find a $\mathrm{Hol}(x)$-invariant positive definite $G_x \in S^2(T_x^*M)$ in one point $x \in M$, and to induce a compatible metric via parallel transport, [780], [555], [901]. The space of all $\mathrm{Hol}(x)$-invariant forms is characterized as a subspace $H(x) \subset S^2(T_x^*M)$ consisting just of all forms G_x satisfying

$$G(AX, Y) + G(X, AY) = 0 \tag{4.164}$$

for all $A \in \underline{hol}(x)$, $X, Y \in T_xM$. Introduce a sequence of subalgebras in $\underline{hol}(x)$ by

$$\underline{h}^{(r)}(x) = \mathrm{span}\,\{\nabla^k R(X, Y, Z_1, \ldots, Z_k) \,|\, 0 \le k \le r\}.$$

Note that $H^0(x)$ consists just of all forms with respect to which all elements $A \in \underline{h}^{(0)}(x)$ are self-adjoint (i.e. satisfy (4.164)); $H(x) \subset H^0(x)$ for all $x \in M$.

Lemma 4.8 ([901, L. 3], [555, p. 3]) *Let M be connected, simply connected manifold endowed with a torsion-free linear connection ∇, $x \in M$. A symmetric bilinear form G_x on T_xM is* Hol*-invariant if and only if $G_x \in H(x)$.*

If the manifold is connected it is sufficient to know the metric form at one point, and to enlarge it by parallel transport, hence the following holds.

Theorem 4.27 [780,901,903] *Let us given (M, ∇), M connected, ∇ symmetric. If there is a (non-degenerate) symmetric bilinear form $G_x \in H(x)$ in one point $x \in M$ then there exists on M a metric of the same signature and compatible with ∇.*

If $\dim \underline{h}^{(r)}(x)$ attains its maximum in some nbd U_x of $x \in M$ for all r, the point is called $\mathrm{Hol}(x)$*-regular*. If this is the case, there exists $N \in \mathbb{N}$ such that $\underline{h}^{(N)}(x) = \underline{h}^{(N+1)}(x) = \ldots$, and the same holds in some neighbourhood $U_x \ni x$. Consequently, for all $y \in U_x$, $\underline{h}^{(N)}(y) = \underline{hol}(y)$. Hence in a local chart, we are able to decide whether the point is $\mathrm{Hol}(x)$-regular and to calculate $\underline{hol}(y)$ if the answer is affirmative; the algorithm proceeds as follows:
Step (1). Choose a local chart (U, x^i). Calculate the curvature and its covariant derivatives at a $\mathrm{Hol}(x)$-regular point x up to the lowest order N for which the sequence $\underline{h}^{(r)}(x)$, $r \in \mathbb{N}$, stabilizes.
Step (2). Calculate $H^0(x)$, $H(x)$. If $\dim H(x) = 0$ the connection is not metrizable, [555]. In the Riemannian metrizable case, $\dim H^0(x) = 1$ must be satisfied according to the above. Hence the only case favourable for Riemannian metrizability is $\dim H(x) = \dim H^0(x) = 1$.
Step (3). If $H(x) = H^0(x) = \mathrm{span}\,\{G\}$ for some positive definite form G take $\tilde{g} = G$ (if not ∇ is not Riemannian).

The rest of the algorithm from [555] is trivial: the only endomorphism is identical, $S = id_{T_xM}$, with $N_x = T_xM$ being the null-space of the trivial commutant $C_x = \{0\}$, and $\tilde{g}|N_x = G$. We have the only generator $S = S^{(1)}$ with T_xM as its eigenspace, hence the required decomposition of the tangent bundle is trivial, $L = N_x = T_xM$, $\tilde{g}|L = G$ is positive definite. By [555], G must be recurrent, with the corresponding 1-form exact.
Step (4). We determine a function f with $\nabla G = df \otimes G$ if possible. In case there is no such function the connection ∇ is not metrizable.
Step (5). Compatible metrics are of the form $g = c \cdot \exp(-f) \cdot G$, $c > 0$.

Recall that if Hol of (M, g) is reducible then the universal cover of M is a Riemannian product; it is never the case if R is regular on M. The irreducible Hol_0 for Riemannian manifolds are listed and discussed in [21, pp. 643–647].

4.7 Harmonic diffeomorphisms and transformations

This section is devoted to a review of the results of the local and global geometries of harmonic diffeomorphisms of Riemannian manifolds obtained by S.E. Stepanov, I. G. Shandra [854, 867, 868].

4.7.1 Harmonic diffeomorphisms

Let (M, g) be a connected, smooth Riemannian manifold, ∇ be its Levi-Civita connection, and R be the corresponding curvature tensor defined by $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for smooth vector fields X and Y on (M, g). Denote by Ric and R the Ricci tensor and the scalar curvature, respectively.

Let $f : (M, g) \to (\overline{M}, \overline{g})$ be a smooth map between two Riemannian manifolds and let $f^*T\overline{M}$ be the pull-back bundle. The Levi-Civita connections on TM and $T\overline{M}$ induce a connection $\overline{\nabla}$ in the bundle of one-forms on M with values in $f^*T\overline{M}$. Then $\wp f := \overline{\nabla} df$ is a symmetric bilinear form on TM which is called the *second fundamental form of* f (see [721]). The trace of $\wp f$ with respect to g is called the tension field of $\wp f$ and is denoted by $\tau(\wp f)$. The map f is said to be *harmonic* if it satisfies the *Euler-Lagrange equations* (see, e.g., [395] for more details and references), that is,

$$\tau(\wp f) := \operatorname{trace}_g \overline{\nabla} df = 0,$$

where trace_g denotes the trace operator with respect to g.

Now, let $U \subset M$ be a domain with local coordinates $x_1, \ldots, x_m$, $m = \dim M$, and $V \subset M$ be a domain with local coordinates $y_1, \ldots, y_n$, $n = \dim \overline{M}$, such that $f(U) \subseteq V$. Assume that $f\colon (M, g) \to (\overline{M}, \overline{g})$ is locally represented by the equations

$$y^\alpha = y^\alpha(x^1, \ldots, x^m), \quad \alpha = 1, \ldots, n.$$

Then we have

$$\tau(\wp f)^\alpha_{ij} = \partial_i \partial_j f^\alpha - \Gamma^k_{ij} \partial_k f^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma}(f) \partial_i f^\beta \partial_j f^\gamma,$$

where $\partial_i = \partial/\partial x^i$ and $\Gamma^k_{ij}, \Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of (M, g), $(\overline{M}, \overline{g})$ respectively. So $f\colon (M, g) \to (\overline{M}, \overline{g})$ is a harmonic if and only if

$$g^{ij}(\partial_i \partial_j f^\alpha - \Gamma^k_{ij} \partial_k f^\alpha + \overline{\Gamma}^\alpha_{\beta\gamma} \partial_i f^\beta \partial_j f^\gamma) = \triangle f^\alpha + g^{ij} \overline{\Gamma}^\alpha_{\beta\gamma} \partial_i f^\beta \partial_j f^\gamma = 0, \quad (4.165)$$

where $\triangle$ is the Laplace operator of the manifold (M, g).

Let $f\colon \mathbb{V}_n (M, g) \to \overline{\mathbb{V}}_n (\overline{M}, \overline{g})$ be a diffeomorphism of Riemannian manifolds with equal dimensions. Then $\dim M = \dim \overline{M} = n$ and in terms of local coordinate systems $x^1, \ldots, x^n$ of $\mathbb{V}_n$ and $y^1, \ldots, y^n$ of $\overline{\mathbb{V}}_n$, we may assume that f is represented by the equations $y^i = x^i$. In this case, the differential df of the diffeomorphism $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ is represented by the matrix (δ^i_j), where δ^i_j is the Kronecker delta.

Let the diffeomorphism $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ be harmonic. Then the Euler-Lagrange equations (4.165) are equivalent to the following equations (see also [868]):

$$g^{ij}P^k_{ij} = 0, \tag{4.166}$$

where $P^k_{ij} = \overline{\Gamma}^k_{ij} - \Gamma^k_{ij}$ are components of the deformation tensor $P = \overline{\nabla} - \nabla$ in a common coordinate system $x^1, \ldots, x^n$ (see [29]). Therefore, the deformation tensor $P = \overline{\nabla} - \nabla$ is a section of the vector bundle $f^*T\overline{M} \otimes S^2_0M$, where S^2_0M is the space of traceless symmetric 2-tensor fields on $\mathbb{V}_n$.

Theorem 4.28 *The diffeomorphism $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ of a Riemannian manifold $\mathbb{V}_n$ with the Levi-Civita connection ∇ onto a Riemannian manifold $\overline{\mathbb{V}}_n$ with the Levi-Civita connection $\overline{\nabla}$ is harmonic if and only if*

$$g^{ij}P^k_{ij} = 0.$$

In [867], it was specified a method of classification of harmonic diffeomorphisms between Riemannian manifolds. This method is based on group representation theory. The authors of the paper have studied seven distinguished classes of harmonic diffeomorphisms (using the result of Theorem 4.28).

4.7.2 Harmonic transformations

A harmonic mappings of $\mathbb{V}_n$ onto itself is called a *harmonic transformations* on $\mathbb{V}_n$.

A vector field ξ on $\mathbb{V}_n$ is called an *infinitesimal harmonic vector fields*, if this field generates a local one-parameter group f_t of harmonic transformations in a neighbourhood U of any point $p \in M$. As a result of an infinitesimal transformation, the Christoffel symbols Γ^h_{ij} of the Levi-Civita connection ∇ take the form: $\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + t\,(\mathcal{L}_\xi\Gamma^h_{ij})$. Therefore, a local one-parameter group of infinitesimal point transformations generated by a vector field ξ is a *group of harmonic transformations (local harmonic diffeomorphisms)* if and only if ξ satisfies the equation [725]:

$$\mathrm{trace}_g(\mathcal{L}_\xi\nabla) = 0.$$

Definition 4.5 A vector field ξ is called an *infinitesimal harmonic transformation* in $\mathbb{V}_n$ if the local one-parameter group of infinitesimal point transformations generated by the vector field ξ is a group of harmonic transformations.

The following theorem is valid.

Theorem 4.29 [868] *A vector field ξ is an infinitesimal harmonic transformation in $\mathbb{V}_n$ if and only if $\triangle\xi = 2S\xi$.*

Now we present three examples of infinitesimal harmonic transformations.

Example 4.1 A vector field ξ' is a projective Killing vector field (see p. 273) if it satisfies the condition (6.51): $(\mathcal{L}_{\xi'}\nabla)(Y,Z) = \pi(Y)Z + \pi(Z)Y$ for any vector fields $X, Z \in TM$ and $\pi = d(\mathrm{div}X)/(n+1)$.

On the other hand, a vector field ξ'' is a conformal Killing vector field (see p. 250) if it satisfies the condition (5.44): $\mathcal{L}_{\xi'' g} = 2\varrho\, g$, where $\varrho = 1/n\ \mathrm{div}\xi''$.

For a conformal Killing vector field ξ'' we have

$$(\mathcal{L}_{\xi''}\nabla)(Y,Z) = (d\varrho)(Y)Z + (d\varrho)(Z)Y - g(Y,Z)(D\varrho)$$

for any vector fields $X, Z \in TM$ and $g(D_\varrho, Y) = (d\varrho)(Y)$.

Let $\xi = \xi' + \xi''$ is harmonic. We obtain $\operatorname{div}\xi' = \frac{1}{2}\, n(n-2)(n+1)\operatorname{div}\xi'' + \mathrm{const}$ which implies the following theorem.

Theorem 4.30 *For any projective Killing vector field ξ' and any conformal Killing vector field ξ'', the vector field $\xi = \xi' + \xi''$ is an infinitesimal harmonic transformation in $\mathbb{V}_n$ if and only if*

$$\operatorname{div}\xi' = 1/2\ n(n-2)(n+1)\operatorname{div}\xi'' + \ \mathrm{const}\,.$$

Example 4.2 A vector field ξ on a compact orientable proper Riemannian manifold $\mathbb{V}_n$ is an infinitesimal isometric transformation if and only if (see [197]) $\triangle\xi = 2S\xi \quad \text{and} \quad \nabla^*\xi = 0$. Therefore, any infinitesimal isometric transformation in $\mathbb{V}_n$ is an infinitesimal harmonic transformation.

Now let ξ be an infinitesimal conformal transformation in $\mathbb{V}_n$. Then

$$g(\triangle\xi, Y) = 2\, Ric(\xi, Y) + 1/n\ (n-2)\, Y(\operatorname{div}\xi)$$

for an arbitrary vector field Y on $\mathbb{V}_n$ (see [197]). Therefore, an infinitesimal conformal transformation in $\mathbb{V}_n$ is harmonic if $n = 2$ or if this transformation is an infinitesimal isometric transformation.

Example 4.3 Let ξ be a holomorphic vector field with local components ξ^k on an almost Kählerian manifold (M, g, J). Then

$$L_\xi J^i_j = \xi^k \nabla_k J^i_j - J^k_j \nabla_k \xi^i + J^i_k \nabla_j \xi^k = 0, \tag{4.167}$$

where (see [443]) $\nabla_k J^i_j + \nabla_j J^i_k = 0$. Applying the operator $\nabla^j = g^{jl}\nabla_l$ to (4.167), we find

$$J^{il} R_{kl} \xi^k + J^i_k \nabla^j \nabla_j \xi^k = 0. \tag{4.168}$$

From (4.168), by the convolution with J^h_i, we obtain $R^h_k \xi^k + \nabla^j \nabla_j \xi^h = 0$. This is a condition for ξ to be an infinitesimal harmonic transformation

Then we can state the following assertion.

Theorem 4.31 *An arbitrary holomorphic vector field on an almost Kähler manifold (M, g, J) is an infinitesimal harmonic transformation in (M, g, J).*

Example 4.4 The concept of Ricci solitons was introduced by R. Hamilton [457] in mid 80s. They are natural generalizations of Einstein metrics. It is well known

It is well known that a *Ricci soliton* (g, ξ, λ) is a Riemannian metric g together with a vector field ξ on M and a constant λ that satisfies the equation

$$-2\,Ric = \mathcal{L}_\xi g + 2\lambda\, g.$$

The Lie derivative of ∇ has the form $\mathcal{L}_\xi \Gamma^k_{ij} = \frac{1}{2}\, g^{kl}(\nabla_i \mathcal{L}_\xi g_{jl} + \nabla_j \mathcal{L}_\xi g_{il} - \nabla_l \mathcal{L}_\xi g_{ji})$, see [197]. From this equation and the equation of a Ricci soliton we obtain

$$g^{ij}\mathcal{L}_\xi \Gamma^k_{ij} = g^{kj}(-2\nabla_l R^l_j + \nabla_j R) = 0$$

for the scalar curvature $R = g^{ij}R_{ij}$. Here we used the Schur lemma $2\nabla_j R^j_k = \nabla_k R$.

Theorem 4.32 [870] *A vector field ξ that makes a Riemannian metric g into a metric of a Ricci soliton is necessarily and infinitesimal harmonic transformation.*

A $(0,2)$-tensor field φ on a Riemannian manifold $\mathbb{V}_n$ is said to be harmonic (see [438]) if φ has zero divergence (i.e. $\delta\varphi = 0$). We can give a geometric characterization of infinitesimal harmonic transformations in terms of harmonic tensors. The following theorem is proved directly.

Theorem 4.33 *A vector field ξ on a Riemannian manifold $\mathbb{V}_n$ is an infinitesimal harmonic transformation in $\mathbb{V}_n$ if and only if $\varphi = \mathcal{L}_\xi g - (\operatorname{div}\xi)\, g$ is a harmonic tensor.*

A. Lichnerowicz proved the following remarkable theorem (see [868]): A holomorphic vector field ξ in a compact Kählerian manifold (M, g, J) with constant scalar curvature can be decomposed into the sum $\xi = \xi' + J\,\xi''$ of two infinitesimal isometric transformations ξ' and ξ''. Then combining the result of Yano and the Lichnerowicz theorem we arrive at the following theorem.

Theorem 4.34 *If a vector field ξ is a harmonic transformation in a compact Kählerian manifold (M, g, J) with constant scalar curvature, then ξ can be decomposed into the sum $\xi = \xi' + J\,\xi''$ of two infinitesimal isometric transformations ξ' and ξ''.*

The following integral formula is well known (see [199]):

$$\int_M \left[g(\triangle\xi, \xi) - 1/2\ \|d\xi\|^2 - (\nabla^*\xi)^2\right] dV = 0. \tag{4.169}$$

Substituting $\triangle\xi = 2S\xi$ into (4.169) we obtain the following assertion.

Theorem 4.35 *An orientable compact Riemannian manifold $\mathbb{V}_n$ with negative-definite Ricci curvature does not admit infinitesimal harmonic transformations.*

Let $\mathbb{V}_n$ be an oriented compact Riemannian manifold and S^2M be the bundle of symmetric bilinear differential forms on $\mathbb{V}_n$. Using the linear differential operator

$$\delta^*:\ T^*M \to S^2M, \qquad \delta^*\xi = L_\xi g,$$

we define the linear differential operator

$$\delta\colon\ S^2M \to T^*M$$

as the adjoint operator of δ^* with respect to the global product

$$\langle\varphi,\varphi'\rangle = \int_M \frac{1}{2}\ g(\varphi,\varphi')dV.$$

The main purpose of the final part of this section is to develop the geometry of the second-order self-adjoint differential operator

$$\Box : T^*M \to T^*M, \qquad \Box = \delta\delta^* - \delta^*\delta.$$

Applying the Ricci identities, we obtain the following Weitzenböck decomposition:

$$\Box\theta = \triangle\theta - 2S\xi,$$

where $\xi = \theta^{\#}$ (see [23, p. 30]). This operator is called the *Yano operator* (see also [866]).

Theorem 4.36 [868] *A vector field ξ on a Riemannian manifold is an infinitesimal harmonic transformation in $\mathbb{V}_n$ if and only if $\xi \in \ker\Box$.*

The symbol σ of the Yano operator $\Box$ satisfies the condition

$$\sigma(\Box)(\theta,x)\varphi_x = -g(\theta,\theta)\varphi_x$$

for arbitrary $\theta \in T_x^* - \{0\}M$ and $x \in M$. Consequently, the Yano operator $\Box$ is the Laplacian operator and its kernel is a finite-dimensional vector space (see [23, p. 464]). We denote by $H(M,\mathbb{R})$ the vector space of infinitesimal harmonic transformations in $\mathbb{V}_n$. In particular, this implies the following theorems.

Theorem 4.37 *Let $\mathbb{V}_n$ be a compact Riemannian manifold. Then the vector space $H(M,\mathbb{R})$ of infinitesimal harmonic transformations in $\mathbb{V}_n$ is finite-dimensional.*

Theorem 4.38 *If the Ricci curvature Ric of an orientable compact property Riemannian manifold is negative definite, then dim $H(M,\mathbb{R}) = 0$.*

5 CONFORMAL MAPPINGS AND TRANSFORMATIONS

5.1 Conformal and isometric mappings

5.1.1 Introduction to the theory of conformal and isometric mappings

A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation that preserves local angles. An analytic function is conformal at every point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives, is analytic. Conformal mappings are extremely important in complex analysis, as well as in many areas of physics and engineering.

Definition 5.1 Let $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ be Riemannian manifolds. A diffeomorphism $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ is called *conformal* if it preserves angles between any pair of curves.

Note that in the above definition, diffeomorphism can be substituted by a bijection of sufficiently high class of differentiability.

A mapping that preserves the magnitude of angles, but not their orientation is called an isogonal mapping (Churchill and Brown 1990, p. 241).

Several important cartographic projections, including the Mercator projection, are conformal maps. In 1779, the first non-trivial example of conformal mappings were discovered by Lagrange [103], namely the *stereographic projection* of a sphere. In geometry, the stereographic projection is a particular mapping that projects a sphere onto a plane (usually, from the "north pole" either to the tangent plane in the "south pole", or to the equatorial plane). The projection is defined on the entire sphere, except at one point, the projection point. The mapping is smooth and bijective on its definition domain. It is conformal, meaning that it preserves angles. It is neither an isometry nor area-preserving: that is in general, it preserves neither distances nor the areas of figures.

A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo-)Riemannian metric tensors, in which two metrics g and $\overline{g}$ are equivalent if and only if $\overline{g} = \sigma \cdot g$, where $\sigma > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class. Thus a conformal metric may be regarded as a metric that is only defined "up to scale". Often, conformal metrics are treated by selecting a metric in the conformal class, and applying only "conformally invariant" constructions to the chosen metric.

A conformal metric is locally conformally flat (or often just conformally flat) if there is a metric representing it, that is, flat in the usual sense - the Riemann

tensor vanishes on an open subset. A manifold with a (locally) conformally flat metric is called a (locally) conformally flat manifold. A locally conformally flat manifold is locally conformal to a Möbius geometry meaning that there exists an angle preserving local diffeomorphism from the manifold into a Möbius geometry. In dimension two, every metric is locally conformally flat. Thus two-dimensional conformal geometry is identical with the study of two-dimensional Möbius geometry. In dimension $n > 3$, a conformal metric is locally conformally flat if and only if its Weyl tensor vanishes; in dimension $n = 3$, if and only if the Cotton tensor vanishes.

Conformal geometry has a number of features which distinguish it from (pseudo-)Riemannian geometry. The first is that although in (pseudo-)Riemannian geometry one has a well-defined metric at each point, in conformal geometry, one has a class of metrics only. Thus the angle between two vectors can be still defined, while the length of a tangent vector cannot be defined. Another feature is, that there is no Levi-Civita connection, because if g and $\overline{g}$ are two representatives of the conformal structure, then the Christoffel symbols of g and $\overline{g}$ would not agree in general. Those associated with $\overline{g} = \sigma \cdot g$ would involve derivatives of the function σ whereas those associated with g would not.

Despite these differences, conformal geometry is still tractable. The Levi-Civita connection and curvature tensor, although only being defined once a particular representative of the conformal structure has been singled out, do satisfy certain transformation laws involving σ and its derivatives when a different representative is chosen. In particular (in dimension higher than 3), the Weyl tensor turns out not to depend on σ, and so it is a conformal invariant. Moreover, even though there is no Levi-Civita connection on a conformal manifold, one can work with a conformal connection instead, which can be handled either as a type of Cartan connection modeled on the associated Möbius geometry, or as a Weyl connection. This allows one to define conformal curvature, as well as other invariants of the conformal structure.

Later on, when the tensor approach started to be more common in differential geometry, H. Weyl[76], L.P. Eisenhart[77] and others gave a new, invariant and more flexible theory of conformal mappings. A modification of such a viewpoint can be found e.g. in [78] V.F. Kagan[78].

Conformal mappings of the space on itself are called *conformal transformation.* Conformal transformations of Riemannian manifolds form a Lie group.

The investigation of these groups was started by L. Fubini [422] see [8, 11, 45–47, 87, 89, 176, 254, 422, 423, 846].

[76] Hermann Klaus Hugo Weyl, 1885–1955, a German mathematician who contributed to theoretical physics as well as to pure disciplines including number theory.

[77] Luther Pfahler Eisenhart, 1876–1965, a mathematician in Princeton, USA, an author of famous textbooks, [50–52].

[78] Veniamin Fedorovich Kagan, 1869–1953, a Russian mathematician, Professor in Odessa and at Moscow State University; an author of a two-volume textbook on geometry [78] published in Russia after the second war.

5.2 Main properties of conformal mappings

5.2.1 Fundamental equations of conformal mappings

Now let us pay attention to conformal mappings of Riemannian manifolds.

So let us consider Riemannian manifolds $\mathbb{V}_n = (M, g)$ and $\overline{\mathbb{V}}_n = (\overline{M}_n, \overline{g})$ endowed with metrics g and $\overline{g}$, respectively. All geometric objects in $\overline{\mathbb{V}}_n$ will be denoted by analogous letters as in $\mathbb{V}_n$, but with "bar".

Assume a conformal mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$. Since f is a diffeomorphism we can suppose that local coordinate maps on M or $\overline{M}$, respectively, are chosen as described in (4.2). That is, locally, $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ maps points onto points with the same coordinates, and $\overline{M} = M$, which considerably simplifies the technicalities: we suppose that all objects under consideration (as connections, tensor fields etc.), with bar or without, are defined on the same underlying manifold.

Recall that if X and Y are tangent vectors of two intersecting curves in the intersection point then their angle is $\alpha = \angle(X, Y)$.

Conformal mappings were defined by Definition 5.1 as angle-preserving diffeomorphisms, i.e.

$$\cos(\alpha)_g = \cos(\alpha)_{\overline{g}}$$

or in more details,

$$\frac{g(X,Y)}{\sqrt{g(X,X)\,g(Y,Y)}} = \frac{\overline{g}(X,Y)}{\sqrt{\overline{g}(X,X)\,\overline{g}(Y,Y)}}.$$

This formula is satisfied in any point of $\mathbb{V}_n$ and for arbitrary tangent vectors X and Y, and by examining it we obtain

Theorem 5.1 *A mapping f of $\mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$ is conformal if and only if the metric tensors satisfy $\overline{g} = \mathrm{e}^{2\sigma} \cdot g$, where σ is a function.*

As already mentioned, a conformal mapping is geometrically characterized as a mapping preserving angles, but is often equivalently described just by the condition $\overline{g} = \mathrm{e}^{2\sigma} \cdot g$ which in local notation reads

$$\overline{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)} \cdot g_{ij}(x). \tag{5.1}$$

Remark 5.1 The formulas (5.1) are often taken as a definition of conformal mappings. It follows naturally from the fact that the proof of Theorem 5.1 is not trivial, and for most applications we can restrict ourselves to this condition.

Remark 5.2 If $\sigma = \text{const}$, then the conformal mapping f is *homothetic* (Definition 2.2, p. 69), and if $\sigma = 0$, then f is *isometric* (Definition 4.3, p. 198).

5.2.2 Equivalence classes of conformal mappings

It follows from the equations for conformal mappings that the class of manifolds with affine connection is decomposed into equivalence subclasses with respect to conformal mappings.

A conformal mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$, which is characterized by the equations (5.1) will be denoted by

$$\mathbb{V}_n \xrightarrow{c.m.\ (\sigma)} \overline{\mathbb{V}}_n.$$

The properties of an equivalence relation can be easily checked:

1. The identity is conformal since we have $\mathbb{V}_n \xrightarrow{c.m.\ (0)} \overline{\mathbb{V}}_n$.
2. If $\mathbb{V}_n \xrightarrow{c.m.\ (\sigma)} \overline{\mathbb{V}}_n$, then $\overline{\mathbb{V}}_n \xrightarrow{c.m.\ (-\sigma)} \mathbb{V}_n$.
3. If $\mathbb{V}_n \xrightarrow{c.m.\ (\sigma)} \overline{\mathbb{V}}_n$ and $\overline{\mathbb{V}}_n \xrightarrow{c.m.\ (\overline{\sigma})} \overline{\overline{\mathbb{V}}}_n$, then $\mathbb{V}_n \xrightarrow{c.m.\ (\sigma+\overline{\sigma})} \overline{\overline{\mathbb{V}}}_n$.

Two manifolds $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ belong to the same *conformal class* iff there is a conformal mapping of $\mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$. Spaces from the same conformal class are also called *conformal equivalent*. We can apply this concept locally as well as globally.

5.3 Some geometric objects under conformal mappings

5.3.1 Christoffel symbols under conformal mappings

Consider two Riemannian manifolds $\mathbb{V}_n = (M_n, g)$ and $\overline{\mathbb{V}}_n = (\overline{M}_n, \overline{g})$ (with the corresponding affine connections ∇ and $\overline{\nabla}$, respectively). Suppose there exists a conformal map $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$. Let us identify $\overline{M}$ with M via f, so that the deformation tensor $P = \overline{\nabla} - \nabla$ of f can be defined. Then (5.1) hold.

Under conformal mappings the following relation holds:

$$\overline{\nabla}(X,Y) = \nabla(X,Y) + \sigma(X)\,Y + \sigma(Y)\,X - g(X,Y)\,\boldsymbol{\sigma}, \tag{5.2}$$

where $\boldsymbol{\sigma}$ is a vector field and σ is a gradient one-form: $\sigma(X) = \nabla_X \sigma(x)$ and $\sigma(X) = g(X, \boldsymbol{\sigma})$, in local transcriptions we have

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h\, g_{ij}, \tag{5.3}$$

where $\sigma_i = \partial_i \sigma(x)$ and $\sigma^h = \sigma_\alpha g^{\alpha h}$ are components of the form σ and the vector $\boldsymbol{\sigma}$.

We can get condition (5.2) as follows. First we write down the Christoffel symbols of type I for the metric $\overline{g}$, $\overline{\Gamma}_{ijk} = \frac{1}{2}\left(\partial_i \overline{g}_{jk} + \partial_j \overline{g}_{ik} - \partial_k \overline{g}_{ij}\right)$, now we insert (5.1) and after simple computations, we get

$$\overline{\Gamma}_{ijk} = \mathrm{e}^{2\sigma(x)} \left(\tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) + \sigma_i g_{jk} + \sigma_j g_{ik} - \sigma_k g_{ij}\right),$$

or equivalently, by (3.7):

$$\overline{\Gamma}_{ijk} = \mathrm{e}^{2\sigma(x)} \left(\Gamma_{ijk} + \sigma_i g_{jk} + \sigma_j g_{ik} - \sigma_k g_{ij}\right).$$

Christoffel symbols of type II arise as in (5.2). Here we find that

$$\overline{g}^{ij} = \mathrm{e}^{-2\sigma(x)}\, g^{ij} \tag{5.4}$$

where g^{ij} and $\overline{g}^{ij}$ are inverse matrices of g_{ij} and $\overline{g}_{ij}$.

5.3.2 Riemannian and Ricci tensor under conformal mappings

Applying the formula (4.3) and the expression of the deformation tensor P under a conformal mapping $\mathbb{V}_n \to \overline{\mathbb{V}}_n$: (5.2) ((5.3), respectively) we find a relationship between the Riemannian tensors (or curvature tensors):

$$\begin{aligned}\overline{R}(Y,Z)X = R(Y,Z)X + \sigma(X,Y)Z - \sigma(X,Z)Y \\ +g(X,Y)\boldsymbol{\sigma}(Z) - g(X,Z)\boldsymbol{\sigma}(X) + (g(X,Y)Z - g(X,Z)X)\,\Delta_1\sigma\end{aligned} \tag{5.5}$$

where $\Delta_1\sigma = g(\boldsymbol{\sigma},\boldsymbol{\sigma})$, $\sigma(X,Y) = g(X,\boldsymbol{\sigma}(Y))$, and

$$\sigma(X,Y) = \nabla_Y\sigma(X) - \sigma(X)\sigma(Y) \tag{5.6}$$

which in components reads

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\sigma_{ij} - \delta^h_j\sigma_{ik} + \sigma^h_k g_{ij} - \sigma^h_j g_{ik} + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\,\Delta_1\sigma \tag{5.7}$$

where

$$\sigma_{ij} = \sigma_{i,j} - \sigma_i\sigma_j, \quad \sigma^h_k = g^{h\alpha}\sigma_{\alpha k}, \quad \Delta_1\sigma = g^{\alpha\beta}\sigma_\alpha\sigma_\beta, \tag{5.8}$$

the comma is a symbol of covariant derivation on $\mathbb{V}_n$.

If we contract (5.7) over h and k we obtain the following relation for the Ricci tensors:

$$\overline{Ric}(X,Y) = Ric(X,Y) - (n-2)\,\sigma(X,Y) - [\Delta_2\sigma + (n-2)\Delta_1\sigma]\,g(Y,X) \tag{5.9}$$

since

$$\overline{R}_{ij} = R_{ij} - (n-2)\sigma_{ij} - [\Delta_2\sigma + (n-2)\Delta_1\sigma]\,g_{ij} \tag{5.10}$$

where $\Delta_2\sigma = g^{\alpha\beta}\sigma_{\alpha,\beta}$; Δ_1 and Δ_2 are the first and second Beltrami operators.

If we contract (5.10) with respect to $\overline{g}^{ij}$ we obtain the following relation for the scalar curvatures:

$$\mathrm{e}^{2\sigma}\,\overline{R} = R - 2(n-1)\,\Delta_2\sigma - (n-1)(n-2)\,\Delta_1\sigma, \tag{5.11}$$

where $\overline{R}$ and R denote the scalar curvature of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$.

5.3.3 Weyl tensor of conformal curvature

From (5.10) and (5.11) we obtain

$$\overline{L}_{ij} = L_{ij} - \sigma_{ij} - \frac{1}{2}\,\Delta_1\sigma\, g_{ij}, \tag{5.12}$$

where

$$L_{ij} = \frac{1}{n-2}\,\left(R_{ij} - \frac{R}{2(n-1)}\,g_{ij}\right) \tag{5.13}$$

is the Brinkmann tensor.

Substituting σ_{ij} into (5.7) we find, after a simple computation, $\overline{C} = C$ which in components reads

$$\overline{C}^h_{ijk} = C^h_{ijk} \tag{5.14}$$

where

$$C^h_{ijk} = R^h_{ijk} - \delta^h_j L_{ik} + \delta^h_k L_{ij} - L^h_j g_{ik} + L^h_k g_{ij}, \tag{5.15}$$

and $L^h_i = g^{h\alpha}L_{\alpha i}$. Analogously we define $\overline{C}$ on $\overline{\mathbb{V}}_n$.

Definition 5.2 The tensor field C of the type $(1,3)$ introduced by (5.15) is called the *tensor of conformal curvature* of the Riemannian manifold $\mathbb{V}_n$.

It is also called the *Weyl tensor of conformal curvature* after Weyl who discovered it in 1921, see [941].

We realize that the tensor of conformal curvature C on a Riemannian manifold $\mathbb{V}_n$, as well as the curvature and the Ricci tensor, is *uniquely defined* by its metric tensor. As well known, C is trace-less.

From the formula (5.14) above the following theorem follows.

Theorem 5.2 *The Weyl tensor of conformal curvature is an invariant of conformal mappings of (pseudo-) Riemannian manifolds.*

In the paper by Rapczak [765] a construction of other geometric objects of intrinsic character that are invariants of conformal mappings is presented.

5.4 Conformally flat manifolds

An important class of Riemannian manifolds are those that are conformally mapped onto (pseudo-) Euclidean spaces.

Definition 5.3 A Riemannian space $\mathbb{V}_n$ admitting locally a conformal mapping onto a (pseudo-) Euclidean space is called *conformally flat*.

From the definition of flat manifolds (p. 85) and conformal mappings, particularly the formula (5.1), it follows that in a neighborhood of any point of a conformally flat manifold there is a coordinate system in which

$$ds^2 = f(x)(e_1(dx^1)^2 + e_2(dx^2)^2 + \ldots + e_n(dx^n)^2), \quad e_i = \pm 1 \tag{5.16}$$

where $f \neq 0$ is a function.

In a conformally flat manifold the Weyl tensor of conformal curvature C is always zero. It follows from the fact that for flat manifolds (that are characterized by zero Riemannian tensor), from (5.15) follows $C = 0$. And since conformal curvature is an invariant of conformal mappings the assertion follows.

We can ask what are the necessary and sufficient conditions for $\mathbb{V}_n$ to be conformally flat? First recall that in two- and three-dimensional $\mathbb{V}_n$ the tensor C always vanishes. The following holds:

Theorem 5.3 *An arbitrary 2-dimensional $\mathbb{V}_2$ is always conformally flat, i.e. in $\mathbb{V}_2$ there is always a metric form*

$$ds^2 = f(x)(e_1 dx^{1^2} + e_2 dx^{2^2}), \quad e_1, e_2 = \pm 1. \tag{5.17}$$

The above coordinate system is called *isothermic*. It follows that any two 2-dimensional Riemannian manifolds with the same signature of metric are locally conformally equivalent. It also follows, among others, that any two surfaces in $\mathbb{E}_3$ are locally conformally equivalent.

Theorem 5.4 *A 3-dimensional $\mathbb{V}_3$ is locally conformally flat if and only if*

$$L_{ij,k} = L_{ik,j}, \tag{5.18}$$

where L_{ij} is the Brinkmann tensor.

Theorem 5.5 *An n-dimensional Riemannian manifold $\mathbb{V}_n$ $(n > 3)$ is locally conformally flat if and only if the Weyl tensor of conformal curvature vanish, i.e.*

$$C = 0. \tag{5.19}$$

We omit the general proof of Theorem 5.3 here; we restrict ourselves to the case $n > 2$ only. So assume a locally conformally flat $\mathbb{V}_n$, $n > 2$, and admit a conformal mapping onto a flat space $\overline{\mathbb{V}}_n$. Then the metrics g and $\overline{g}$ are connected by (5.1) and the formulae (5.5)-(5.15) hold.

From the formula (5.14) we get that the Riemannian tensor takes the form

$$R_{hijk} = g_{hj}L_{ik} - g_{hk}L_{ij} + g_{ik}L_{hj} - g_{ij}L_{hk}. \tag{5.20}$$

From the formulas (5.12) and (5.13) it follows that

$$\sigma_{,ij} = \sigma_{,i}\sigma_{,j} - \frac{1}{2}\,\Delta_1\sigma g_{ij} + L_{ij}, \tag{5.21}$$

where $\Delta_1\sigma = \sigma_{,\alpha}\sigma_{,\beta}g^{\alpha\beta}$. We ask under what conditions there is a function $\sigma(x)$ for which (5.21) holds. Using the Ricci identity $\sigma_{i,jk} - \sigma_{i,kj} = \sigma_\alpha R^\alpha_{ijk}$ and the conditions (5.20) we find that the integrability conditions for (5.21) read

$$L_{ij,k} = L_{ik,j}. \tag{5.22}$$

The following can be easily checked:

Theorem 5.6 *A Riemannian space $\mathbb{V}_n$ is locally conformally flat if and only if in $\mathbb{V}_n$, the conditions (5.20) and (5.22) hold.*

Because in $\mathbb{V}_3$ the tensor C is zero, as a consequence we get the Theorem 5.4.

The condition that $C = 0$ is equivalent to the formula (5.20). Substituting to the corollary (3.13a) of the Bianchi identity we get, under the assumption $n > 3$, the conditions (5.18); this finishes the proof of the Theorem 5.5. □

Note that the conditions (5.22) are satisfied is spaces of constant curvature, hence we get:

Theorem 5.7 *A space of constant curvature is locally conformally flat.*

Among others, a metric in a space of constant curvature K is of the form (3.18).

5.5 Conformal mappings onto Einstein spaces

5.5.1 Linear equations of conformal mappings onto Einstein spaces

J. Mikeš, M.L. Gavril'chenko and E.I. Gladysheva [660, 661] obtained that the main equations for a conformal mapping of a Riemannian space $\mathbb{V}_n$ onto an Einstein space has been expressed as a Cauchy-type homogeneous linear system of differential equations in covariant derivatives with respect to $(n+2)$ unknown functions. The conformal mappings of symmetric spaces onto Einstein spaces have been studied in detail. In particular, it has been proved that compact conformally Euclidean symmetric spaces do not admit non homothetic conformal mappings "on the whole" onto Einstein spaces.

Conformal mappings of arbitrary Riemannian spaces $\mathbb{V}_n$ onto Einstein spaces were studied by Brinkmann [314]. A detailed treatment of his investigations is given in the monograph by Petrov [139]. The question whether a Riemannian space $\mathbb{V}_n$ admits a conformal mapping onto some Einstein space $\mathbb{V}_n$ was reduced by Brinkmann to the question about the existence of a solution to a Cauchy-type nonlinear system of differential equations in covariant derivatives with respect to $(n+1)$ unknown functions.

In this section we shall develop a method for reducing the Brinkmann nonlinear system to a Cauchy-type linear system. This method is helpful for evaluating the degree of parametric arbitrariness of the solution of the problem mentioned above and for solving the problem completely for symmetric Riemannian spaces $\mathbb{V}_n$.

Two Riemann spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ are said to be conformal to each other if in a coordinate system common to the mapping, their metric tensors g and $\overline{g}$ are related by the following expression (5.1): $\overline{g} = \mathrm{e}^{2\sigma}\, g$ where σ is a function.

As is known, the Ricci tensors R_{ij} and $\overline{R}_{ij}$ of the spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$, conformally corresponding to each other, are related by the expression (5.9):

$$\overline{R}_{ij} = R_{ij} - (n-2)\psi_{ij} - [\Delta_2\psi + (n-2)\Delta_1\psi]\, g_{ij}.$$

We assume that $\mathbb{V}_n$ admits a conformal mapping onto an Einstein space $\overline{\mathbb{V}}_n$ in which the conditions $\overline{Ric} = \overline{R}/n \cdot \overline{g}$, where $\overline{R}$ is the scalar curvature of $\overline{\mathbb{V}}_n$, are satisfied. Then, putting $s = \mathrm{e}^{-\sigma}$, from (5.9) we obtain

$$s_{,ij} = -s\, L_{ij} + u\, g_{ij} \tag{5.23}$$

where u is a function and L_{ij} is the Brinkmann tensor (5.12).

On the other hand, if Eqs. (5.23) is satisfied in $\mathbb{V}_n$, then a space $\overline{\mathbb{V}}_n$ with a metric tensor of the structure

$$\overline{g}_{ij}(x) = s^{-2}(x)\, g_{ij}(x) \tag{5.24}$$

is an Einstein space. This assertion is implied by the relations (5.1) and (5.23). Thus, we have

Theorem 5.8 *A Riemannian space $\mathbb{V}_n$ admits a conformal mapping onto an Einstein space $\overline{\mathbb{V}}_n$ if and only if there exists in $\mathbb{V}_n$ a solution to Eqs. (5.23) relative to the unknown functions $s\,(>0)$ and u. The metric tensor of the space $\overline{\mathbb{V}}_n$ is of the form (5.24).*

Let us differentiate (5.23) covariantly with respect to x^k and then alternate with respect to j and k. Let us then contract the expression thus obtained with g^{ij}. Finally, we obtain $u_{,i} = -s_\alpha\, L_i^\alpha$, where $L_i^h = g^{h\alpha}\, L_{\alpha i}$. Thus, by virtue of (5.23), using the notation $s_i = s_{,i}$, we readily find that the following theorem is true.

Theorem 5.9 *A Riemann space $\mathbb{V}_n$ admits a conformal mapping onto an Einstein space $\overline{\mathbb{V}}_n$ if and only if there exists in $\mathbb{V}_n$ a solution to a Cauchy-type system of linear homogeneous differential equations in covariant derivatives with respect to the functions $s\,(>0)$ and u, and the vector s_i:*

$$\begin{aligned} &\text{(a)} \quad s_{,i} \;&&= s_i;\\ &\text{(b)} \quad s_{i,j} \;&&= -s\, L_{ij} + u\, g_{ij};\\ &\text{(c)} \quad u_{,i} \;&&= -s_\alpha\, L_i^\alpha. \end{aligned} \tag{5.25}$$

Since this system is linear, its solution is regular. The conditions of integrability of Eqs. (5.25) and their differential prolongations represent a system of linear homogeneous algebraic equations in s, u and s_i with coefficients determined by the metric tensor g.

The integrability conditions for Eqs. (5.25b) are

$$s_\alpha C_{ijk}^\alpha = -s\, S_{ijk} \tag{5.26}$$

where C_{ijk}^h is the Weyl tensor of conformal curvature and $S_{ijk} = L_{ij,k} - L_{ik,j}$. By virtue of (5.25b), the integrability conditions for (5.25a) are satisfied identically, and the integrability conditions for Eq. (5.25c) follow from (5.26).

Remark 5.3 The system (5.25) was found in 1992 by Mikeš, Gavrilchenko and Gladysheva [660, 661]. The theory of a similar type of linear partial differential equations is well-known, see Section 1.5, p. 100. This theory is a useful tool in geometry since the times of of Levi-Civita and Fubini [107, 422], see [50–52].

Recently, the equalities has been derived is the so-called tractor theory, by A.R. Gower and P. Nurowski [441], and used in the paper [442].

5.5.2 On the quantity of the solution's parameters

Equations (5.25) are completely integrable (just like the Brinkmann equations [314]) in conformally flat spaces and, consequently, their solutions exist for arbitrary initial values

$$s(x_o) = \overset{\circ}{s}\,(>0), \quad u(x_o) = \overset{\circ}{u}, \quad s_i(x_o) = \overset{\circ}{s}_i.$$

Hence, the general solution of (5.26) in a conformally flat space depends on $(n+2)$ essential parameters.

Brinkmann [314] also noticed that three-dimensional nonconformally flat spaces do not admit conformal mappings onto Einstein spaces.

For spaces $\mathbb{V}_n$ $(n>3)$, we have

Theorem 5.10 *For nonconformally flat spaces $\mathbb{V}_n$ $(n>3)$, the solution of Eqs. (5.25) depends on not more than $(n-1)$ parameters.*

Proof. The differential prolongation of conditions (5.26) is of the form

$$s_\alpha C^\alpha_{ijk,l} + s_l S_{ijk} + s\,(L_{\alpha l} C^\alpha_{ijk} + S_{ijk,l}) + u\,C_{lijk} = 0.$$

From this expression it follows that in a nonconformally flat space the function u is expressed in terms of the invariant (function) s, the vector s_i and objects in the space $\mathbb{V}_n$. Closely examining conditions (5.26), we find that at least two components of the vector s_i are expressed through the other components of this vector and the invariant s. Hence, the general solution of (5.25) depends on not more that $(n-1)$ parameters. This completes the proof of Theorem 5.10. □

5.5.3 Conformal mappings onto 4-dimensional Einstein spaces

In [314] it has been demonstrated that if in the space $\mathbb{V}_4$ the conditions $a_\alpha C^\alpha_{ijk} = 0$ are satisfied for a nonzero vector a_i, then either the vector a_i is null, or C^h_{ijk} is zero. Therefore, in proper Riemannian nonconformally flat spaces $\mathbb{V}_4$ we obtain the condition $s_i = s \cdot c_i$ from (5.26), where c_i is a vector defined by the objects in $\mathbb{V}_4$. By virtue of (5.25), differentiating the last relation covariantly, we obtain $u = s \cdot c$, where c is a function defined in $\mathbb{V}_4$. Hence we have [661]:

Theorem 5.11 *In four-dimensional proper Riemannian nonconformally flat spaces, the solution of Eqs. (5.25) depends on not more than one parameter.*

Hence the spaces in Theorem 5.11 either do not admit conformal mappings on Einstein spaces because the trivial solution of system (5.25) does not generate conformal mappings, or admit conformal mappings onto a one-parameter family of Einstein spaces homothetic to one another. This is in agreement with the results derived by Brinkmann [314].

We now consider the case when the integrability conditions (5.26) have a one-parameter solution in the unknowns s and s_i. We shall solve the equations $S_\alpha C^\alpha_{ijk} + S_{ijk} = 0$ for the unknowns S_1, S_2, $\ldots$, S_n.

Since $s_{,i} = s\,S_i$, we have $s = \exp(\int S_\alpha dx^\alpha)$.

Whether or not the function s constructed in the described way is a solution of Eqs. (5.23), is to be assessed by direct verification. If it is a solution, then it generates a conformal mapping onto an Einstein space.

5.5.4 Conformal mappings from symmetric spaces onto Einstein spaces

Let us now study the conformal mappings of symmetric spaces onto Einstein spaces. Certain specificities of such mappings were stated by M.L. Gavril'chenko.

Suppose that $\mathbb{V}_n$ is a nonconformally flat symmetric space (in which the Riemann tensor R is covariantly constant, as well known) that admits a conformal mapping onto an Einstein space $\overline{\mathbb{V}}_n$. The conditions (5.26) take a simpler form

$$s_\alpha C^\alpha_{ijk} = 0. \tag{5.27}$$

The differential prolongation of (5.27) can be represented as $s\,L_{\beta l} C^\beta_{ijk} - u\,C_{lijk} = 0$. Since $C_{hijk} \neq 0$, we have

$$u = \alpha \cdot s \tag{5.28}$$

where α is a function.

By virtue of (5.28), the previous conditions take the form

$$\mathcal{L}_{\alpha h} C^{\alpha}_{ijk} = 0 \tag{5.29}$$

where $\mathcal{L}_{ij} = L_{ij} - \alpha\, g_{ij}$. Differentiating (5.29), we readily find that α is a constant. Substituting (5.28), let us express conditions (5.25c) as

$$s_\beta L^{\beta}_{i} = -\alpha\, s_i. \tag{5.30}$$

Differentiating (5.30), we obtain

$$\mathcal{L}_{\beta i}(L^{\beta}_{j} + \alpha\, \delta^{\beta}_{j}) = 0 \tag{5.31}$$

where δ^{h}_{i} is the Kronecker delta.

Contracting (5.31) with respect to g^{ij}, we find $\alpha^2 = \frac{1}{n}\, L^{\gamma}_{\beta} L^{\beta}_{\gamma}$. Thus, the constant α is defined by objects in $\mathbb{V}_n$.

The system of equations (5.25) in nonconformally flat symmetric spaces $\mathbb{V}_n$ takes the form

$$\text{a)}\ s_{,i} = s_i; \quad \text{b)}\ s_{i,j} = -s\, \mathcal{L}_{ij}. \tag{5.32}$$

It is a simple matter to verify that the mixed differential-algebraic system of equations (5.27), (5.30), and (5.32) is completely integrable in a symmetric space $\mathbb{V}_n$ where conditions (5.29) and (5.31) are satisfied. Analyzing the symmetric spaces where such conditions are satisfied, we obtain the following theorems.

Theorem 5.12 *A non conformally flat symmetric Einstein space $\mathbb{V}_n$ admits a nontrivial conformal mapping onto an Einstein space $\overline{\mathbb{V}}_n$ if and only if it is a Ricci flat space and* $\operatorname{rank} \|R^{h}_{ijk}\| = r < n$. *The general solution of Eqs. (5.32) depends on $(n - r + 1)$ essential parameters.*

Proof. Indeed, in the space $\mathbb{V}_n$ under consideration, we have $L_{ij} = \beta\, g_{ij}$. From (5.30) it follows that $\beta = \alpha$. Substituting L_{ij} into (5.29), we obtain $\alpha = 0$ and $L_{ij} = 0$. By virtue of the structure of this tensor, we find that the Ricci tensor is also zero, i.e. $\mathbb{V}_n$ is a Ricci flat space. Equations (5.32) take the form $s_{,i} = s_i$ and $s_{i,j} = 0$, and their integrability conditions are of the form $s_\alpha R^{\alpha}_{ijk} = 0$. Since the differential continuations are satisfied identically, these equations have a solution under arbitrary initial conditions $\overset{\circ}{s}_i$, $\overset{\circ}{s}$ (> 0) satisfying the relations $\overset{\circ}{s}_\alpha R^{\alpha}_{ijk}(x_o) = 0$. This completes the proof of Theorem 5.12.

Theorem 5.13 *A nonconformally flat non-Einstein symmetric space $\mathbb{V}_n$ admits nontrivial conformal mappings onto Einstein spaces if and only if conditions (5.29) and (5.31) are satisfied in $\mathbb{V}_n$. The general solution of Eqs. (5.32) depends on at least* $\operatorname{rank} \|\mathcal{L}_{ij}\| + 1$ *essential parameters.*

Proof. Indeed, analyzing (5.29) and (5.31), we find that $r = \operatorname{rank} \|\mathcal{L}_{ij}\|$ satisfies the inequalities $0 < r < n$, and the solutions of Eqs. (5.27) and (5.30) in the unknown variables $\overset{\circ}{s}_i$ at the point x_o depend on not less than r essential parameters. □

For the symmetric spaces $\mathbb{V}_n$ we can distinguish two cases: Case 1) $\alpha \neq 0$ and Case 2) $\alpha = 0$. For Case 1, we have

Theorem 5.14 *For $\alpha \neq 0$, a nonconformally flat symmetric space in which conditions (5.29) and (5.31) are satisfied is an even-dimensional ($n = 2m$) reducible space with the metric $ds^2 = d\tilde{s}^2 + d\tilde{\tilde{s}}^2$, where $d\tilde{s}^2$ is the metric of a space $\tilde{\mathbb{V}}_m$ of constant curvature with nonzero scalar curvature R, and $d\tilde{\tilde{s}}^2$ is the metric of a symmetric Einstein space $\tilde{\tilde{\mathbb{V}}}_m$ of variable curvature whose scalar curvature is $(-R)$. The general solution of (5.32) depends on $(m+1)$ essential parameters.*

Proof. Indeed, conditions (5.31) can be expressed as $L_{i\alpha}L_j^{\alpha} = \alpha^2 g_{ij}$. The last condition and the relation $L_{ij} = 0$ are, according to the Shirokov's theorem [167], the test of a reducible space. The proof is completed by demonstrating conditions (5.27), (5.29), (5.30), and (5.31). □

Proper Riemann four-dimensional non conformally flat symmetric spaces, as noted by Brinkmann [314], do not admit nontrivial conformal mappings.

5.5.5 Conformal mappings onto Einstein spaces "in the large"

Finally, let us study conformal mappings "in the large" onto compact Einstein spaces. We have

Theorem 5.15 *A compact space $\mathbb{V}_n$ in which the tensor S_{ijk} vanishes in at most one point does not admit nontrivial conformal mappings onto Einstein spaces.*

Proof. The proof follows from the fact that, if a conformal mapping is nontrivial, there must exist a maximum and a minimum for the function (invariant) s. But at these points $s_i = 0$. Since $s > 0$, from (5.26) we find that the tensor S_{ijk} vanishes at least at two points in the space $\mathbb{V}_n$.

Mikeš [634] has proved that compact Einstein spaces with an alternating metric do not admit nontrivial conformal mappings "on the whole" onto Einstein spaces. The following theorem holds:

Theorem 5.16 *Compact nonconformally flat symmetric Riemannian spaces do not admit nontrivial conformal mappings "on the whole" onto Einstein spaces.*

Proof. As shown in Theorem 5.12, if a symmetric space $\mathbb{V}_n$ is an Einstein space, then $s_{,ij} = 0$. On a compact space this equation has only a constant solution $s = \text{const}$ [634].

Consequently, what remains is the case where $\mathbb{V}_n$ is a non-Einstein space. Then there would exist a vector ε^i such that $\varepsilon^i\varepsilon^j\mathcal{L}_{ij}(x_o) > 0$ or $\varepsilon^i\varepsilon^j\mathcal{L}_{ij}(x_o) < 0$ at some point x_o.

Since the space is symmetric, we have $L_{ij,k} = 0$ and similar relations hold at any point in a connected component of the space $\mathbb{V}_n$. Consequently, from (5.32) we find that the relations $s_{,ij}\varepsilon^i\varepsilon^j > 0$ (or accordingly $s_{,ij}\varepsilon^i\varepsilon^j < 0$) hold at any point in $\mathbb{V}_n$. The last assertion contradicts the existence of a maximum (or minimum) for the function s. This completes the proof of Theorem 5.15. □

5.6 Concircular mappings

H.W. Brinkmann [314], see [139, 568], and others, e.g. [661], investigated the existence of conformal mappings of Riemannian spaces onto Einstein spaces.

As far as other special types of conformal mappings are concerned, let us mention the following. V.F. Kagan [79] studied spaces admitting such a mapping onto Euclidean space under which geodesics are mapped onto curves that are embedded in two-dimensional subspaces passing through a general point. The case when this point is an ideal point (point "at infinity"), that is, the case when all two-dimensional subspaces are paralell to some direction, was studied by Vranceanu [933]. The spaces with this property are called subprojective spaces by Kagan, and the mapping mentioned above is a composition of a conformal and geodesic mapping.

Under a geodesic circle we understand a curve for which the first curvature is constant and the second curvature is zero. A conformal mapping preserving geodesic circles is called concircular. When K. Yano [949] investigated such mappings he introduced the concept of the tensor of concircular curvature, invariant under concircular mappings. It turns out that invariance of the tensor of concircular curvature under a conformal mapping is a necessary and sufficient condition for the mapping to be concircular.

In this respect, we can also mention the works by Oboznaya and Fedishenko [398] who studied conformal mappings preserving the Ricci tensor and the scalar curvature, respectively. In the paper [333] is to pay attention to conformal mappings preserving the Einstein tensor.

Concircular mappings as wel as other aspect have been studied for Riemannian manifolds, see [8, 9, 11, 565–567].

5.6.1 Concircular mappings

Recall that geodesic circles are curves for which the first curvature is constant and the second curvature is zero (that is, its tangent vector λ and the main normal ν satisfy: $\nabla_t\lambda = k\,\nu$; $\nabla_t\nu = -k\,e_\lambda e_\nu\,\lambda$, where $k = \text{const}$, and $g(\lambda,\lambda) = e_\lambda = \pm 1$, $g(\nu,\nu) = e_\nu = \pm 1$.

Analyzing the equation (5.1) of conformal mappings, K. Yano [949] found necessary and sufficient conditions for concircular mappings which usually serve as a contemporary definition.

Definition 5.4 A conformal mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ defined by (5.1): $\overline{g} = \mathrm{e}^{2\sigma}\,g$ is *concircular*, if

$$\sigma_{ij} = \omega\,g_{ij} \tag{5.33}$$

where ω is a function and σ_{ij} is the tensor defined by (5.8): $\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}$.

We put $\phi = -\exp(-\sigma)$, $\varphi_i = \phi_{,i}$. Using definition (5.8) of the tensor σ_{ij}, we can easily verify that the formula (5.33) is equivalent with

$$\varphi_{i,j} = \varrho\,g_{ij}. \tag{5.34}$$

Vector fields φ_i satisfying the conditions (5.34) are called concircular, by K. Yano [121, 170, 949], and the spaces $\mathbb{V}_n$ themselves are called equidistant, by N.S. Sinyukov [170], see Section 2.3, p. 140.

In this case the formulae (5.7) are reduced to

$$\overline{R}^h_{ijk} = R^h_{ijk} - \omega(g_{ik}\delta^h_j - g_{ij}\delta^h_k). \tag{5.35}$$

Contracting the equation (5.35) with respect to the indices h and k we have

$$\overline{R}_{ij} = R_{ij} - (n-1)\,\omega\, g_{ij}. \tag{5.36}$$

Another contraction with g^{ij} $(= e^{2\sigma}\,\overline{g}^{ij}$, see (5.4)) gives

$$\omega = \frac{1}{n(n-1)}\,(R - e^{2\varphi}\overline{R})$$

where R and $\overline{R}$ are the scalar curvature of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$, respectively.

Let us substitute ω into the equation (5.35). We easily find

$$\overline{Y}^h_{ijk}(x) = Y^h_{ijk}(x), \tag{5.37}$$

where

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\left(g_{ik}\delta^h_j - g_{ij}\delta^h_k\right). \tag{5.38}$$

Analogously, in $\overline{\mathbb{V}}_n$ we introduce the tensor $\overline{Y}$. Y is called the *Yano's tensor of concircular curvature*, and is invariant under concircular mappings.

Let us note that $\mathbb{V}_n$ $(n > 2)$ has constant curvature if and only if $Y = 0$. By contraction of (5.37) over h and k we verify that

$$\overline{E}_{ij}(x) = E_{ij}(x), \tag{5.39}$$

where E is the Einstein tensor.

The Einstein tensor is the tensor given by $E = Ric - \frac{R}{n}\cdot g$, i.e.

$$E_{ij} = R_{ij} - \frac{R}{n}\,g_{ij} \tag{5.40}$$

where R_{ij} are components of the Ricci tensor and R is the scalar curvature on $\mathbb{V}_n$.

We easily check the following:

Theorem 5.17 *Any concircular mapping $f : \mathbb{V}_n \to \overline{\mathbb{V}}_n$ preserves Yano's tensor of concircular curvature and the Einstein tensor.*

5.6.2 Conformal mappings preserving the Einstein tensor

In what follows, we present results obtained by Z.G. Bai [284].

Definition 5.5 If the conformal mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ satisfies $\overline{E} = E$ we say that the *conformal mapping preserves the Einstein tensor.*

In a common coordinate system (x) relative to the mapping f, this condition reads (5.39).

Now we are going to study a conformal mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ preserving the Einstein tensor. From formula (5.10) it follows

$$\sigma_{ij} = \frac{-1}{n-2}(\overline{R}_{ij} - R_{ij}) + \alpha\, g_{ij} \tag{5.41}$$

where α is a function.

On the other hand, from (5.40) and (5.39) we obtain

$$\overline{R}_{ij} - R_{ij} = \frac{\overline{R}}{n}\ \overline{g}_{ij} - \frac{R}{n}\ g_{ij}. \tag{5.42}$$

Using formulae (5.1) and (5.42), the formula (5.41) reads (5.33). It means that f is a concircular mapping. Using the Theorem 5.17 and (5.33) we can easily see that the following is satisfied. □

Theorem 5.18 *A conformal mapping $f\colon \mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$ is concircular if and only if f preserves the Einstein tensor.*

The formula (5.37) gives (5.39), and we have an analogous theorem:

Theorem 5.19 *A conformal mapping $f\colon \mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$ is concircular if and only if f preserves the Yano tensor of concircular curvature.*

It is known that concircular vector fields are a special class of torse-forming fields. It was proved that on compact pseudo-Riemannian manifolds, there do not exist constant torse-forming fields [652]. From this result it follows that on such spaces, concircular vector fields do not exist as well. Hence we have

Theorem 5.20 *The only conformal mappings on a compact pseudo-Riemannian manifold $\mathbb{V}_n$ $(n > 2)$, preserving the Einstein tensor, are just homotheties.*

From [568] we may deduce that from compactness follows completeness.

5.6.3 Conformal mappings between Einstein spaces

The problem of conformal mappings between Einstein spaces was formulated and solved by H.W. Brinkmann [314]. In a detailed form, this theory can be found in the monographs by A.Z. Petrov [139, 140].

As a consequence of Theorem 5.18, we get the result by Brinkmann [314], see [139, 140]:

Theorem 5.21 *An Einstein space $\mathbb{V}_n$ $(n > 2)$ admits a conformal mapping f onto some Einstein space $\overline{\mathbb{V}}_n$ if and only if either $\mathbb{V}_n$ is equidistant (and f is concircular), or f is homothetic.*

In the work [314] by Brinkmann, metrics of equidistant Einstein spaces are found, see Section 2.3, p. 151.

5.7 Conformal transformations

In the present section we study the question of the distribution of the maximal dimensions of the Lie groups of conformal transformations of Riemannian spaces $\mathbb{V}_n$, see J. Mikeš and D. Moldobaev [674].

5.7.1 Groups of conformal transformations

Conformal mappings of a Riemannian space $\mathbb{V}_n = (M, g)$ onto itself are called *conformal transformations* on $\mathbb{V}_n$ or *conformal motions* of $\mathbb{V}_n$.

A vector field X on $\mathbb{V}_n$ is called an *infinitesimal conformal* or a *conformal Killing vector field* if for each point $p \in M$ there is a neighborhood U of p such that the local one-parameter group f_t determined by the vector field preserves the metric with functional ratio, that is, the mapping $f_t \colon M \to M$ is a conformal transformation. In a special coordinate system (x^i) in which $X = \partial_1$, the conformal transformation G_1 is characterized by

$$\partial_1 g_{ij}(x) = \sigma(x)\, g_{ij}(x) \tag{5.43}$$

where $\sigma(x)$ is a function. By (4.33) and (4.36), these conditions take the form

$$\mathcal{L}_\xi g_{ij} \equiv \xi_{i,j} + \xi_{j,i} = \sigma\, g_{ij} \tag{5.44}$$

where $\xi_i = g_{i\alpha}\xi^\alpha$, $X = \xi^\alpha \partial_\alpha$. Let us recall that ξ^h is called a conformal Killing vector. The transformation G_1 is homothetic if $\sigma \equiv \text{const}$, G_1 is a motion on $\mathbb{V}_n$ if $\sigma \equiv 0$.

Since the commutator of any two solutions of (5.44) is a solution of the same equation (5.44), the linearly independent solutions $\xi^h_1, \xi^2_2, \ldots, \xi^h_r$ over $\mathbb{R}$ form a basis of the conformal transformation Lie group. The dimensions of the mentioned groups are bounded (see [139, 140]):

- $r_{con} \le n(n+3)/2 + 1$ for the conformal transformations,
- $r_{hom} \le n(n+1)/2 + 1$ for the homothetic transformations,
- $r_{mot} \le n(n+1)/2$ for the motions.

In these bounds, the maxima are attained for conformally flat, flat and constant curvature spaces, respectively.

The motions in Riemannian spaces have attracted many authors (I.P. Egorov [45–47], G. Vranceanu [933], G. Kruchkovich [99], K. Yano [197] and others).

Homothetic transformations are similar to motions, because an r-dimensional group of homothetic transformations either contains an (r–1)-dimensional invariant subgroup of motions, or is the r-dimensional group of motions (M. Knebelman, [549]). Conformal transformations are less investigated. A. Taub in [890] proved that conformally flat spaces (and only these) admit a conformal transformation group of the maximal dimension $r = (n+1)(n+2)/2$.

H. Hiramatu (see [491]) had found the following estimation of the lacuna in the distribution of possible dimensions of conformal transformation groups for spaces with positive definite metrics:

1. there are no Riemannian spaces $\mathbb{V}_n$ $(n \geq 3, \neq 4)$ admitting an r_{con}-dimensional conformal transformation group, where

$$n(n+1)/2 + 2 < r_{con} < (n+1)(n+2)/2;$$

2. if the space $\mathbb{V}_n$ (here $n \geq 3$, a finite set of the dimension n is excluded, this set depends on dimensions of special series of simple Lie groups) admits an r-dimensional conformal transformation group, where

$$(n-1)(n-2)/2 + 2 < r_{con} < n(n-1)/2 + 1,$$

then $\mathbb{V}_n$ is a conformally flat space.

For Riemannian spaces $\mathbb{V}_n$ of arbitrary signature, Mikeš and Moldobaev [674] proved the following

Theorem 5.22 *There is no Riemannian space $\mathbb{V}_n$ $(n \geq 4)$ which admits a complete r-dimensional conformal transformation group, where*

$$(n-1)(n-2)/2 + 6 < r < (n+1)(n+2)/2.$$

Let us notice that the estimate is strict, since I. Egorov ([45]) had found Riemannian spaces with r-dimensional homothetic transformation group, where $r = (n-1)(n-2)/2 + 6$. See the Section 4.7.4 for more detailed information.

5.7.2 Criterion of conformal flatness

In order to prove Theorem 5.22 let us begin with the following theorems ascending to a result proved by I. Schouten and D. Struik in [163]: a space with a linear connection is a projectively-flat space if and only if all the components of the Riemann tensor $R^H_{IJK} = R^H_{IJK} = 0$, in arbitrary system of coordinates, where H, I, J and K are mutually distinct. We had strengthened this result (see [673]) in another direction.

Theorem 5.23 *For an arbitrary point $M \in \mathbb{V}_n$ $(n > 3)$ let the components of the conformal curvature tensor $C_{HIJK}(M)$ be equal to zero in each coordinate system orthonormal at the point M, where H, I, J and K are fixed distinct indices. Then the space $\mathbb{V}_n$ is conformally flat.*

Proof. Recall that a coordinate system such that $g_{ij}(M) = e_i\,\delta_{ij}$, where δ_{ij} equals 1 when $i = j$ and 0 when $i \neq j, e_i = \pm 1$, is said to be orthonormal at the point M.

Let I and J be fixed distinct indices. It is easy to verify that the coordinate transformations preserve the orthonormality of the coordinate system at the point M:

$$\begin{array}{cc}
\text{if } e_I e_J = 1, & \text{if } e_I e_J = -1, \\
x'^I = x^I \cos(\alpha) - x^J \sin(\alpha), & x'^I = x^I \cosh(\alpha) + x^J \sinh(\alpha), \\
x'^J = x^I \sin(\alpha) + x^J \cos(\alpha), & x'^J = x^I \sinh(\alpha) + x^J \cosh(\alpha), \\
x'^h = x^h \ (h \neq I, J); & x'^h = x^h \ (h \neq I, J).
\end{array} \tag{5.45}$$

where α is a real number.

If $x'^h = x'^h(x)$ is a coordinate transformation, then the components of the conformal curvature tensor are being transformed by the rule:

$$C'_{hijk} = C_{abcd}A^a_h A^b_i A^c_j A^d_k, \tag{5.46}$$

where $A^a_h = \partial x^a/\partial x^{h\prime}$. By the Theorem's conditions, via (5.45) and (5.46), it is easy to show that $C_{HIJK}(M) = 0$ in any system orthonormal at the point M (H, I, J and K are as above). Henceforth, without additional mentioning, we shall work in orthonormal coordinate systems.

Having computed (5.46) for $h = H, i = I, j = J$ and $k = K$ (indices are distinct), we obtain that $(C_{HIIK}e_I - C_{HJJK}e_J)\, A^I_I A^J_J = 0$. But since, in the general case, $A^I_I A^J_J \neq 0$, we find $C_{HIIK}e_I - C_{HJJK}e_J = 0$. This condition is satisfied for all distinct indices H, I, J and K. Therefore, $C_{HIIK} = e_I T_{HK}$, where T_{HK} are real numbers. Now let us compute $\vartheta_1 = C_{H\alpha\beta K}g^{\alpha\beta}$. To compute this let us take advantage of well-known properties of the conformal curvature tensor: $C_{hijk} = -C_{ihjk} = -C_{hikj}$. We obtain that $\vartheta_1(M) = (n-2)T_{HK}$. But since $C_{h\alpha\beta k}g^{\alpha\beta} = 0$, it follows that $T_{HK} = 0$ and, therefore, $C_{HIIK} = 0$ for all distinct indices H, I and K.

Having evaluated (5.46) for $h = k = H$, $i = I$ and $j = J$ (here the indices H, I and J are distinct), we obtain that $e_I C_{HIIH} - e_J C_{HJJH} = 0$. It follows that $C_{HIIH} = e_I T_H$ for $I \neq H$. Now compute $\vartheta_2 = C_{H\alpha\beta H}g^{\alpha\beta}$. Obviously, $\vartheta_2(M) = (n-1)T_H$. Thus, $T_H = 0$ and $C_{HIIH} = 0$.

So, we have convinced ourselves that all the components of the conformal curvature tensor vanish. Theorem 5.23 has been proved.

Further, the following result is valid.

Theorem 5.24 *For an arbitrary point $M \in \mathbb{V}_n$ ($n > 3$), let the components of the conformal curvature tensor $C_{HIIK}(M)$ be equal to zero in each orthonormal coordinate system at the point M, where H, I and K are fixed distinct indices. The space $\mathbb{V}_n$ then is conformally flat.*

Proof. For $h = H, i = I, j = J$ and $k = K$ let us compute (5.46) for the coordinate transformations (5.45). Taking into account the assumptions of the Theorem, we obtain at the point M, in an orthonormal coordinate system centred at the same point M, that

$$(C_{HIJK} + C_{HJIK})A^I_I A^J_I + C_{HJJK}A^J_I A^J_I = 0,$$

where H, I, J and K are distinct indices, again. By virtue of arbitrariness of α and due to the latter equation, we have $C_{HIJK} + C_{HJIK} = 0$ in the transformations (5.45). One can check easily that the latter equalities hold with distinct indices H, I, J and K. Alternating this correlation with respect to J and K, we obtain $C_{HIJK} = 0$ due to the well-known property of the conformal curvature tensor. Theorem 5.24 then is true due to Theorem 5.23.

By contraposition of Theorem 5.24 we come to the following.

Corollary. *Let $\mathbb{V}_n$ $(n > 3)$ be a nonconformally flat space. Then there is an orthonormal coordinate system at a certain point M such that $C_{HIIK}(M) \neq 0$ for certain distinct indices H, I and K.*

The Corollary can be improved in the following way:

Theorem 5.25 *Let $\mathbb{V}_n$ be a nonconformally flat space. Then there is a point $M \in \mathbb{V}_n$ and a coordinate system orthonormal at the point M such that both $C_{HIIK}(M) \neq 0$ and $C_{HIIJ}(M) \neq 0$ for certain distinct indices H, I, J and K.*

Proof. Following the Corollary, we can choose a coordinate system orthonormal at the point M, for which $C_{HKKI}(M) \neq 0$ with distinct indices H, I and K. We show how the following components of conformal curvature tensor vary, according to (5.46), under the transformation (5.45):

$$\begin{aligned} C'_{HKKI} &= C_{HKKJ}A^I_I + C_{HKKJ}A^J_I \\ C'_{HKKJ} &= C_{HKKI}A^I_J + C_{HKKJ}A^J_J. \end{aligned}$$

Since $C_{HKKI} \neq 0$, it is easy to show that there is an α for which we have both $C'_{HKKI} \neq 0$ and $C_{HKKJ} \neq 0$. Theorem 5.25 has been proved. □

5.7.3 On the lacunarity of the degree of conformal motions

It is well-known that the fundamental equations of conformal transformations in $\mathbb{V}_n$ are:

$$\begin{aligned} &a)\ \xi_{i,j} = \xi_{ij}; \\ &b)\ \sigma_{,i} = \sigma_i; \\ &c)\ \xi_{ij} + \xi_{ji} = \sigma g_{ij}; \\ &d)\ \xi_{i,jk} = \xi_\alpha R^\alpha_{kji} + \tfrac{1}{2}\left(\sigma_k g_{ij} + \sigma_j g_{ik} - \sigma_i g_{jk}\right); \\ &e)\ \sigma_{j,i} = \tfrac{-2}{n-2}\{\xi^\alpha(R_{ij,\alpha} - \tfrac{1}{2(n-1)}R_{,\alpha}g_{ij}) + \\ &\qquad\qquad \xi_{\alpha i}R^\alpha_j + \xi_{\alpha j}R^\alpha_i - \tfrac{1}{n-1}\xi_{\alpha\beta}R^{\alpha\beta}g_{ij}\}. \end{aligned} \tag{5.47}$$

The formulas (5.47) are a mixed system of algebraic and partial differential equations (of Cauchy type) with respect to the unknown tensors ξ_{ij}, ξ_i, σ_i and σ. Therefore the general solution depends on $r_{con} \leq (n+1)(n+2)/2$ parameters.

Lemma 5.1 *If the space $\mathbb{V}_n (n > 3)$ is nonconformally flat, then the vector σ_i can be expressed via ξ_{ij}, ξ_i, σ and intrinsic objects of $\mathbb{V}_n$.*

Proof. The expression $\mathcal{L}_\xi C^h_{ijk} = 0$ is, as known, conditions of integrability for equations (5.47). These and (5.43) imply

$$\mathcal{L}_\xi C_{hijk} = \sigma C_{hijk}. \tag{5.48}$$

We write out these formulas in details:

$$\xi_{\alpha h}C^\alpha_{ijk} + \xi_{\alpha i}C^\alpha_{hjk} + \xi_{\alpha j}C^\alpha_{hik} + \xi_{\alpha k}C_{hij}{}^\alpha = \sigma C_{hijk} \tag{5.49}$$

Here ξ_{hi} equals $\xi_{h,i}$, as before. Derivating covariantly (5.49) with respect to x^l and taking into account (5.47), we have

$$\begin{gathered}2\sigma_l C_{hijk}+\sigma_h C_{lijk}+\sigma_i C_{hljk}+\sigma_j C_{hilk}+\sigma_k C_{hijl}-g_{hl}\sigma_\alpha C^\alpha_{ijk}- \\ g_{il}C^\alpha_{hjk}-g_{ij}\sigma_\alpha C_{hi}{}^\alpha_k-g_{kl}\sigma_\alpha C_{hij}{}^\alpha=0,\ \ \mathrm{mod}(\xi_{ij},\xi_i,\sigma).\end{gathered} \tag{5.50}$$

Here and in what follows we denote by $\mathrm{mod}(\xi_{ij},\xi_i,\sigma)$ the rejected summands; in this case, these depend on ξ_{ij},ξ_i,σ and intrinsic objects of $\mathbb{V}_n$. We contract (5.50) with the tensor g^{hl}, then alternate the result with respect to the indices j and k. Taking the properties of the conformal tensor into account, we obtain that $\sigma_\alpha C^\alpha_{ijk}=0,\ \mathrm{mod}(\xi_{ij},\xi_i,\sigma)$. Correlations (5.50) then have the more simple form:

$$2\sigma_l C_{hijk}+\sigma_h C_{lijk}+\sigma_i C_{hljk}+\sigma_j C_{hilk}+\sigma_k C_{hijl}=0,\ \ \mathrm{mod}(\xi_{ij},\xi_i,\sigma). \tag{5.51}$$

Since $\mathbb{V}_n$ is nonconformally flat, there are two noncollinear vectors a^i and b^i such that $a^h b^i b^j c^k C_{hijk}=e=\pm1$, (otherwise, $a^h b^i b^j a^k C_{hijk}=0$ for arbitrary vectors a^i and b^i; this implies $C_{hijk}=0$). Contracting (5.51) with $a^h b^i b^j a^k$, we obtain

$$e\sigma_l+\sigma_i a^i c_l+\sigma_l b^l\tilde{c}_l=0,\ \ \mathrm{mod}(\xi_{ij},\xi_i,\sigma), \tag{5.52}$$

where $c_l=C_{lijk}b^i b^j a^k$ and $\tilde{c}_l=C_{hljk}a^h b^j a^k$. Contracting (5.52) first with a^l we have $\sigma_i a^i=0,\ \mathrm{mod}(\xi_{ij},\xi_i,\sigma)$. Contracting then the result with b^l, we obtain $\sigma_i b^i=0,\ \mathrm{mod}(\xi_{ij},\xi_i,\sigma)$. Now (5.52) has the form $\sigma_l=0,\ \mathrm{mod}(\xi_{ij},\xi_i,\sigma)$.

As a result, the vector σ_l can be linearly expressed, for nonconformally flat spaces, via the tensors $\xi_i,\xi_j,\xi_{ij},\sigma$ and intrinsic objects of $\mathbb{V}_n$. This proves Lemma 5.1. □

Lemma 5.2 *Let $\mathbb{V}_n$ $(n>3)$ be a nonconformally flat space. Among the components of the tensor ξ_{ij} there are at least $2n-6$ components linearly dependent on other components of the tensor ξ_{ij}, the vector ξ_i and the function σ. These components depend also on objects of $\mathbb{V}_n$.*

Proof. Let us consider a nonconformally flat space $\mathbb{V}_n$. Following Theorem 5.25, there is an orthonormal coordinate system at the point M in which, for instance, $C_{1223}\neq0$ and $C_{1224}\neq0$. In the chosen coordinate system, ξ_{ij} $(i\geq j)$ can be expressed, via (4c) through ξ_{ij} $(i<j)$ and σ. In addition, $\xi_{ij}=-\xi_{ji}$ and $\xi_{ii}=\frac{1}{2}\sigma e_i$.

Setting $i=j=2$ and $k=3$ in (5.49), we have

$$\xi_{\alpha h}C^\alpha_{223}+\xi_{\alpha 2}(C^\alpha_{h23}+C_{h2}{}^\alpha{}_3)+\xi_{\alpha 3}C_{h22}{}^\alpha=0,\ \ \mathrm{mod}(x_i,\sigma). \tag{5.53}$$

Now, setting $h=3$, we have from (5.53) $\xi_{\alpha 3}C^\alpha{}_{223}+\xi_{\alpha 2}C^\alpha{}_{332}=0\ \mathrm{mod}(\xi_i,\sigma)$. In the chosen coordinate system the latter can be written in the form

$$e_1\xi_{13}C_{1223}+\xi_{p3}C^p{}_{223}+\xi_{\alpha 2}C^\alpha{}_{332}=0,\ \ \mathrm{mod}(\xi_i,\sigma).$$

Hereafter $p=4,5,\ldots,n$. So, this expression can be written as follows: $\xi_{13}=0,\ \mathrm{mod}(\xi_i,\sigma)$. In a similar way, setting in (5.53) that $h=4,5,\ldots,n$, we obtain

$\xi_{ip} = 0$, $\mathrm{mod}(\xi_i, \sigma, \xi_{12}, \xi_{24}, \xi_{jk}\ (j,k \neq 1,4))$. Further, setting $i = j = 2$ and $k = 1$ in (5.49), one can easily see that

$$\xi_{4h} = 0, \ \ \mathrm{mod}(\xi_i, \sigma, \xi_{jk}(j,k \neq 1,4), \xi_{12}, \xi_{24}, \xi_{34})$$

for $h = 4, 5, \ldots n$. Thus, we have established that $2n-6$ components of the tensor $\xi_{ij}(i<j)$ are expressed through the remaining components of the tensor ξ_{ij}, the vector ξ_i and the function σ. □

Finally, by virtue of Lemmas 5.1 and 5.2, we can see that n components of the vector σ_i and $2n-6$ components of the tensor ξ_{ij} can be expressed through the other components of the tensor ξ_{ij}, the vector ξ_i and the function σ. Thus, the dimension of the complete conformal transformation group of nonconformally flat spaces is reduced by $3n-6$, being compared with the case of conformally flat spaces. Therefore, the dimension r of the conformal transformation group for these spaces does not exceed $(n-1)(n-2)/2+6$. This completes Theorem 5.22.

5.7.4 Riemannian space of second lacunarity of conformal motions

Let us consider a Riemannian space of the second order (see [677, 747, 748, 825]), associated with the space $\mathbb{V}_n$ in a neighborhood of an arbitrary point $M_0 \in \mathbb{V}_n$. The space $\tilde{V}_n^2$ has the metric tensor

$$\tilde{g}_{ij}(y) = \overset{o}{g}_{ij} + \frac{1}{3}\underset{o}{R}_{i\alpha\beta j} y^\alpha y^\beta,$$

where $\overset{o}{g}_{ij}$ and $\underset{o}{R}_{i\alpha\beta j}$ are values of the components of the metric tensor and the Riemannian tensor, respectively, at the point M_0.

An infinitesimal transformation $y'^h = y^h + t\xi^h(y)$ in the space $\tilde{\mathbb{V}}_n^2$ is called a *transformation of the second degree* (see [748]) if the displacement vector of these transformations has the form:

$$\xi^h(y) = C^h + C_i^h y^i + C_{ij}^h y^i y^j,$$

where C^h, C_i^h and C_{ij}^h are some constants.

For the proof of the following theorem, methods from prevoius sections are invoked.

Theorem 5.26 *The dimension r of the second degree motion Lie group (of the conformal transformation, respectively) in a space associated with the non-conformally flat $\mathbb{V}_n$ $(n > 3)$ satisfies the inequality $r \leq n(n-3)/2+6$ (or $r \leq n(n-3)/2+7$, respectively).*

The estimates given by Theorem 5.26 are strict. In fact, the I.P. Egorov spaces (see [46]), corroborating the exactness of the estimates given by I.P. Egorov for motions and those from Theorem 5.22 for conformal transformations, are the spaces $\tilde{V}_n^2$, indeed. Admitted by the latter groups of motions and conformal transformation consist of the second order transformations.

In a certain coordinate system y^h, the metric tensor of these spaces has the components (see [633]):

$$g_{ij}(y) = \overset{o}{g}_{ij} + em_i m_j, \tag{5.54}$$

where $\overset{o}{g}_{ij} = \text{const}$; in addition, $|\overset{o}{g}_{ij}| \neq 0$, $e = \pm 1$, $m_i = ab_i - ba_i \equiv 0$, and the constants a_i and b_i satisfy conditions

$$a_i a_j \overset{o}{g}^{ij} = a_i b_j \overset{o}{g}_{ij} = b_i b_j \overset{o}{g}^{ij} = 0.$$

Here $\overset{o}{g}^{ij}$ are elements of the matrix inverse $|\overset{o}{g}_{ij}|, a = a_i y^i$ and $b = b_i y^i$.

One can show that the spaces $\mathbb{V}_n^*$ are associated to themselves. They are namely the spaces $\tilde{\mathbb{V}}_n^2$.

Let us consider a vector field $\xi^h(y)$ defined by the formulas:

$$\xi^h(y) = C^h + C_i^h y^i + e\overset{o}{g}^{hi} C^j m_i m_j,$$

where the constants C^h and C_i^h are connected via the following relations:

$$C_i^h a_h = \alpha a_i + \beta b_i;\ \ C_i^h b_h = \gamma a_i - (\alpha - c/2) b_i,\ \ C_i^h \overset{o}{g}_{hj} + C_j^h \overset{o}{g}_{hi} = c\overset{o}{g}_{ij}. \tag{5.55}$$

Here α, β, γ and c are some constants. Due to conditions (5.55), one can easily show that there are $n(n-3)/2+7$ independent components among the constants C^h and C_i^h. If we set $c = 0$, there are then $n(n-3)/2 + 6$ independent components. The vector $\xi^h(y)$ determines a proper second degree transformation. By a direct computation, one can find that $\mathcal{L}_\xi g_{ij} = cg_{ij}$ on $\mathbb{V}_n^*$, so the vector $\xi^h(y)$ satisfies the generalized Killing equations. Due to the above we have

Theorem 5.27 *In the space $\mathbb{V}_n^*$, there are transitive groups of conformal second degree transformations. These groups are homothetic groups of the dimension $n(n-3)/2+7$, and motion groups of the dimension $n(n-3)/2+6$.*

The dimension of the second degree motion group (of the space) gives a counterexample to the Theorem 5.26 from [825]. Let us notice that the vector field $\eta^h = C^h + C_i^h y^i$, where the constants C^h and C_i^h satisfy (5.55) and also $C^h a_h = C^h b_h = 0$, generates in $\mathbb{V}_n^*$ an intransitive group of first degree motions with dimension $n(n-3)/2+4$.

6 GEODESIC MAPPINGS OF MANIFOLDS WITH AFFINE CONNECTION

6.1 Geodesic mappings

6.1.1 Introduction to geodesic mappings theory

Given the manifolds $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}, \overline{\nabla})$ with affine connection, a C^2-mapping $f\colon M \to \overline{M}$ is called *connection-preserving* if the covariant derivative is preserved under the tangent mapping, $Tf(\nabla_X Y) = \overline{\nabla}_{Tf(X)} Tf(Y)$, and *geodesic-preserving* if $f \circ c$ is geodesic in $\overline{M}$ for each geodesic c in M.

If $f\colon M \to \overline{M}$ is a diffeomorphism and $\overline{\nabla}$ is a fixed connection on $\overline{M}$ then there is a unique connection ∇ on M for which f is connection-preserving, [70, p. 60]. Trivially, any connection-preserving map is geodesic-preserving. This is a motivation for examining geodesic-preserving mappings themselves.

Definition 6.1 Let $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ be manifolds with affine connection. A diffeomorphism $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is called a *geodesic mapping of* $\mathbb{A}_n$ *onto* $\bar{\mathbb{A}}_n$ if f maps any geodesic curve in $\mathbb{A}_n$ onto a geodesic curve in $\bar{\mathbb{A}}_n$.

Note that in the above definition, "diffeomorphism" can be substituted by a (geodesic-preserving) bijection of sufficiently high class of differentiability.

As far as the history of the problem of geodesic mappings is concerned, it was Beltrami[79)] in 1865 who first posed the question, of course, not in the full generality but in the particular case when a surface $\mathbb{V}_2$ (i.e. a Riemannian two-manifold) is mapped onto a Euclidean two-plane $\mathbb{E}_2$. These results can be considered as an initial impulse for later respecting and acknowledgement of Non-Euclidean Geometry founded by Lobachevsky[80)], Bolyai[81)] and Gauss.

[79)] Eugenio Beltrami, 1835–1899, an Italian mathematician, notable for his work in non-Euclidean geometry (the Beltrami-Klein model of a hyperbolic plane), magnetism and elasticity.

[80)] Nikolai Ivanovich Lobachevsky, 1792–1856, a Russian mathematician and physicist, the rector of the Kazan University from 1827 to 1846, who formulated (in 1826) and published (in 1829) basic ideas of non-Euclidean hyperbolic geometry (called also Lobachevsky geometry). He was influenced by J.Ch.M. Bartels, a former teacher and friend of Gauss.

[81)] János Bolyai, 1802–1860, a Hungarian soldier and mathematician, a son of the mathematician Farkas Bolyai who also studied at Bartels. Between 1820 and 1823 J. Bolyai prepared a treatise on a complete system of non-Euclidean geometry, independently of the results of Lobachevsky. Bolyai's work was published in 1832 as an Appendix to a mathematics textbook by his father. As a matter of interest, let us mention that as a soldier, during his military service, J. Bolyai was for a short time (1832–1833) a member of a garrison also in Olomouc (Czech Republic, Olmütz), this fact is reminded by his memory desk under his bust at the Army House in Olomouc.

In 1779, first non-trivial examples of geodesic mappings were discovered by J. Lagrange [103].

In 1869, U. Dini[82)] posed a more general problem of existence of possible geodesic mappings of $\mathbb{V}_2$ onto $\overline{\mathbb{V}}_2$ and gave, in principle, its full solution. His solution, which was constructed in a rather complicated way, was modified, simplified and specified many times. Finally in 1896, T. Levi-Civita[83)] in his work on transformation of equations of dynamics, examined the question in their more general setting, and obtained the basic equations for the solution of geodesic mappings between classical Riemannian spaces.

Later on, when tensor approach started to be more common in differential geometry, H. Weyl, L.P. Eisenhart, and others gave a new, invariant and more flexible theory of geodesic mappings. A modification of such a view-point can be found e.g. in [78] V.F. Kagan. Geodesic mappings for Finsler spaces were formulated by H. Rund[84)] in [156].

It is also significant to mention those papers in which important results on applications of the methods of Lobachevsky geometry to analysis and integration of nonlinear equations of modern mathematical physics are obtained [750–752].

Geodesic mappings of the space on itself are called *projective* (*geodesic*) *transformations*. Projective transformations of Riemannian manifolds and also manifolds with affine connections form a Lie group.

Investigation of these groups was started by L. Fubini [422] see [8, 11, 45–47, 89, 118, 176, 254, 422, 423, 846]. Projective transformations as well as other aspects have been studied for projective spaces, Klingenberg spaces, see: [110, 511, 564].

6.1.2 Examples of geodesic mappings

Let us give some elementary examples of geodesic mappings $S \to \overline{S}$ of surfaces in $\mathbb{E}_3$.

Let us consider a surface S as a "rigid body" in the three-space, and apply on S any Euclidean motion (i.e. a particular type of isometry, a composition of rotations and translations in $\mathbb{E}_3$), and denote by $\overline{S}$ the resulting surface arising as an image of S under the motion. It is almost obvious that on the transported surface, the geodesics are preserved (are "the same as before"). Moreover, if the surfaces are "correspondingly coordinatized", i.e. coordinates of each point are the same as before the transport, there arises a geodesic mapping of S onto $\overline{S}$ in this way.

More generally, if $\overline{S}$ is obtained from S by some isometry, or by infinitesi-

[82)] Ulisse Dini, 1845–1918, was an Italian mathematician and politician in Pisa, who worked calculus and function theory.

[83)] Tullio Levi-Civita, 1875–1941, an Italian mathematician, most famous for his work on absolute differential calculus and its applications to the relativity theory (A. Einstein used his results in tensor calculus, an important tool in Einstein's development of the theory of general relativity). He influenced the future developement in differential geometry.

[84)] Hanno Rund, 1925–1993, a mathematician who received his PhD from the University in Cape Town, South Africa, and published numerous publications. The most famous is *The Hamilton-Jacobi theory in the calculus of variations. Its role in mathematics and physics.* D. Van Nostrand Comp., 1966.

mal isometry, then the intrinsic geometry is not changed, as we already know from basic courses of differential geometry. Hence geodesics are mapped onto geodesics, arc-length parameters of geodesic arcs are preserved, and such a mapping is again a geodesic one.

Now suppose that $\overline{S}$ is an image of S under a homothety in a three-space, and let k be the coefficient of the homothety. Then the metric tensors of the surfaces are related by

$$\overline{g} = k^2 \cdot g, \qquad \text{i.e.} \quad \overline{g}_{ij} = k^2 \cdot g_{ij},$$

and hence also the Christoffel symbols coincide, $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij}$. Consequently, a geodesic ℓ on S is mapped onto a geodesic $\overline{\ell}$ on $\overline{S}$, where the length parameters s on ℓ and $\overline{s}$ on $\overline{\ell}$ are related by $\overline{s} = k \cdot s + \text{const}$. So we can easily see that the canonical parameters of geodesics are preserved.

Two concentric spheres represent a simple example. Let us realize one common feature of all the above examples: one surface is mapped geodesically onto another surface if (under the condition of the corresponding coordinatization) the surfaces have the same Christoffel symbols of second kind in the corresponding points. Such mappings can be found, at least locally, for any pair of surfaces. That is why they are called *trivial geodesic mappings*. Naturally, non-trivial geodesic mappings are of more interest.

6.2 Fundamental properties of geodesic mappings

6.2.1 Levi-Civita equations of geodesic mappings

Now let us pay attention to geodesic mappings of manifolds with affine connection. Because geodesics on manifolds are characterized by the symmetric part of the connection only, we can restrict ourselves to torsion-free manifolds with affine connection, i.e. from now on, we assume that all connections under consideration are symmetric (= torsion-free).

So let us consider manifolds with affine connection $\mathbb{A}_n = (M, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}_n, \overline{\nabla})$ endowed with symmetric connections ∇ and $\overline{\nabla}$, respectively. All geometric objects in $\bar{\mathbb{A}}_n$ will be denoted by analogous letters as in $\mathbb{A}_n$, but with "bar".

Let there exist a geodesic mapping $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$. Since f is a diffeomorphism we can suppose that local coordinate maps on M or $\overline{M}$, respectively, are chosen as described in (4.2). That is, locally, $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ maps points onto points with the same coordinates, and $\overline{M} = M$, which considerably simplifies the technicalities: we suppose that all objects under consideration (as connections, tensor fields etc.), with bar or without, are defined on the same underlying manifold.

If $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is a geodesic mapping and $\ell\colon I \to M$, $\ell(t) = (x^1(t), \dots, x^n(t))$ is a geodesic in $\mathbb{A}_n$ then its image in $\bar{\mathbb{A}}_n$ has the same parametrization (with the same parameter), and hence a system of equations, similar to (2.25) must be satisfied, namely

$$\overline{\nabla}_\lambda \lambda = \overline{\varrho}(t)\lambda. \tag{6.1}$$

If we add (2.25) to (6.1) and subtract (4.1) we get $P(\lambda,\lambda) = (\overline{\varrho}(t) - \varrho(t))\,\lambda$, where $P(X,Y) = (\overline{\nabla} - \nabla)_X Y$ is a deformation tensor of affine connections, see (4.1) and (4.2).

In a fixed point x on M for all tangent vector λ the following formula hold

$$P(\lambda,\lambda) = \varrho(\lambda)\,\lambda, \tag{6.2}$$

where ϱ is a function which depended on λ.

Hence, $P(\lambda,\lambda) \wedge \lambda = 0$, i.e. $P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta\lambda^i - P^i_{\alpha\beta}\lambda^\alpha\lambda^\beta\lambda^h = 0$, and finally

$$(P^h_{\alpha\beta}\delta^i_\gamma - P^i_{\alpha\beta}\delta^h_\gamma)\lambda^\alpha\lambda^\beta\lambda^\gamma = 0. \tag{6.3}$$

Obviously, in the case of a trivial geodesic mapping, $P = 0$ holds, and (6.3) is satisfied identically. But also in the case $P \neq 0$, (6.3) must be satisfied identically. In fact, since there passes a geodesic through any point in any direction, and due to the definition of geodesic mapping, geodesics of the second manifold correspond to geodesics on the first one, the equations (6.3) must hold at any point independently of the vectors λ (which represent coordinates of the particular tangent direction). It means that the coefficients of the cubic form from (6.3) must vanish, that is, for any $i,j,k,\ \alpha,\beta \in \{1,\dots,n\}$,

$$P^\alpha_{ij}\delta^\beta_k + P^\alpha_{jk}\delta^\beta_i + P^\alpha_{ki}\delta^\beta_j = P^\beta_{ij}\delta^\alpha_k + P^\beta_{jk}\delta^\alpha_i + P^\beta_{ki}\delta^\alpha_j. \tag{6.4}$$

The equations (6.4) has a tensorial character. Contracting (6.4) in β and k we get $(n+1)\,P^\alpha_{ij} = P^\beta_{j\beta}\delta^\alpha_i + P^\beta_{i\beta}\delta^\alpha_j$. If we put

$$\psi_i = \frac{1}{n+1}\,P^\beta_{i\beta} \tag{6.5}$$

we introduce a 1-form ψ on $\mathbb{V}_n$ with components ψ_i, i.e. $\psi = \psi_j dx^j$ (in another notation, $\psi(X) = \frac{1}{n+1}\,Tr\,(Y \mapsto P(X,Y))$, in short $\psi = \frac{1}{n+1}\,Tr\,(Y \mapsto P(*,Y))$, where Tr denotes the trace of a linear map), and the formula for P can be given as follows:

$$P^k_{ij} = \psi_i\delta^k_j + \psi_j\delta^k_i, \tag{6.6}$$

or, in an invariant form,

$$P(X,Y) = \psi(X)Y + \psi(Y)X, \qquad X,Y \in \mathcal{X}(M), \tag{6.7}$$

eventually $P = \psi \otimes I + I \otimes \psi$.

It follows from (6.6) and (4.2) that

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \psi_i\delta^k_j + \psi_j\delta^k_i, \tag{6.8}$$

or equivalently, for any vector fields $X,Y \in \mathcal{X}(M)$:

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + X\psi(Y). \tag{6.9}$$

We will call (6.8) *the equations of Levi-Civita*. Although these relationship was discovered already by T. Levi-Civita, [107], for classical Riemannian spaces, the

formula was first written in the form (6.8) rather later on, by H. Weyl in 1918, [941], see [50, 51, 170, 892] etc.

We can easily check that the conditions (6.8) are not only necessary, but also sufficient. In fact, if the equations $x^i = x^i(t)$ determine a solution of (2.25), that is, geodesics of $\mathbb{A}_n$, and if (6.9) holds in any coordinate system which is common for $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ (w.r.t. f) we can rewrite (3.18) in the form

$$\overline{\nabla}_\lambda \lambda = \overline{\sigma}(t)\lambda \quad \text{where} \quad \overline{\sigma}(t) = \sigma(t) + 2\,\psi(\lambda).$$

Hence it follows that $x^k = x^k(t)$ represent a parametrization of geodesics even in $\bar{\mathbb{A}}_n$, and consequently the mapping is a geodesic one.

Therefore the following theorem holds.

Theorem 6.1 *A manifold with affine connection $\mathbb{A}_n$ admits a geodesic mapping onto $\bar{\mathbb{A}}_n$ if and only if the Levi-Civita equations (6.8) hold.*

Due to the fact that the equations (6.8) characterize geodesic mappings $\mathbb{A}_n \to \bar{\mathbb{A}}_n$, they are called *fundamental*.

Geodesic mappings with $\psi \equiv 0$ will be called *trivial* or *affine*.

Under an affine mapping $f\colon\ \mathbb{A}_n \to \bar{\mathbb{A}}_n$, not only geodesics are preserved, but also their canonical parameter remains canonical. Furthermore, any affine mapping is connection-preserving. That is, from the manifold view-point, we can suppose for simplicity that the manifolds with affine connection, the image and the pre-image, coincide.

Remarks.
For completeness we present different two proofs of the previous theorem. The second one is using vector forms and operators and is applicable for infinite dimension spaces. The third one is easier and applicable for finite dimension spaces.

Note. Formulae (6.2) and (6.7) are equivalent if and only if $\varrho(\lambda) = 2\psi(\lambda)$.

Formula (6.2) follows for $\varrho(\lambda) = 2\psi(\lambda)$ from (6.7). On the oter hand if $\varrho(\lambda)$ is linear, that for $\psi = 1/2\varrho$, from (6.2) it follows formula (6.7).

We only need to prove that the formula (6.2) implies linearity of $\varrho(\lambda)$.

Proof 2.
The second method of proof is based on the operator calculus. If we suppose that $\lambda = \alpha \cdot v$ in (6.2) we can see that ϱ is homogeneous, i.e $\varrho(\alpha \cdot v) = \alpha \cdot \varrho(v)$. After that we calculate for the linearly independent vectors v and w:

$$P(v+w, v+w) = P(v,v) + 2P(v,w) + P(w,w),$$
$$P(v-w, v-w) = P(v,v) - 2P(v,w) + P(w,w).$$

After summation and application of formula (6.2) we have the following

$$\varrho(v+w)\cdot(v+w) + \varrho(v-w)\cdot(v-w) = 2\varrho(v)\cdot v + 2\varrho(w)\cdot w.$$

Because v and w are non-collinear it follows additivity of the operator ϱ, i.e. $\varrho(v+w) = \varrho(v) + \varrho(w)$. This means that ϱ is linear. □

Proof 3.

We can rewrite formula (6.2) in local form $P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho(\lambda)\cdot\lambda^h$.

For $\lambda^1 = 0$ and arbitrary λ^a, $a > 1$, it follows $P^1_{ab} = 0$ for $a, b > 1$, and for $h = 1$ has the following expression $P^1_{11}(\lambda^1)^2 + 2\sum\limits_{a=2}^{n} P^1_{1a}\lambda^a\cdot\lambda^1 = \varrho(\lambda)\cdot\lambda^1$, this means $\varrho(\lambda) = P^1_{11}\lambda^1 + 2\sum\limits_{a=2}^{n} P^1_{1a}\lambda^a$ and the function $\varrho(\lambda)$ is linear.

Note that the above formula is valid for all λ, $\lambda^1 \neq 0$. We can do analogous calculations sequentially for $\lambda^2, \ldots, \lambda^n$ and therefore linearity of ϱ is true for all $\lambda \neq 0$. The case $\lambda = 0$ is excluded. □

Remark. The problem in which each geodesic $\ell(s)$ with canonical parameter on $\mathbb{A}_n$ $(\nabla_{\lambda(s)}\lambda(s) = 0)$ is mapped onto a geodesic $\overline{\ell} = f(\ell(s))$ on $\overline{\mathbb{A}}_n$ $(\overline{\nabla}_{\lambda(s)}\lambda(s) = \varrho(s)\lambda(s))$ has the same solution – the Levi-Civita equation (6.8).

The mapping $\mathbb{A}_n \to \overline{\mathbb{A}}_n$ in which each geodesic $\ell(s)$ with canonical parameter on $\mathbb{A}_n$ $(\nabla_{\lambda(s)}\lambda(s) = 0)$ is mapped onto a geodesic with the same canonical parameter s on $\overline{\mathbb{A}}_n$ $(\overline{\nabla}_{\lambda(s)}\lambda(s) = 0)$ is an *affine mapping* which is characterized by the vanishing of deformation tensor P, i.e. the equation (4.37) holds. This mapping is also called *trivial geodesic mapping*.

6.2.2 Equivalence classes of geodesic mappings

It follows from the Levi-Civita equations that the class of manifolds with affine connection is decomposed into equivalence subclasses with respect to geodesic mappings.

A geodesic mapping $f\colon \mathbb{A}_n \to \overline{\mathbb{A}}_n$, which is characterized by the equations (6.9), will be denoted by

$$\mathbb{A}_n \xrightarrow{g.m.\ (\psi)} \overline{\mathbb{A}}_n.$$

The properties of equivalence can be easily checked:

1. The identity is geodesic since we have $\mathbb{A}_n \xrightarrow{g.m.\ (0)} \overline{\mathbb{A}}_n$.
2. If $\mathbb{A}_n \xrightarrow{g.m.\ (\psi)} \overline{\mathbb{A}}_n$, then $\overline{\mathbb{A}}_n \xrightarrow{g.m.\ (-\psi)} \mathbb{A}_n$.
3. If $\mathbb{A}_n \xrightarrow{g.m.\ (\psi)} \overline{\mathbb{A}}_n$ and $\overline{\mathbb{A}}_n \xrightarrow{g.m.\ (\overline{\psi})} \overline{\overline{A}}_n$, then $\mathbb{A}_n \xrightarrow{g.m.\ (\psi+\overline{\psi})} \overline{\overline{A}}_n$.

Two manifolds $\mathbb{A}_n$ and $\overline{\mathbb{A}}_n$ belong to the same *geodesic class* iff there is a geodesic mapping of $\mathbb{A}_n$ onto $\overline{\mathbb{A}}_n$. Spaces from the same geodesic class are also called *geodesically equivalent*. We can apply this concept locally as well as globally.

6.2.3 Thomas projective parameter

Consider two manifolds with affine connection $\mathbb{A}_n = (M_n, \nabla)$ and $\bar{\mathbb{A}}_n = (\overline{M}_n, \overline{\nabla})$ (with the corresponding affine connections ∇ and $\overline{\nabla}$, respectively). Suppose there exists a geodesic map $f : \mathbb{A}_n \to \bar{\mathbb{A}}_n$. Let us identify $\overline{M}$ with M via f, so that the deformation tensor $P = \overline{\nabla} - \nabla$ of f can be defined. Then (4.2), (6.5) and (6.8) hold.

Assume geodesic mappings between manifolds with affine connection $\mathbb{A}_n \to \bar{\mathbb{A}}_n$. Then the equations (6.9) are satisfied. Contracting these equations we can express the covector ψ_i (i.e. find the one-form ψ):

$$\psi_i = \frac{1}{n+1}\,(\overline{\Gamma}^{\alpha}_{i\alpha} - \Gamma^{\alpha}_{i\alpha}), \tag{6.10}$$

or

$$\psi(X) = \frac{1}{n+1}\mathrm{Tr}\,(Y \to \overline{\nabla}(X,Y) - \nabla(X,Y)), \tag{6.11}$$

respectively.

Expressing ψ_i (or $\psi(X)$, respectively) from the Levi-Civita equations we get

$$\overline{\Pi}^{h}_{ij}(x) = \Pi^{h}_{ij}(x), \tag{6.12}$$

or

$$\overline{\Pi}(X,Y) = \Pi(X,Y) \quad \text{for all } X, Y \in \mathcal{X}(M), \tag{6.13}$$

respectively, where

$$\Pi^{k}_{ij} = \Gamma^{k}_{ij} - \frac{1}{n+1}\left(\delta^{k}_{i}\Gamma^{\alpha}_{\alpha j} + \delta^{k}_{j}\Gamma^{\alpha}_{\alpha i}\right), \tag{6.14}$$

or

$$\Pi(X,Y) = \nabla(X,Y) - \frac{1}{n+1}\left(X \cdot \mathrm{Tr}\,(\nabla(Y,*)) - Y \cdot \mathrm{Tr}\,(\nabla(X,*))\right), \tag{6.15}$$

and $\overline{\Pi}$ is introduced analogously in $\bar{\mathbb{A}}_n$.

An object Π defined by (6.15) is called a *Thomas projective parameter*, or *Thomas object of projective connection* of $\mathbb{A}_n$. It was found by T.Y. Thomas in 1925, [892] (as we shall see, they are *invariant under geodesic maps*). This parameter is a geometric object, and under the coordinate change (2.8), its coordinate transformation reads

$$\Pi'^{h}_{ij}(x')\frac{\partial x^h}{\partial x'^{\alpha}} = \Pi^{h}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{i}}\frac{\partial x^{\beta}}{\partial x'^{j}} + \frac{1}{n+1}\left(\frac{\partial \ln\Delta}{\partial x'^{i}}\frac{\partial x^h}{\partial x'^{j}} + \frac{\partial \ln\Delta}{\partial x'^{j}}\frac{\partial x^h}{\partial x'^{i}}\right) + \frac{\partial^2 x^h}{\partial x'^{i}\partial x'^{j}}$$

where $\Delta = \det\left(\frac{\partial x^i}{\partial x'^j}\right)$.

A simple verification shows that the formula (6.13) is not only necessary for $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ to be geodesic, but also sufficient. The following holds:

Theorem 6.2 *A manifold with affine connection $\mathbb{A}_n$ admits a geodesic mapping f onto $\bar{\mathbb{A}}_n$ if and only if Thomas projective parameter Π is invariant under f, i.e. the formulas (6.13) (or (6.12) in a common coordinate system x, respectively) are satisfied.*

Theorem 6.3 *Given a diffeomorphism of manifolds with affine connection, invariance of the Thomas projective parameters Π is a necessary and sufficient condition for the map to be geodesic.*

6.2.4 Manifold with projective connection

A geometric object Π that transforms according to a similar transformation law as Thomas projective parameters is called a *projective connection*. Manifolds on which an object of projective connection is defined is called a *manifold with projective connection* and denoted by $\mathbb{P}_n$. Such manifolds represent an obvious generalization of affine connection manifolds.

A projective connection on $\mathbb{P}_n$ will be denoted by $\blacktriangledown$. Obviously, $\blacktriangledown$ is a mapping $TP_n \times TP_n \to TP_n$, i.e. $(X, Y) \mapsto \blacktriangledown_X Y$. Thus, we denote a manifold M_n with projective connection $\blacktriangledown$ by $\mathbb{P}_n = (M_n, \blacktriangledown)$. See [51, p. 99], [55].

We restricted ourselves to the study of a coordinate neighbourhood (U, x) of the points $p \in \mathbb{A}_n$ $(\mathbb{P}_n)$ and $f(p) \in \bar{\mathbb{A}}_n$ $(\overline{\mathbb{P}}_n)$. The points p and $f(p)$ have the same coordinates $x = (x^1, \dots, x^n)$.

We assume that $\mathbb{A}_n$, $\bar{\mathbb{A}}_n$, $\mathbb{P}_n$, $\overline{\mathbb{P}}_n \in C^r$ $(\nabla, \overline{\nabla}, \blacktriangledown, \overline{\blacktriangledown} \in C^r)$ if their components $\Gamma^h_{ij}(x), \overline{\Gamma}^h_{ij}(x), \Pi^h_{ij}(x), \overline{\Pi}^h_{ij}(x) \in C^r$ on (U, x), respectively. Here C^r is the smoothness class. On the other hand, the manifold M_n in which these structures exist, must have a class smoothness C^{r+2}. This means that the atlas on M_n is of class C^{r+2}, i.e. for the non disjunct charts (U, x) and (U', x') on $(U \cap U')$ it is true that the transformation $x' = x'(x) \in C^{r+2}$.

Formulae (6.9) and (6.13) in the common system (U, x) have the local form:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i \quad \text{and} \quad \overline{\Pi}^h_{ij}(x) = \Pi^h_{ij}(x),$$

respectively, where ψ_i are components of ψ and δ^h_i is the Kronecker delta.

It is seen that in a manifold $\mathbb{A}_n = (M_n, \nabla)$ with affine connections ∇ there exists a projective connection $\blacktriangledown$ (i.e. Thomas projective parameter) with the same smoothness. The opposite statement is not valid, for example if $\nabla \in C^r$ $(\Rightarrow \blacktriangledown \in C^r$ and also $\overline{\blacktriangledown} \in C^r)$ and $\psi(x) \in C^0$, then $\overline{\nabla} \in C^0$.

Let $\mathbb{P}_n = (M_n, \blacktriangledown)$ be given. Put a question on existence of affine connection ∇ on M_n with common geodesics.

This question is locally solved in papers by È. Cartan [322] and J.M. Thomas [891], see [51, p. 105], where the construction of a normal affine connection ∇ is presented:

$$\Gamma^h_{ij}(x) = \Pi^h_{ij}(x). \tag{6.16}$$

This connection ∇ is equiaffine, because $\Gamma^\alpha_{i\alpha} = 0$.

In the paper [480] we presented global construction of ∇ on M_n. Moreover, the following theorem holds:

Theorem 6.4 *An arbitrary manifold $\mathbb{P}_n = (M_n, \blacktriangledown) \in C^r$ admits a global geodesic mapping onto a manifold $\bar{\mathbb{A}}_n = (M_n, \overline{\nabla}) \in C^r$ and, moreover, for which a formula* $\operatorname{trace}(V \to \overline{\nabla}_V)X = \nabla_X G$ *holds for arbitrary X and a function G on M_n, i.e. $\bar{\mathbb{A}}_n$ is an equiaffine manifold and $\overline{\nabla}$ is an equiaffine connection. Moreover, if $r \geq 1$ the Ricci tensor on $\bar{\mathbb{A}}_n$ is symmetric.*

Proof. It is known that on the whole manifold $M_n \in C^{r+2}$ exists globally a sufficiently smooth metric $\hat{g} \in C^{r+1}$. For our purpose it is sufficient if $\hat{g} \in C^{r+1}$, i.e. the components $\hat{g}_{ij}$ of $\hat{g}$ in a coordinate domain of M_n are functions of type C^{r+1}. We denote by $\hat{\nabla}$ the Levi-Civita connection of $\hat{g}_{ij}$, and, evidently, $\hat{\nabla} \in C^r$.

We define $\tau(X) = \frac{1}{n+1}\, trace(V \mapsto \hat{\nabla}_V X)$ and we construct $\overline{\nabla}$ in the following way

$$\overline{\nabla}_X Y = \blacktriangledown_X Y + \tau(X) \cdot Y + \tau(Y) \cdot X.$$

It is easily seen that $\overline{\nabla}$ constructed in this way is an affine connection on M_n. The components of the object $\overline{\nabla}$ in the coordinate system (U, x) can be written in the form: $\overline{\Gamma}^h_{ij}(x) = \Pi^h_{ij}(x) + \tau_i(x) \cdot \delta^h_j + \tau_j(x) \cdot \delta^h_i$ where Π^h_{ij} and $\overline{\Gamma}^h_{ij}$ are components of the projective connection $\blacktriangledown$ and the affine connection $\overline{\nabla}$, respectively, and $\tau_i = \frac{1}{n+1}\, \partial G / \partial x^i$, $G = \ln \sqrt{|\det \|\hat{g}_{ij}\||}$. It is obvious that $\mathbb{P}_n$ is geodesically mapped onto $\bar{\mathbb{A}}_n = (M_n, \overline{\nabla})$, and, evidently because $\overline{\Gamma}^h_{ij} \in C^r$, $\bar{\mathbb{A}}_n \in C^r$.

Insofar as $\Pi^\alpha_{\alpha i}(x) = 0$, then $\overline{\Gamma}^\alpha_{\alpha i}(x) = \partial G / \partial x^i$, i.e. $\operatorname{trace}(V \to \overline{\nabla}_V)X = \nabla_X G$. Hence follows that $\bar{\mathbb{A}}_n$ has an equiaffine connection [135, p. 151]. Moreover, if $\nabla \in C^1$ then the Ricci tensor Ric is symmetric [122, p. 35], [135, p. 151]. □

6.2.5 Riemannian and Ricci tensor under geodesic mappings

Applying the formula (4.3) and the expression of the deformation tensor P under a geodesic mapping $\mathbb{A}_n \to \bar{\mathbb{A}}_n$: (6.6) ((6.7), respectively) we find a relationship between Riemannian tensors (or curvature tensors):

$$\overline{R}(Y, Z)X = R(Y, Z)X + X \cdot (\psi(Z, Y) - \psi(Y, Z)) + Z \cdot \psi(X, Y) - Y \cdot \psi(X, Z),$$

which in components reads

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_i(\psi_{kj} - \psi_{jk}) + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}, \tag{6.17}$$

where $\psi(X, Y) = \nabla_Y X - \psi(X)\psi(Y)$, or locally,

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j. \tag{6.18}$$

If we contract (6.17) over h and k we obtain the following relation for the Ricci tensors: $\overline{\mathrm{Ric}}\,(X, Y) = \mathrm{Ric}\,(X, Y) - n\psi(X, Y) + \psi(Y, X)$, since

$$\overline{R}_{ij} = R_{ij} - \psi_{[ij]} - (n-1)\psi_{ij}. \tag{6.19}$$

Here $\overline{R}_{ij}$ and R_{ij} denote the Ricci tensors of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively, and $[ij]$ denotes the alternation without division.

6.2.6 Weyl tensor of projective curvature

Alternating of (6.19) with respect to the indices i, j we get

$$(n+1)\psi_{[ij]} = R_{[ij]} - \overline{R}_{[ij]}.$$

Hence from (6.19) we obtain

$$(n+1)(n-1)\psi_{[ij]} = (nR_{ij} + R_{ij}) - (n\overline{R}_{ji} + \overline{R}_{ji}).$$

Substituting ψ_{ij} to (6.17) we find after simple computation

$$\overline{W}^h_{ijk} = W^h_{ijk}, \qquad \overline{W} = W, \tag{6.20}$$

where

$$\begin{aligned} W(Y,Z)X = R(Y,Z)X - \frac{1}{n+1}(\mathrm{Ric}\,(Y,Z) - \mathrm{Ric}\,(Z,Y))\cdot X \\ + \frac{1}{(n+1)(n-1)}\left[(n\,\mathrm{Ric}\,(X,Y) + \mathrm{Ric}\,(Y,X))\cdot Z\right. \\ \left. + (n\,\mathrm{Ric}\,(X,Z) + \mathrm{Ric}\,(Z,X))\cdot Y\right], \end{aligned} \tag{6.21}$$

or locally

$$\begin{aligned} W^h_{ijk} &= R^h_{ijk} - \frac{1}{n+1}\delta^h_i\,(R_{jk} - R_{kj}) \\ &+ \frac{1}{(n+1)(n-1)}\left[(n\,R_{ij} + R_{ji})\delta^h_k - (n\,R_{ik} + R_{ki})\delta^h_j\right]. \end{aligned} \tag{6.22}$$

Analogously we define $\overline{W}$ on $\bar{\mathbb{A}}_n$.

Since the Ricci tensor on an equiaffine manifold $\mathbb{A}_n$ is symmetric, and of course on a Riemannian manifold $\mathbb{V}_n$ as well, the Weyl tensor W is simplified as follows:

$$W^h_{ijk} = R^h_{ijk} - \frac{1}{1-n}(\delta^h_j R_{ik} - \delta^h_k R_{ij}). \tag{6.23}$$

In the invariant form,

$$W(X,Y,Z) = R(Y,Z)X + \frac{1}{1-n}(\mathrm{Ric}\,(X,Y)\cdot Z - \mathrm{Ric}\,(X,Z)\cdot Y).$$

Definition 6.2 The tensor field W of the type $(1,3)$ introduced by (6.22) is called the *tensor of projective curvature* of the manifold $\mathbb{A}_n$ with affine connection, or of the Riemannian manifold $\mathbb{V}_n$, respectively.

It is also called the *Weyl tensor of projective curvature* after Weyl who discovered it in 1921, see [941].[85)]

We realize that the tensor of projective curvature W on a manifold with affine connection $\mathbb{A}_n$, as well as the curvature and the Ricci tensor, is *uniquely defined* by the affine connection ∇, and similarly on a Riemannian manifold $\mathbb{V}_n$, by its metric tensor. As well known, W is trace-less, it is in fact the trace-less part of the curvature tensor R of type $(1,3)$.

From the formulas (6.13) and (6.20) above, the following theorem follows.

[85)] More often, under the Weyl tensor the tensor of conformal curvature is understood.

Theorem 6.5 *The Weyl tensor of projective curvature and the Thomas projective parameter are invariants of geodesic mappings of manifolds with affine connection (of (pseudo-) Riemannian manifolds, respectively).*

In the papers by Thomas [186] and Rapczak [765] a construction of other geometric objects of intrinsic character is presented that are invariants of geodesic mappings. These objects are expressed by means of the Thomas projective parameter, since their invariance is a property characterizing geodesic mappings.

6.2.7 Geodesic mappings of equiaffine manifolds

As we already mentioned above, an equiaffine manifold $\mathbb{A}_n$ [86)] is characterized by the symmetry of its Ricci tensor:

$$\mathrm{Ric}\,(X,Y) = \mathrm{Ric}\,(Y,X).$$

This condition is equivalent to the property that locally, in any coordinate chart (U,φ), $\varphi = (x^i)$, there is a function $f(x)$ such that

$$\mathrm{Tr}\,\nabla(X,*) = \nabla_X f, \quad \text{i.e. } \Gamma^\alpha_{i\alpha}(x) = \partial_i f(x). \tag{6.24}$$

Remark The function $f(x)$ from the previous formula is not a tensor of type (0,0), as it can be different in other coordinate systems. On the other hand, this function is geometric object, but not necessarily induced by a global function defined on the manifold.

Suppose that $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is a geodesic mapping between equiaffine manifolds $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$. Hence the Ricci tensors of $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively, satisfy $\mathrm{Ric}\,(X,Y) = \mathrm{Ric}\,(Y,X)$ and $\overline{\mathrm{Ric}}\,(X,Y) = \overline{\mathrm{Ric}}\,(Y,X)$. Alternating the formula (6.19) we get

$$\psi(X,Y) = \psi(Y,X), \quad \text{locally}, \quad \psi_{ij} = \psi_{ji}. \tag{6.25}$$

Due to this formula, we can simplify (6.17) and (6.19):

$$\overline{R}(Y,Z)X = R(Y,Z)X + Z\cdot\psi(X,Y) - Y\cdot\psi(X,Z),$$

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} \tag{6.26}$$

and

$$\overline{\mathrm{Ric}}\,(X,Y) = \mathrm{Ric}\,(X,Y) - (n-1)\cdot\psi(X,Y),$$

$$\overline{R}_{ij} = R_{ij} - (n-1)\cdot\psi_{ij}. \tag{6.27}$$

As we already mentioned before, also the Weyl tensor of projective curvature is simplified, see (6.22).

Let us go back to the formula (6.25). It follows from the definition (6.18) of the tensor ψ that

$$\nabla_X\psi(Y) = \nabla_Y\psi(X). \tag{6.28}$$

[86)] or manifold with equiaffine connection

Hence we deduce that $\psi(X)$ is locally a gradient vector ($\psi(X)$ is a closed one-form), i.e. for a chart $(U,(x^i))$ there exists a function (invariant) Ψ for which

$$\psi(X) = \nabla_X \Psi, \tag{6.29}$$

that is locally,

$$\psi_i(x) = \partial_i \Psi(x). \tag{6.30}$$

This result can be deduced also directly from the analysis of the Levi-Civita equation (6.8). Contracting (6.8) with respect to h and k we get:

$$\overline{\Gamma}^{\alpha}_{i\alpha} = \Gamma^{\alpha}_{i\alpha} + (n+1)\psi_i.$$

Since $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ are equiaffine, there are functions f and $\overline{f}$ such that

$$\Gamma^{\alpha}_{i\alpha} = \partial_i f(x) \qquad \text{and} \qquad \overline{\Gamma}^{\alpha}_{i\alpha} = \partial_i \overline{f}(x).$$

It follows that

$$\psi_i = \frac{1}{n+1}\, \partial_i(\overline{f}(x) - f(x)). \tag{6.31}$$

That is, ψ_i is locally a gradient.

Remark Note that a one-form $\psi(X)$ determining a geodesic mapping between equiaffine manifolds need not be globally a gradient, i.e. there need not be a global function Ψ such that $\psi(X) = \nabla_X \Psi$.

Let us point out the important role of manifolds with affine connection in the theory of geodesic mappings. We show that globally, any projective class of manifolds with affine connection includes equiaffine manifolds.

In the paper by È. Cartan [322] and J.M. Thomas [891], see [51, p. 105], using normal connection authors have it was proved that any manifold with affine connection is locally projective equiaffine. We show that these properties hold globally, i.e. any arbitrary manifold with affine connection globally admits geodesic mappings onto some equiaffine manifold.

For this reason, the solution of the problem of the projective metrizability of a manifold $\mathbb{A}_n$ (or equivalently of geodesic mappings of $\mathbb{A}_n$ onto (pseudo-) Riemannian manifolds $\overline{\mathbb{V}}_n$) can be realized as a geodesic mapping of the equiaffine manifold $\tilde{\mathbb{A}}_n$, which is projectively equivalent to the given manifold $\mathbb{A}_n$.

Using Theorem 6.4 we may see, that the following theorem holds generally:

Theorem 6.6 (Mikeš, Hinterleitner [663]) *All manifolds $\mathbb{A}_n$ with affine connection are projectively equivalent to equiaffine manifolds.*

In other words, an arbitrary manifold $\mathbb{A}_n$ ($\in C^r$, $r \geq 1$) with affine connection admits a global geodesic mapping onto an equiaffine manifold $\bar{\mathbb{A}}_n$, and moreower $\bar{\mathbb{A}}_n \in C^r$.

The equiaffine connection $\overline{\nabla}$ constructed in this way is constructed explicitly from the original connection ∇.

Similarly it could be shown that a manifold with projective connection admits a geodesic mapping onto a manifold with equiaffine connection.

Simple proof for the local case. Let us construct a covector $\psi_i(x)$ as follows:

$$\psi_i(x) = -\frac{1}{n+1}\,\Gamma^\alpha_{\alpha i}(x). \tag{6.32}$$

From (6.32) and the Levi-Civita equation (6.9), it follows

$$\overline{\Gamma}^\alpha_{\alpha i}(x) = 0. \tag{6.33}$$

The formulae (6.32) and (6.33) hold only in a distinguished coordinate system x, since $\overline{\Gamma}^\alpha_{\alpha i}(x)$ is not a covector. □

6.3 Projectively flat manifolds

6.3.1 Geodesic mappings of projectively flat manifolds

Now let us consider geodesic mappings of $\mathbb{A}_n$ onto flat spaces $\bar{\mathbb{A}}_n$. If such a geodesic mapping exists, the space $\mathbb{A}_n$ will be called a *projectively flat manifold*.

Since the curvature vanishes in any flat manifold $\bar{\mathbb{A}}_n$, $\overline{R} = 0$, the Ricci tensor also vanishes, $\overline{\mathrm{Ric}} = 0$ (see (2.18)), and it follows from (6.21) that in $\bar{\mathbb{A}}_n$, the Weyl tensor of projective curvature vanishes, $\overline{W} = 0$.

The Weyl tensor of projective curvature is invariant under geodesic mappings, hence it must vanish in any projectively flat manifold, that is,

$$W = 0. \tag{6.34}$$

Note that if $n > 2$, the condition (6.34) is also sufficient, more precisely:

Theorem 6.7 *A manifold with affine connection $\mathbb{A}_n$ $(n > 2)$ is locally projectively flat if and only if the Weyl tensor of projective curvature vanishes.*

Proof. Obviously, if $\mathbb{A}_n$ is projectively flat, or locally projectively flat, then $W = 0$. Let us check the converse. Let (6.34) holds in $\mathbb{A}_n\,(n > 2)$. By Theorem 6.6, $\mathbb{A}_n$ can be globally geodesically mapped onto an equiaffine manifold $\tilde{\mathbb{A}}_n$ in which the Weyl tensor of projective curvature also vanishes, $\tilde{W} = 0$. So in what follows, we can suppose that $\mathbb{A}_n$ is (locally, around x) equiaffine, and the condition $W = 0$ can be equivalently expressed in terms of the curvature tensor as

$$R^h_{ijk} = \frac{1}{n-1}\,(\delta^h_j R_{ik} - \delta^h_k R_{ij}). \tag{6.35}$$

By (6.35) and the Bianchi identity (2.17c), we get

$$\delta^h_k R_{i[j,k]} + \delta^h_\ell R_{i[k,j]} + \delta^h_\ell R_{i[j,k]} = 0.$$

Contracting over indices h and ℓ we obtain

$$R_{ij,k} = R_{ik,j}. \tag{6.36}$$

Further, for the covariant derivative of ψ_i we get the formula

$$\psi_{i,j} = \psi_i\psi_j + \frac{1}{n-1}R_{ij}. \tag{6.37}$$

The integrability conditions for (6.37) take the form of the Ricci identity

$$\psi_{i,jk} - \psi_{i,kj} = -\psi_\alpha R^\alpha_{ijk}.$$

The conditions are satisfied identically according to (6.35), (6.36) and (6.37). Hence by the results on systems of PDEs of Cauchy type (see Subsection 1.5, p. 100), there is locally a solution $\psi_i(x)$ for initial data $\psi_i(x_0) = \psi_i^0$ in a neighbourhood of any point x_0. Take the covector $\psi_i(x)$ as a solution of (6.37), and let us construct the geodesic mapping of $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ generated by $\psi_i(x)$. We get from (6.37) that the Ricci tensor $\overline{R}_{ij}$ of $\bar{\mathbb{A}}_n$ vanishes. Since $\overline{W} = 0$ in $\bar{\mathbb{A}}_n$ we get $\overline{R}^h_{ijk} = 0$ (by (6.23)) which indicates that $\bar{\mathbb{A}}_n$ is flat. That is locally, for $\mathbb{A}_n$ there exists a geodesic mapping onto a flat manifold $\bar{\mathbb{A}}_n$. □

6.3.2 Characterization of projectively flat manifolds

In this part we find a new property of projectively-Euclidean manifolds, which can be viewed as a generalization of properties that have been found earlier by Schouten and Struik [163].

First we prove the following theorem [673]:

Theorem 6.8 *Let $A^h_{ijk}(x)$ be a tensor of type $(1,3)$ which satisfies the following equation:*

$$A^h_{ijk} + A^h_{ikj} = A^h_{ijk} + A^h_{jki} + A^h_{kij} = 0. \tag{6.38}$$

Let in each local coordinate system, $A^1_{223} = 0$ (or $A^1_{234} = 0$) hold.

Then the tensor A^h_{ijk} has the form

$$A^h_{ijk} = \delta^h_i(A_{jk} - A_{kj}) + \delta^h_j A_{ik} - \delta^h_k A_{ij}, \tag{6.39}$$

where A_{ij} is a certain tensor of type $(0,2)$.

Proof. By the coordinate transformation $x^{'h} = x^{'h}(x^1, x^2, \cdots, x^n)$, the components of the tensor A^h_{ijk} are transformed according to

$$A'^h_{ijk} = A^\alpha_{\beta\gamma\delta}A^h_\alpha B^\beta_i B^\gamma_j B^\delta_k \tag{6.40}$$

where $A^h_i \overset{\text{def}}{=} \partial_i x^{'h}$; $\left\|B^h_i\right\| \overset{\text{def}}{=} \left\|A^h_i\right\|^{-1}$. It is easy to see that if $A^1_{223} = 0$ (or $A^1_{234} = 0$) holds in any coordinate system then

$$(a)\ A^h_{iij} = 0; \quad (b)\ A^h_{ijk} = 0 \tag{6.41}$$

is satisfied in any coordinate system for any distinct indices h, i, j, k. We use the following transformations of coordinates:

$$x^{'p} = x^p + rx^q, \quad x^{'s} = x^s, \quad s \neq p. \tag{6.42}$$

Here p and q, respectively, are fixed distinct indices, and r is any real constant. Thus A_i^h and B_i^h are in the form (we used (6.40))

$$A_i^h = B_i^h = \delta_i^h; \quad A_q^p = -B_q^p = r, \tag{6.43}$$

provided either $h \neq p$, or $i \neq q$. Let us express the components of the tensor A in a new coordinate system which is determined by the transformations (6.42):

$$\begin{aligned}
&(a) \quad A'^h_{pqk} = A^h_{pqk} + rA^h_{qqk};\\
&(b) \quad A'^h_{ppk} = A^h_{ppk} + r(A^h_{pqk} + A^h_{qpk}) + r^2 A^h_{qqk};\\
&(c) \quad A'^q_{pjk} = A^q_{pjk} + r(A^q_{qjk} - A^p_{pjk}) - r^2 A^p_{qqk};\\
&(d) \quad A'^q_{ipk} = A^q_{ipk} + r(A^q_{iqk} - A^p_{ipk}) - r^2 A^p_{iqk};\\
&(e) \quad A'^q_{ppk} = A^q_{ppk} - r(A^p_{ppk} - A^q_{qpk} - A^q_{pqk}) -\\
&\qquad\qquad - r^3 A^p_{qqk} + r^2(A^q_{qqk} - A^p_{pqk} - A^p_{qpk}).
\end{aligned} \tag{6.44}$$

In the last formulae, differently marked indices are actually distinct. We do not use the Einstein summation convention in the indices p and q neither here nor further. Using (6.44a), it is easy to see that (6.41b) follows from (6.41a).

Now let us prove the converse. Let (6.41a) hold, i.e. $A^h_{iik} = 0$ in any coordinate system for different h, i, k. Then from (6.44b) it follows

$$A^h_{ijk} + A^h_{jik} = 0,$$

where h, i, j, k are different. Let us alternate the last formula in the indices j and k. Using (6.38) we get (6.41b) for the tensor A. Further, from (6.44d) and (6.44) we find

$$A^q_{qjk} = A^p_{pjk} \quad \text{and} \quad A^q_{iqk} = A^p_{ipk},$$

where $p, q \neq j, k$. Hence it follows that

$$A^p_{pjk} = B_{jk}; \quad A^p_{ipk} = A_{ik} \tag{6.45}$$

(for all indices p, j, k with $p \neq j, k$) form a geometric object.

Using (6.41b), (6.45) where r is any real constant, from (6.44e) we get

$$A^p_{ppk} = B_{pk} + A_{pk} \tag{6.46}$$

for the indices p, k with $p \neq k$. We check that (6.41), (6.45) and (6.46) can be expressed as follows (h, i, j, k are arbitrary)

$$A^h_{ijk} = \delta^h_i B_{jk} + \delta^h_j A_{ik} - \delta^h_k A_{ij}. \tag{6.47}$$

We symmetrize in the indices i, j, k and check

$$B_{jk} = A_{jk} - A_{kj}.$$

Now we can see that (6.39) holds. It remains to prove that A_{ij} is a tensor of type $(0, 2)$. Contracting (6.39) we get

$$A_{kj} = \frac{1}{n^2 - 1}(nA^\alpha_{jk\alpha} + A^\alpha_{kj\alpha}). \tag{6.48}$$

Obviously, components of a tensor stand on the right hand side of (6.48), hence A_{kj} is a type $(0,2)$ tensor, which completes the proof of Theorem 6.8. □

The following can be verified:

Theorem 6.9 *Let $\mathbb{A}_n$ be a manifold with affine connection. Let components of its curvature tensor satisfy in any coordinate system the following:*

$$\textit{either} \quad \textit{(a)} \quad R^1_{223} = 0, \qquad \textit{or} \quad \textit{(b)} \quad R^1_{234} = 0. \tag{6.49}$$

Then $\mathbb{A}_n$ is a projectively flat manifold.

Let us note the following:
a) It is possible to replace the formula (6.49) by one of the following formulas:

$$\text{(a)} \quad R^h_{iik} = 0; \qquad \text{or} \quad \text{(b)} \quad R^h_{ijk} = 0, \tag{6.50}$$

where h, i, j, k, respectively, are distinct indices.

b) It is possible to replace the curvature tensor in the Theorem 6.9 by the Weyl tensor of projective curvature, or by the Yano tensor of sectional curvature.

c) If we replace the curvature tensor by the Weyl tensor of conformal curvature then $\mathbb{A}_n$ will be conformally Euclidean ($\mathbb{A}_n$ is a Riemannian space).

Proof. By Theorem 6.8, the curvature tensor of $\mathbb{A}_n$ for which the assumptions of the Theorem 6.9 hold, satisfies

$$R^h_{ijk} = \delta^h_i (A_{jk} - A_{kj}) + \delta^h_j A_{ik} - \delta^h_k A_{ij}.$$

These equations have the character of a tensor of a projectively flat manifold [163], as we wanted to show. □

The Theorem 6.9 strengthens the results by Schouten and Struik [163], and it can be used to reduce the reasoning by I.P. Egorov [46], his concept of the rank of moving groups of Riemannian manifolds and manifolds with affine connection, but also by Sinyukov [170], his concept of degree of mobility groups $\mathbb{V}_n$ with respect to geodesic mappings.

This theorem allows us to find a new exact estimation of the first lacuna in a distribution of degree of mobility groups of $\bar{\mathbb{A}}_n$ with respect to geodesic mappings onto Riemannian manifolds.

6.4 Projective transformations

Towards the end of this Chapter, let us mention the main concepts concerning the projective transformations (or projective motions) of manifolds with affine connection, which were examined for Riemannian manifolds, see [11, 45, 50–52, 139, 170, 422, 423].

Let us consider an n-dimensional manifold $\mathbb{A}_n$ with symmetric affine connection ∇.

Definition 6.3 A geodesic mapping of a manifold $\mathbb{A}_n$ onto itself is called a *projective transformation* or *projective motion*.

The above definition generalizes motions, affine, homothetic and conformal motions of manifolds with affine connection and Riemannian manifolds.

It is known that the following holds.

Theorem 6.10 *An infinitesimal operator $X = \xi^\alpha(x)\partial_\alpha$ determines a one-parameter Lie group of projective transformations of a manifold $\mathbb{A}_n$ with affine connection if and only if it satisfies the following condition*

$$\mathcal{L}_\xi \nabla(X, Y) = \psi(X)\, Y + \psi(Y)\, X \tag{6.51}$$

where ψ is a one-form and $\mathcal{L}_\xi$ is the Lie derivative in the direction ξ.

In local form, the formulae (6.51) read

$$\mathcal{L}_\xi \Gamma^h_{ij} = \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{6.52}$$

and according to (4.34),

$$\xi^h_{,jk} = \xi^\alpha R^h_{ij\alpha} + \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{6.53}$$

where R^h_{ijk} are components of the curvature tensor on $\mathbb{A}_n$.

For a projective connection T (the Thomas object (6.14)), these equations are simplified:

$$\mathcal{L}_\xi T = 0. \tag{6.54}$$

Proof. We choose a special coordinate system x in which the fundamental vector field $\xi^h(x)$ of a one-parameter Lie group has the form $\xi^h = \delta^h_1$. This one-parameter Lie group reads $\overline{x}^h = x^h + \delta^h_1 \tau$, where τ is the canonical parameter.

For projective transformations we obtain easily (see Theorem 6.12, p. 263):

$$T^h_{ij}(\overline{x}) = T^h_{ij}(x),$$

and obviously, it follows $\delta_1 T^h_{ij}(x) = 0$ and (6.54). From the definition of the Thomas object (6.14) we obtain (6.52). □

Note that if the one form $\psi = 0$ in (6.51), then the transformation is affine.

The transformation $\overline{x}^h = x^h + \varepsilon\, \xi^h(x)$, where ε is a *small parameter*, is an *infinitesimal projective transformation* if the equations of the geodesic are preserved within an accuracy up to the second order of smallness; the necessary and sufficient condition for the existence of an infinitesimal projective transformation is the relation (6.51). This fact we demonstrated for a general maping, see 12.7, pp. 405.

It was proved in [45] that equation (6.51) of projective transformations can be written in the form of a closed linear system of PDEs of Cauchy type in the unknown functions $\xi^h(x)$, $\xi^h_i(x)$ and $\psi_i(x)$:

$$\begin{aligned}
&\xi^h_{,i} = \xi^h_i;\\
&\xi^h_{i,j} = \xi^\alpha R^h_{ij\alpha} + \delta^h_i\psi_j + \delta^h_j\psi_i;\\
&\psi_{i,j} = -\tfrac{1}{n-1}\,(\xi^\alpha R_{ij,\alpha} + R_{\alpha j}\xi^\alpha_i + R_{i\alpha}\xi^\alpha_j).
\end{aligned} \tag{6.55}$$

Hence the dimension r_{pt} of the Lie group of projective transformations does not exceed the number n^2+2n. The maximum can be reached only for projectively flat manifolds.

The group of projective transfromations involves the subgroup of of affine, homothetic and isometric motions.

The projective transformations of a manifolds with affine connection and a Riemannian manifolds were investigated by many mathematicians. For example, the investigations of A.V. Aminova [8, 9, 206, 254, 261], G. Fubini [422, 423], L.P. Eisenhart [50–52], K. Yano [197], I.P. Egorov [45–47], A.S. Solodovnikov [176, 178, 843, 846], S. Kobayashi [89], S. Kobayashi, K. Nomizu [90], G.I. Kruchkovich [98, 99, 559–561], and others were devoted to the general laws of this theory.

In the works [45–47] by I.P. Egorov results on projective transformations of manifolds with affine connection from the view-point of the order r_{pt} of projective transformation groups are contained.

The maximum $n(n+2)$ is achieved only in projective flat manifolds. Egorov showed that $r_{pt} \le n(n-2)+5$ for the nonprojective flat $\mathbb{A}_n$.

This bound was reinforced by J. Mikeš [633] for Riemannian manifolds of nonconstant curvature $\mathbb{V}_n$: $r_{pt} \le n(n-3)+8$. We mention more details in Subsection 7.2.3, p. 307, where we present results concerning Riemannian spaces.

7 GEODESIC MAPPINGS ONTO RIEMANNIAN MANIFOLDS

An important part of the theory of geodesic mappings (GM) concerns the case, when one of the manifolds is a Riemannian, i.e. classical (positive definite) Riemannian, possibly a pseudo-Riemannian manifold[87)].

7.1 Fundamental equations of GM: $\mathbb{A}_n \to \overline{\mathbb{V}}_n$

For geodesic mappings between classical Riemannian manifolds, Levi-Civita has found fundamental equations in another, equivalent form. However, these equations hold in a more general case, when a manifold with affine connection $\mathbb{A}_n$ is geodesically mapped onto a Riemannian manifold $\overline{\mathbb{V}}_n$. We suppose that $\overline{\mathbb{V}}_n$ is a (classical) Riemannian space, possibly a pseudo-Riemannian space.

Subsection 3.1.5, p. 183, was devoted to more general relationships, from which the Levi-Civita equations follow as an easy consequence.

7.1.1 Levi-Civita equations of geodesic mappings

The following holds:

Theorem 7.1 *A manifold with affine connection $\mathbb{A}_n$ admits a geodesic mapping onto a Riemannian space $\overline{\mathbb{V}}_n$ with metric $\overline{g}$ if and only if the following equations are satisfied:*

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}, \tag{7.1}$$

or equivalently, for any vector fields $X, Y, Z \in \mathcal{X}(\mathbb{A}_n)$:

$$\nabla_Z \overline{g}(X,Y) = 2\psi(Z)\overline{g}(X,Y) + \psi(X)\overline{g}(Y,Z) + \psi(Y)\overline{g}(X,Z). \tag{7.2}$$

Here ", " is the covariant derivative relative to the connection ∇ on $\mathbb{A}_n$, ψ is a one-form, and ψ_i are its components.

Proof. The proof of this theorem follows from the more general Theorem 4.1 and the shape of the deformation tensor P, which must be in the form (6.6) if the mapping should be geodesic. □

The equation (7.1) was discovered by Levi-Civita [107] for the case of geodesic mappings between Riemannian manifolds. These equations were originally formulated only locally; but later on, their global validity was proved and used many times. The same can be stated for the equations (7.1), which

[87)] Let us note that the problem of geodesic mappings onto Riemannian spaces is equivalent to the so-called metrizability, or projective metrizability of a (projective) connection, see [316, 388, 389]

characterize the case of geodesic mappings of a manifold $\mathbb{A}_n$ onto a (pseudo-) Riemannian manifold $\overline{\mathbb{V}}_n$, see [894].

Let us give yet another equation characterizing a geodesic mapping $\mathbb{A}_n \to \overline{\mathbb{V}}_n$, which is equivalent to the equation (7.1).

For the inverse matrix $\overline{g}^{ij}(x)$ to $\overline{g}_{ij}(x)$, $\overline{g}^{i\alpha}\overline{g}_{\alpha j} = \delta^i_j$ holds. Let us calculate the covariant derivative of the formula with respect to x^k: $\overline{g}^{i\alpha}_{\ ,k}\overline{g}_{\alpha j} + \overline{g}^{i\alpha}\overline{g}_{\alpha j,k} = 0$. Contracting with $\overline{g}^{jj'}$ we find (after getting rid of " ' " in the index j')

$$\overline{g}^{ij}_{\ ,k} = -\,\overline{g}^{i\alpha}\overline{g}^{j\beta}\overline{g}_{\alpha\beta,k}\,. \tag{7.3}$$

Let us substitute from (7.1) to (7.3):

$$\overline{g}^{ij}_{\ ,k} = -\,2\psi_k\overline{g}^{ij} - \psi_\alpha\overline{g}^{\alpha i}\delta^j_k - \psi_\alpha\overline{g}^{\alpha j}\delta^i_k\,. \tag{7.4}$$

Similarly we can show that from (7.4) follows (7.1). In other words, the equations (7.1) and (7.4) are equivalent.

7.1.2 Cauchy type equations of GM of $\mathbb{A}_n$ onto $\overline{\mathbb{V}}_n$

The following theorem holds:

Theorem 7.2 (Mikeš, Berezovski [118,651]) *A manifold with affine connection $\mathbb{A}_n$ admits a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$ with the metric tensor $\overline{g}_{ij}(x)$ if and only if the following system of differential equations of Cauchy type in covariant derivatives has a solution with respect to a symmetric tensor $\overline{g}_{ij}(x)$ $(\det(\overline{g}_{ij}(x)) \neq 0)$, the covector $\psi_i(x)$ and the function $\mu(x)$:*

$$\begin{aligned}
&\text{(a)} && \overline{g}_{ij,k} &&= 2\psi_k\overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k};\\
&\text{(b)} && n\psi_{i,j} &&= n\psi_i\psi_j + \mu\overline{g}_{ij} + R_{ij} + \overline{g}_{i\alpha}\overline{g}^{\beta\gamma}R^\alpha_{\beta\gamma j} - \tfrac{2}{n+1}R^\alpha_{\alpha ij};\\
&\text{(c)} && (n-1)\mu_{,i} &&= 2(n-1)\psi_\alpha\overline{g}^{\beta\gamma}R^\alpha_{\beta\gamma i} - \\
& && && -\psi_\alpha\overline{g}^{\alpha\beta}(5R_{\beta i} - \tfrac{6}{n+1}R^\gamma_{\gamma\beta i} - R_{i\beta}) + \\
& && && +\overline{g}^{\alpha\beta}(R^\gamma_{\alpha\beta i,\gamma} + R_{\alpha i,\beta} - \tfrac{2}{n+1}R^\gamma_{\gamma\alpha i,\beta}).
\end{aligned} \tag{7.5}$$

where the comma denotes the covariant derivative with respect to the connection of $\mathbb{A}_n$, $\overline{g}^{ij}(x)$ are components of the matrix inverse to $(g_{ij}(x))$, R^h_{ijk} and R_{ij} are respectively the Riemannian and the Ricci tensors of the manifold $\mathbb{A}_n$.

Proof. Suppose that $\mathbb{A}_n$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n$ with the metric tensor $\overline{g}_{ij}(x)$. Then the connections of $\mathbb{A}_n$ and $\overline{\mathbb{V}}_n$ obey the relation (7.1) in a common coordinate system relative to the mapping. Taking into account the covariant constancy of $\overline{g}_{ij}(x)$ in $\overline{\mathbb{V}}_n$, the conditions are at the same time sufficient for $\mathbb{A}_n$ to admit a geodesic mapping onto $\overline{\mathbb{V}}_n$.

Let us consider the integrability conditions of the equations (7.5a)

$$\overline{g}_{h\alpha}R^{\alpha}_{ijk} + \overline{g}_{i\alpha}R^{\alpha}_{hjk} = 2\overline{g}_{hi}\psi_{[jk]} + \overline{g}_{j(h}\psi_{i)k} - \overline{g}_{k(h}\psi_{i)j}, \tag{7.6}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, and by $[i\,j]$ denote alternation over i and j.

Contracting (7.6) and $\overline{g}^{ij}$, we get $\psi_{[jk]} = \frac{1}{n+1}R^{\alpha}_{\alpha jk}$. Expressing $\psi_{[jk]}$ from (7.6), we obtain

$$\overline{g}_{\alpha(h}R^{\alpha}_{i)jk} - \frac{2}{n+1}\overline{g}_{hi}R^{\alpha}_{\alpha jk} = \overline{g}_{j(h}\psi_{i)k} - \overline{g}_{k(h}\psi_{i)j}. \tag{7.7}$$

After contraction of (7.7) with $\overline{g}^{ik}$ one easily obtains the conditions (7.5b) with $\mu \equiv \psi_{\alpha\beta}\overline{g}^{\alpha\beta}$.

We covariantly differentiate the conditions (7.5b) with respect to x^k and then alternate the result with respect to the indices j and k taking into account (7.5a), (7.5b), (7.4) and contract with $\overline{g}^{ik}$, and finally we get equations (7.5c). The theorem has been proved. □

The system (7.5) has not more than one solution for initial conditions in the point $x_\circ$:

$$\overline{g}_{ij}(x_\circ) = \overset{\circ}{\overline{g}}_{ij}, \quad \psi_i(x_\circ) = \overset{\circ}{\psi}_i, \quad \mu(x_\circ) = \overset{\circ}{\mu}. \tag{7.8}$$

The general solution of Eqs. (7.5) depends on a finite number of essential parameters

$$r_{gm} \le r_0 \equiv \frac{(n+1)(n+2)}{2}.$$

Hence we may conclude from Theorem 7.2 that the set of all Riemannian spaces, onto which the given manifold with affine connection $\mathbb{A}_n$ admits geodesic mappings, depends on $r_{gm} \le r_0$ parameters.

To find all the solutions of (7.5) requires a consideration of their integrability conditions and differential extensions, which form a set of algebraic equations with respect to the unknown functions $\overline{g}_{ij}$, ψ_i and μ with coefficients from $\mathbb{A}_n$. But this set is not linear and its solution is certainly difficult.

These results were generalized for the case of geodesic mappings from Finsler spaces onto Riemannian spaces [648].

7.1.3 On the mobility degree with respect to geodesic mappings

The number r_{gm} of significant parameters on which the general solution of (7.5) depends will be called the *mobility degree with respect to geodesic mappings onto Riemannian spaces* $\overline{\mathbb{V}}_n$ (by analogy with [170]).

It is easy to prove that the maximal degree of mobility $r = \frac{1}{2}(n+1)(n+2)$ with respect to geodesic mappings onto Riemannian spaces is admitted by projective Euclidean spaces, and only by them.

The following estimation is obtained for a distribution of the degrees of mobility with respect to the geodesic mappings onto Riemannian spaces.

Theorem 7.3 *The degree of mobility of a manifold with affine connection $\mathbb{A}_n$, which is not projectively flat, with respect to geodesic mappings onto Riemannian spaces does not exceed the number $n(n-1)/2$.*

The results under discussion are generalizations of analogous theorems by Sinyukov [170] for geodesic mappings of Riemannian spaces.

Remark The accuracy of this number is confirmed by an example given by Mikeš for geodesic mappings of Riemannian manifolds and published in the monograph [170], pp. 139-142.

Proof. In $\mathbb{A}_n$ let us choose local coordinate systems in which $R^3_{112} \neq 0$. Since $\mathbb{A}_n$ is not projectively flat, such a coordinate system can be always found (about any point) [673], see Theorem 6.9, p. 272.

In formulas (7.7) let us raise the indices h and i by mens of $\overline{g}^{ij}$, we get

$$\overline{g}^{\alpha(h}R^{i)}_{\alpha jk} - \frac{2}{n+1}\overline{g}^{hi}R^{\alpha}_{\alpha jk} = \delta^{(h}_j\psi^{i)}_k - \delta^{(h}_k\psi^{i)}_j, \tag{7.9}$$

where $\psi^i_j = \psi_{\alpha j}\overline{g}^{\alpha i}$. Let us insert $h = i = 3$, $j = 1$, $k = 2$ to (7.9), and check that $\overline{g}^{13}$ depends on the components $\overline{g}^{i3}$ $(i \neq 1)$ and on the objects of $\mathbb{A}_n$:

$$\overline{g}^{13} = \mathcal{Z}(\overline{g}^{i3}, i \neq 1),$$

where $\mathcal{Z}$ indicates that the expression depends on the argument in brackets and on objects of tensor character which are generated by $\mathbb{A}_n$, that is, can be constructed by means of the components of the affine connection ∇ of $\mathbb{A}_n$.

If we set in (7.9) $h = i = j = 3$, $k = 2$, and then $h = i = k = 3$, $j = 1$, we get

$$\psi^3_1,\ \psi^3_2 = \mathcal{Z}(\overline{g}^{ij},\ i, j \neq 1).$$

According to the above, for $i = 3$, $j = 1$ and $k = 2$, we get:

$$\overline{g}^{1h} = \mathcal{Z}(\overline{g}^{ij},\ i, j \neq 1).$$

It follows that among the components of the tensor $\overline{g}_{ij}$, at most n are dependent. Differentiating (7.9) we get

$$-\psi^{(h}R^{i)}_{ljk} = \delta^{(h}_l\Phi^{i)}_{jk} + \delta^{(h}_j\Psi^{i)}_{kl} - \delta^{(h}_k\Psi^{i)}_{jl} - \overline{g}^{\alpha(h}R^{i)}_{\alpha jk,l} + \frac{2}{n+1}\overline{g}^{hi}R^{\alpha}_{\alpha jk,l}, \tag{7.10}$$

where

$$\Psi^i_{kl} = \psi^i_{k,l} + 2\psi^i_k\psi_l; \quad \Phi^i_{jk} = R^i_{\alpha jk}\psi^\alpha - \frac{2}{n+1}\psi^i R^{\alpha}_{\alpha jk}.$$

Step by step, we insert into (7.10):

$$\begin{array}{l} i = 1, \ldots, n,\ j = 1,\ m = k = 2,\ l = 3; \\ i = j = k = 1,\ l = 3,\ m = 2; \\ i = j = m = 1,\ l = 3,\ k = 2; \\ i = j = k = 1,\ l = m = 2 \end{array}$$

and we can see that

$$\psi_i = \mathcal{Z}(\overline{g}_{ij}) \tag{7.11}$$

holds. The system of equations is reduced to the equations (7.5a) ($\equiv$ (7.1)) where ψ_i is determined by (7.11). The system (7.5a) is closed. Finally, we can see that for not projectively flat spaces $\mathbb{A}_n$ the mobility degree r is reduced by $2n+1$. Hence the Theorem 7.3 is proved. □

From the proof of the above Theorem, among others follows:

Theorem 7.4 *Let $\mathbb{A}_n$ be a manifold with affine connection which is not locally projectively flat about a point $p \in \mathbb{A}_n$. Then in a neigborhood of p, $\mathbb{A}_n$ admits a geodesic mapping onto a Riemannian space $\overline{\mathbb{V}}_n$, if and only if the complete set of differential equations of Cauchy type in the covariant derivatives (7.1) in $\mathbb{A}_n$, with (7.11), has a solution with respect to an unknown symmetric regular tensor $\overline{g}_{ij}(x)$.*

Since the manifolds $\mathbb{A}_n$ (and $\mathbb{V}_n$ as well) form equivalence classes closed with respect to geodesic mappings, most aspects concerning the concept of the degree of mobility r_{gm} of manifolds with affine connection $\mathbb{A}_n$ with respect to geodesic mappings onto Riemannian manifolds follow from the result on the degree of mobility r_{gm} of Riemannian manifolds $\mathbb{V}_n$ with respect to geodesic mappings onto Riemannian manifolds.

Below, let us formulate some results on the mobility degree which are treated in details by Kiosak and Mikeš in [542] (see [118]) and in the Ph.D. thesis by Kiosak [220, 221].

For Riemannian manifolds $\mathbb{V}_n$ with non constant curvature, and naturally for manifold with affine connection $\mathbb{A}_n$ which are not projectively flat, the following can be verified:

- for $\mathbb{V}_3$ ($\mathbb{A}_3$), $r_{gm} \leq 2$;
- for $\mathbb{V}_4$ ($\mathbb{A}_4$), r_{gm} can take the values 1, 2, 4;
- there exist neither $\mathbb{V}_n$ nor $\mathbb{A}_n$ with

$$n(n+3)/2 + 1 > r_{gm} > n(n-3)/2 + 2,\ n > 2;$$
$$n(n-3)/2 + 2 > r_{gm} > n(n-5)/2 + 8,\ n > 4.$$

Riemannian spaces $\mathbb{V}_n$ ($n > 2$), with $r_{gm} = \frac{n(n-3)}{2} + 2$, admit exactly $n-2$ linearly independent concircular fields (see Theorem 2.1 [542]); see Sec. 6.1.2.

Riemannian manifolds $\mathbb{V}_n$, admitting non trivial geodesic mappings with $r_{gm} > 2$, are precisely the spaces $\mathbb{V}_n(B)$, $B \equiv$ const (see Sec. 6.2). For many classes of Riemannian manifolds $\mathbb{V}_n$ it was proved that $r_{gm} \leq 2$. Among them are, for example, conformally flat spaces $\mathbb{V}_n$ and three-dimensional Riemannian manifolds $\mathbb{V}_3$ with non constant curvature, the spaces, where the conditions $L_{ij,k} = L_{ik,j}$ hold, and the spaces L_n (see Sec. 9.4 and also [19, 22, 35, 117]).

7.2 Linear equations of the theory of geodesic mappings

7.2.1 Mikeš-Berezovski equations of geodesic mappings

Sinyukov started from the following problem: *find all Riemannian spaces* $\overline{\mathbb{V}}_n$ *which admit geodesic mappings onto an a priori defined Riemannian space* $\mathbb{V}_n$, see [118, 170].

This means: we must find all metric tensors $\overline{g}$ which are solutions of the Levi-Civita equations (6.8), or (7.1), respectively. These equations are non-linear with respect to the components of the metric tensor $\overline{g}$ and for their solution no standard methods exist. Sinyukov (see [118, 170]) obtained a set of linear equations of Cauchy type for this problem.

Mikeš and Berezovski started from the generalized problem: *find all Riemannian spaces* $\overline{\mathbb{V}}_n$ *which admit geodesic mappings onto an a priori given manifold* $\mathbb{A}_n$, see [118, 208, 651]. First let us prove the following.

Theorem 7.5 (Mikeš, Berezovski [118, 208, 651]) *The equiaffine space* $\mathbb{A}_n$ *admits a geodesic mapping onto a Riemannian space* $\overline{\mathbb{V}}_n$ *if and only if the set of differential equations in covariant derivatives in* $\mathbb{A}_n$

$$a^{ij}_{\ ,k} = \lambda^i\,\delta^j_k \ + \ \lambda^j\,\delta^i_k \tag{7.12}$$

has a solution with respect to an unknown symmetric regular tensor $a^{ij}(x)$ *and a vector* $\lambda^i(x)$*. Solutions of this system and solutions of (7.1) are related by the equalities*

$$a^{ij}(x) = \exp(2\Psi(x))\,\overline{g}^{ij}(x), \tag{7.13}$$

$$\lambda^i(x) = -\exp(2\Psi(x))\,\overline{g}^{i\alpha}(x)\psi_\alpha(x), \tag{7.14}$$

where ψ_i *is a gradient vector of the function* Ψ*,* $\overline{g}^{ij}$ *are components of the dual tensor of the metric tensor of* $\overline{\mathbb{V}}_n$*, and the comma "," denotes the covariant derivative in* $\mathbb{A}_n$*.*

Proof. Let an equaffine $\mathbb{A}_n$ admit a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$. Then the equation (7.4) is satisfied and moreover, the covector ψ_i is locally a gradient, i.e. there exists a function $\Psi(x)$ such that $\psi_i(x) = \partial_i\Psi(x)$. Let us put $a^{ij}(x) = \exp(2\Psi(x))\,\overline{g}^{ij}(x)$, i.e. (7.13) holds, and we can calculate $a^{ij}_{\ ,k}$. After a short evaluation, we get according to (7.4) the equations (7.12) and the formula (7.14).

Vice versa, let us verify that the equation (7.12) is not only a necessary, but also a sufficient condition for $\mathbb{A}_n$ to admit geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$. Let $\tilde{a}_{ij}$ be an inverse of a^{ij}. Let us set

$$\psi_i = -\,\tilde{a}_{i\alpha}\lambda^\alpha. \tag{7.15}$$

Check that ψ_i is necessarily (locally) gradient-like. Since the tensor $\tilde{a}_{ij}$ is regular we can construct to it the corresponding Christoffel symbols of second type $\tilde{\Gamma}^h_{ij}$ that satisfy the Voss-Weyl formula (3.11):

$$\tilde{\Gamma}^\alpha_{\alpha k} = \frac{1}{2}\,\partial_k \ln|\tilde{a}|, \quad \tilde{a} = \det(\tilde{a}_{ij}).$$

We evaluate

$$\begin{aligned}\tilde{\Gamma}^{\alpha}_{\alpha k} &= \tfrac{1}{2}\,(\partial_\alpha \tilde{a}_{\beta k} + \partial_k \tilde{a}_{\alpha\beta} - \partial_\beta \tilde{a}_{\alpha k})\, a^{\alpha\beta} &&= \tfrac{1}{2}\,\partial_k \tilde{a}_{\alpha\beta}\, a^{\alpha\beta}\\ &= \tfrac{1}{2}\,(\tilde{a}_{\alpha\beta,k} + \tilde{a}_{\alpha\gamma}\Gamma^{\gamma}_{\beta k} + \tilde{a}_{\beta\gamma}\Gamma^{\gamma}_{\alpha k})\, a^{\alpha\beta} &&= -\tfrac{1}{2}\,\tilde{a}_{\alpha\beta} a^{\alpha\beta}_{\ ,k} + \Gamma^{\alpha}_{\alpha k}\\ &\overset{(7.12)}{=} \psi_k + \Gamma^{\alpha}_{\alpha k}.\end{aligned}$$

It follows that $\psi_k = \tilde{\Gamma}^{\alpha}_{\alpha k} - \Gamma^{\alpha}_{\alpha k}$. Since $\Gamma^{\alpha}_{\alpha k} = \partial_k f(x)$ holds in an equiaffine manifold $\mathbb{A}_n$, we obtain $\psi_k = \partial_k \Psi(x)$ where we used the notation $\Psi(x) = \frac{1}{2}\,\ln|\tilde{a}| - f(x)$. Accounting (7.12) and (7.15), we get for the tensor

$$\overline{g}^{ij}(x) = \exp(-2\Psi(x)) \cdot a^{ij}(x)$$

the equation (7.4) which is equivalent to the Levi-Civita equation (7.1). These equations guarentee that the mapping $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ is geodesic. □

Remark The Theorem 7.5 holds under the assumptions $\mathbb{A}_n \in C^0$ and $\overline{\mathbb{V}}_n \in C^1$.

Further, we have

Theorem 7.6 (Mikeš, Berezovski [118, 651]) *An equiaffine space $\mathbb{A}_n$ admits a geodesic mapping onto a Riemannian space $\overline{\mathbb{V}}_n$, if and only if the complete set of linear differential equations of Cauchy type in covariant derivatives in $\mathbb{A}_n$*

$$\begin{aligned}&(a) && a^{ij}_{\ ,k} = \lambda^i\,\delta^j_k \ + \ \lambda^j\,\delta^i_k;\\ &(b) && n\,\lambda^i_{\ ,j} = \mu\,\delta^i_j - a^{i\alpha}R_{\alpha j} - a^{\alpha\beta}R^i_{\alpha\beta j};\\ &(c) && (n-1)\,\mu_{,i} = -2(n+1)\lambda^\alpha R_{\alpha i} - a^{\alpha\beta}(2R_{\alpha i,\beta} - R_{\alpha\beta,i})\end{aligned} \tag{7.16}$$

has a solution with respect to an unknown symmetric nondegenerate tensor a^{ij}, a vector λ^i, and a function μ. The solutions of this system and (7.1) are related by the equalities (7.13) and (7.14).

Here R^h_{ijk} and $R_{ij} = R^\alpha_{i\alpha j}$ are components of the Riemannian and Ricci tensors of $\mathbb{A}_n$, the comma ", " denotes the covariant derivative in $\mathbb{A}_n$.

Proof. The first formula (7.16) gives a necessary and sufficient condition for the existence of a geodesic mapping: $\mathbb{A}_n \to \overline{\mathbb{V}}_n$, see Theorem 7.5. This mapping is nontrivial if and only if $\lambda_i \neq 0$. The integrability conditions for the first formula (7.16) follow from the Ricci identity for the tensor a^{ij}: $a^{ij}_{\ ,kl} - a^{ij}_{\ ,lk} = -a^{\alpha i}R^j_{\alpha kl} - a^{\alpha j}R^i_{\alpha kl}$. After substitution we get

$$-a^{\alpha i}R^j_{\alpha kl} - a^{\alpha j}R^i_{\alpha kl} = \delta^i_k\lambda^j_{,l} + \delta^j_k\lambda^i_{,l} - \delta^i_l\lambda^j_{,k} - \delta^j_l\lambda^i_{,k}\,. \tag{7.17}$$

If we denote $\mu = \lambda^\alpha_{,\alpha}$, then after contraction of the formula (7.17) with respect to j and l we get the equations (7.16b). The equation (7.16c) can be obtained analogously if we contract the integrability conditions of the equations (7.16b) (as an important tool, we used the Bianchi identities for the curvature and the Ricci tensors during evaluations). It finishes the proof. □

By a detailed analysis it can be shown that Theorem 7.6 holds for $\mathbb{A}_n \in C^2$, i.e. in the case that all the components of the affine connection ∇ satisfy $\Gamma^h_{ij}(x) \in C^2$, and $\overline{\mathbb{V}}_n \in C^2$.

In this case, the set of equations (7.16) is linear and its solution is reduced to investigation of the integrability conditions and their differential prolongations, which form a set of algebraic (homogeneous with respect to the unknown tensors a^{ij}, λ^i, and μ) equations with coefficients from $\mathbb{A}_n$ (i.e. coefficients formed from objects defined on $\mathbb{A}_n$). Thus, in principle, we can solve the following problem, if the given equiaffine space $\mathbb{A}_n$ admits geodesic mappings onto the Riemannian space $\overline{\mathbb{V}}_n$ and if the choice of this mapping is arbitrary.

This system has not more than one solution for initial conditions in the point x_0:

$$a^{ij}(x_0) = \overset{\circ}{a}{}^{ij}, \quad \lambda^i(x_0) = \overset{\circ}{\lambda}{}^{i}, \quad \mu(x_0) = \overset{\circ}{\mu}. \tag{7.18}$$

The general solution of Eqs. (7.16) depends on a finite number of substantial parameters

$$r \le N_0 \equiv \frac{(n+1)(n+2)}{2}.$$

Due to (7.13) and (7.14) we can check that the number r is also the degree r_{gm} of mobility of $\mathbb{A}_n$ with respect to geodesic mappings onto Riemannian spaces, see Sec. 6.1.3.

Remark 7.1 Later on the equations (7.12) and the system (7.16) has been derived in other notation in [316, 388, 389].

7.2.2 Linear equations of geodesic mappings $\mathbb{A}_n \to \overline{\mathbb{V}}_n$

In the previous section we studied a geodesic mappings of equiaffine manifold $\mathbb{A}_n$ onto a Riemannian manifolds $\overline{\mathbb{V}}_n$. After the interchange $\Gamma^h_{ij}(x) \mapsto \Pi^h_{ij}(x)$ in formulae (7.12) and (7.16) we obtain fundamental formulae for geodesic mappings of $\mathbb{P}_n = (M_n, \blacktriangledown)$ onto $\overline{\mathbb{V}}_n = (M_n, \overline{g})$, where $\Pi^h_{ij}(x)$ are components of projective connection $\blacktriangledown$.

Our assertion follows from properties of the normal affine connection Γ^h_{ij}, see by È. Cartan [322] and J.M. Thomas [891], see [51, p. 105]. If $\Gamma^h_{ij}(x) = \Pi^h_{ij}(x)$, then $\mathbb{A}_n$ (with affine connection Γ^h_{ij}) and $\mathbb{A}_n$ (with projective connection Π^h_{ij}), have common geodesics. Because $\Gamma^\alpha_{i\alpha} = 0$, manifold $\mathbb{A}_n$ is equiaffine.

The manifold $\mathbb{P}_n = (M_n, \blacktriangledown)$ admits a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n = (M_n, \overline{g})$ if and only if the following set of differential equations in projective derivatives in $\mathbb{P}_n$ has a solution with respect to an unknown symmetric regular tensor $a^{ij}(x)$ and functions $\lambda^i(x)$:

$$\blacktriangledown_k a^{ij} = \lambda^i \delta^j_k \ + \ \lambda^j \delta^i_k, \tag{7.19}$$

where $\blacktriangledown_k a^{ij} = \partial_k a^{ij} + a^{i\alpha}\Pi^j_{\alpha k} + a^{j\alpha}\Pi^i_{\alpha k}$.

Solutions of this system and solutions of (7.1) are related by the equalities

$$\begin{gathered} a^{ij}(x) = e^{2\Psi(x)}\,\overline{g}^{ij}(x), \ \lambda^i(x) = -e^{2\Psi(x)}\,\overline{g}^{i\alpha}(x)\psi_\alpha(x), \\ \Psi = \tfrac{1}{2(n+1)} \ \ln|\det \overline{g}_{ij}|, \ \psi_i = \partial_i \Psi. \end{gathered} \tag{7.20}$$

A projective manifold $\mathbb{P}_n$ admits a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$, if and only if the complete set of linear differential equations of Cauchy type in projective derivatives in $\mathbb{P}_n$

$$\begin{aligned}
&(a) \qquad \blacktriangledown_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k;\\
&(b) \qquad n \blacktriangledown_j \lambda^i = \mu \delta^i_j - a^{i\alpha}\mathcal{R}_{\alpha j} - a^{\alpha\beta}\mathcal{R}^i_{\alpha\beta j};\\
&(c) \quad (n-1)\blacktriangledown_i \mu = -2(n+1)\lambda^\alpha \mathcal{R}_{\alpha i} - a^{\alpha\beta}(2\blacktriangledown_\beta \mathcal{R}_{\alpha i} - \blacktriangledown_i \mathcal{R}_{\alpha\beta}),
\end{aligned} \tag{7.21}$$

has a solution with respect to an unknown symmetric nondegenerate tensor a^{ij}, a vector λ^i, and a function μ. The solutions of this system and (7.1) are related by the equalities (7.13) and (7.14). Here

$$\mathcal{R}^h_{ijk} = \partial_j \Pi^h_{ik} - \partial_k \Pi^h_{ij} + \Pi^\alpha_{ik}\Pi^h_{\alpha j} - \Pi^\alpha_{ij}\Pi^h_{\alpha k} \quad \text{and} \quad \mathcal{R}_{ij} = \mathcal{R}^\alpha_{i\alpha j}.$$

By direct computation the previously mentioned equations were introduced [389] in 2008, these authors appeal to paper [651].

7.2.3 Geodesic mappings $\mathbb{P}_n \to \overline{\mathbb{V}}_n$ for $\mathbb{P}_n \in C^2$ and $\overline{\mathbb{V}}_n \in C^1$

Let $\mathbb{A}_n = (M, \nabla)$ and $\mathbb{P}_n = (M, \blacktriangledown)$ be manifolds with affine and projective connection, respectively.

The formulae (7.12) and (7.19) for $\nabla, \blacktriangledown \in C^0$ and $\overline{g} \in C^1$ hold. The systems of fundamental equations (7.16) and (7.21) for $\nabla, \blacktriangledown \in C^2$ and $\overline{g} \in C^3$ hold, too. Moreover, the following theorems are true.

Theorem 7.7 *If $\mathbb{P}_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a Riemannian manifold $\overline{\mathbb{V}}_n \in C^1$, then $\overline{\mathbb{V}}_n \in C^{r+1}$.*

Theorem 7.8 *If $\mathbb{A}_n \in C^r$ $(r \geq 2)$ admits geodesic mappings onto a Riemannian manifold $\overline{\mathbb{V}}_n \in C^1$, then $\overline{\mathbb{V}}_n \in C^{r+1}$.*

Based on the previous comments (pp. 265 and 282), it will be sufficient to prove the validity of the second Theorem. The proof of the Theorem 7.7 follows from the following lemmas.

By analysis formula (7.20) we check that, the following lemma holds:

Lemma 7.1 *A condition $a^{ij} \in C^r$ holds if and only if $\overline{g}_{ij} \in C^r$.*

Lemma 7.2 *Let $\mathbb{P}_n \in C^2$ admit geodesic mapping onto $\overline{\mathbb{V}}_n \in C^1$. Then $a^{ij} \in C^2$.*

Proof. We analyze equations (7.19), can be written in the form:

$$\partial_k a^{ij} = \lambda^j \delta^i_k + \lambda^i \delta^j_k - a^{\alpha i}\Pi^j_{\alpha k} - a^{\alpha j}\Pi^i_{\alpha k}. \tag{7.22}$$

After simple analysis of equations (7.22) we obtain all two partial derivatives of $a^{ij}(x)$, excluding of $\partial_{ii}a^{ii}$ and $\partial_{ij}a^{ij}$. Analogically all partial derivatives of $\lambda^i(x)$ exist, excluding $\partial_i \lambda^i(x)$.

Further we integrate $\partial_2 a^{11} = -2a^{11}\Pi^1_{12} - G$, where $G = \sum_{\alpha=2}^{n} a^{1\alpha}\Pi^1_{\alpha 2}$, we find

$$a^{11} = A \cdot B + C, \tag{7.23}$$

where A is a function, that does not depend on the coordinate system x^2,

$$B = \exp\left(-2\int_{x_0^2}^{x^2} \Pi^1_{12}(x^1, \tau^2, x^3, \dots)\, \mathrm{d}\tau^2\right), \quad C = -2B\int_{x_0^2}^{x^2} \sum_{\alpha=2}^{n} a^{1\alpha}\cdot\Pi^1_{\alpha 2}\cdot B\cdot \mathrm{d}\tau^2.$$

These presented functions B and C are differentiating of x^1. This assertion follows from differentiating of $a^{1\alpha}$, Π^h_{ij} and from properties of integrals with parameters, see [100].

Hence $\partial_1 A$ exists. Differentiating (7.23) we find:

$$\partial_1 a^{11} = \partial_1 A \cdot B + G, \quad \text{where} \quad G = A \cdot \partial_1 B + \partial_1 G. \tag{7.24}$$

From the other side using equations (7.24) we get

$$\partial_1 a^{11} = 2\lambda^1 - 2a^{1\alpha}\Pi^1_{\alpha 1} \quad \text{and} \quad \partial_2 a^{12} = \lambda^1 - a^{1\alpha}\Pi^2_{\alpha 2} - a^{2\alpha}\Pi^1_{\alpha 2}. \tag{7.25}$$

Using (7.24) and (7.25) can be obtained

$$2\partial_2 a^{12} = \partial_1 A \cdot B + H, \tag{7.26}$$

where $H = G + 2\left(a^{1\alpha}\Pi^1_{\alpha 1} - a^{1\alpha}\Pi^2_{\alpha 2} - a^{2\alpha}\Pi^1_{\alpha 2}\right)$.

With subsequent integration we get

$$2a^{12} = \tilde{A} + \partial_1 A \cdot \int_{x_0^2}^{x^2} B(x^1, \tau^2, x^3, \dots)\, \mathrm{d}\tau^2 + \int_{x_0^2}^{x^2} H(x^1, \tau^2, x^3, \dots)\, \mathrm{d}\tau^2.$$

where $\tilde{A}$ is a function, that does not depend on the coordinate system x^2. Because ∂_1 of the functions a^{12}, $\tilde{A}$, B and H exist, $\partial_{11} A$ exists too. Then, from (7.24) and (7.25) existence of $\partial_{11} a^{11}$ and $\partial_{21} a^{12}$ follow. Elementary, $a^{ij} \in C^2$. From this and Lemma 7.1 follows that also $\overline{g}_{ij} \in C^2$. □

Lemma 7.3 *If $\mathbb{P}_n \in C^2$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n \in C^2$, then $\overline{\mathbb{V}}_n \in C^3$.*

Proof. In this case equations (7.21a) and (7.21b) hold. According to the assumptions, $\Pi^h_{ij} \in C^2$ and $\overline{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a^{ij} \in C^2$, $\lambda^i \in C^1$ and $\mathcal{R}^h_{ijk}, \mathcal{R}_{ij} \in C^1$.

From the above-mentioned conditions we easily convince ourselves that we can write equation (7.21b) in the form $\partial_i \lambda^h - \varrho\, \delta^h_i = f^h_i$, where

$$\varrho = \mu/n \quad \text{and} \quad f^h_i = (-\lambda^\alpha \Pi^h_{\alpha i} - a^{j\alpha}\mathcal{R}_{\alpha i} - a^{\alpha\beta}\mathcal{R}^j_{\alpha\beta i})/n \in C^1.$$

From Lemma 3.4, see p. 142, follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda^i \in C^2$. Differentiating (7.21a) twice we convince ourselves that $a^{ij} \in C^3$. From this and Lemma 7.1 follows that also $\overline{g}_{ij} \in C^3$. □

Further we notice that for geodesic mappings from $\mathbb{P}_n \in C^2$ onto $\overline{\mathbb{V}}_n \in C^3$ holds the third set of equations (7.21c).

If $\mathbb{P}_n \in C^r$ and $\overline{\mathbb{V}}_n \in C^2$, then by Lemma 7.3, $\overline{\mathbb{V}}_n \in C^3$ and (7.21) hold. Because system (7.21) is closed, we can differentiate equations (7.19) r times. So we convince ourselves that $a^{ij} \in C^{r+1}$, and also $\overline{g}_{ij} \in C^{r+1}$ ($\equiv \overline{V}_n \in C^{r+1}$). It finishes the proof of Theorem 7.7. □

7.2.4 Example

The general solution of equations (7.12):

$$a^{ij}{}_{,k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k$$

in the affine space $\mathbb{R}^n$ in affine coordinates reads

$$\begin{aligned} a^{ij} &= \mu \cdot x^i x^j + c^i x^j + c^j x^i + c^{ij}, \\ \lambda^i &= \mu \cdot x^i + c^i, \end{aligned} \tag{7.27}$$

where μ, c^i and c^{ij} are constants.

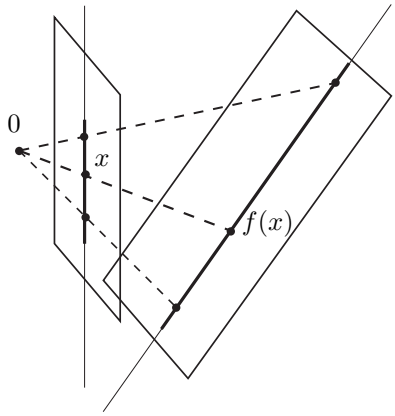
These solutions in $\mathbb{R}^n$ are defined globally, but it is not guaranteed that $\det(a^{ij}) \neq 0$ holds, which is a condition for a mapping to be geodesic.

Note that a gnomonic projection, see 8.2.1, generates a regular solution (7.27) in $\mathbb{R}^n$ of the equation (7.12).

Let us have a transformation of the affine space $\mathbb{R}^n$ given in affine coordinates $(x^1, x^2, \ldots, x^n)$ by the formula:

$$\overline{x}^h = \frac{a^h_\alpha x^\alpha + b^h}{c_\alpha x^\alpha + d} \tag{7.28}$$

where a^h_i, b^i, c_α, d are constants, $\det(a^h_i) \neq 0$.



Because all straight lines in $\mathbb{R}^n$ are mapped onto straight lines, this transformation realizes a geodesic mapping (central projection). This case is demonstrated by the Figure below.

For $c_\alpha = 0$ this transformation is an affine, and for $c_\alpha \neq 0$ it is a non trivial geodesic mapping.

Obviously, this non trivial geodesic mapping (7.28) is not defined for the hyperplane $c_\alpha x^\alpha + d = 0$, and there exists no global nontrivial geodesic mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$. That is why in this case, there is no regular solution (7.27) of the equation (7.12).

On the other hand, the transformation (7.28) in the half-space $c_\alpha x^\alpha + d > 0$, $c_i \neq 0$, generates a regular solution (7.27) of (7.12), which corresponds to some nontrivial geodesic mapping of the half-space of $\mathbb{R}^n$ onto a half-space of $\mathbb{R}^n$.

7.3 Geodesic mappings of special manifolds

7.3.1 Geodesic mappings of semisymmetric manifolds

The present part studies geodesic mappings of Riemannian manifolds onto semisymmetric equiaffine manifolds and Riemannian manifolds.

Following Sinyukov (see [810] and [170]), by a *semisymmetric manifold* (Ps_n) we mean a manifold with affine connection $\mathbb{A}_n$ whose Riemannian tensor satisfies the condition $R \circ R = 0$, locally

$$R^h_{ijk,[lm]} = 0 \tag{7.29}$$

where by the comma we denote the covariant derivative in $\mathbb{A}_n$, and by the square brackets $[lm]$ the alternation.

Manifolds satisfying (7.29) were considered in the twenties by P.A. Shirokov [167] and then by É. Cartan [323, 324], Lichnerowicz [108] and others in the studies of symmetric manifolds ($\nabla R = 0$, i.e. $R^h_{ijk,l} = 0$).

In the seventies, a considerable interest in studies of semisymmetric manifolds arose due to Nomizu's conjecture [734] supposing that under certain natural conditions, a semisymmetric manifold is a symmetric one. However, as it became apparent later on, this conjecture fails. The properly Riemannian semisymmetric manifolds were classified by Z. Szabó [183]. There are many other investigations devoted to the mentioned manifolds; they are adequately covered in reviews of Boeckx, Kowalski, Vanhecke [25], Kaigorodov [81] and Lumiste [109].

A manifold with affine connection whose Ricci tensor R_{ij} satisfies the condition $R_{ij,[lm]} = 0$ has been called *Ricci-semisymmetric* $(RicPs_n)$. The literature on Ricci-semisymmetric manifolds was reviewed by Mirzoyan in [125]. These manifolds are natural generalizations of semisymmetric manifolds. (The concept of Ricci-semisymmetric manifolds is a natural generalization of the concept of semisymmetric manifolds).

Geodesic mappings of Riemannian manifolds onto equiaffine semisymmetric manifolds were first studied by Sinyukov in [810] and [170] (1954), and were further explored by Mikeš in [623] (1976), see [117, 118, 170, 226, 227, 626, 640–643, 665] and by Venzi in [917] (1978). In the first Section of the present part we improve the results of local nature, obtained by the authors cited above. Problems of global geodesic transformation theory for compact semisymmetric and Ricci semisymmetric Riemannian manifolds were considered by Sinyukova in [829, 830, 832]. She has given additional sufficient conditions in case when in such manifolds global geodesic mappings do not exist. Related conditions were found by Venzi (see [917]), Prvanović (see [753]), and some other authors.

In this part we prove that compact semisymmetric Riemannian manifolds of nonconstant curvature and non-Einsteinian Ricci-semisymmetric Riemannian manifolds do not admit nontrivial global geodesic mappings. This completes the investigations mentioned above.

On the other hand, among the manifolds under discussion, there are some which do admit local nontrivial geodesic mappings. However, there have been found wide classes of manifolds, namely symmetric, recurrent, m-recurrent,

2-Ricci-recurrent manifolds which do not admit local non-trivial geodesic mappings (see [117, 118, 170, 623, 626, 643, 753, 810, 917]).

Sinyukov had demonstrated in [810] and [170] that projectively Euclidean symmetric and recurrent equiaffine manifolds do not admit non-trivial geodesic mappings onto Riemannian manifolds. We present new classes of equiaffine manifolds which do not admit nontrivial geodesic mappings onto Riemannian manifolds. Manifolds that do not admit nontrivial geodesic mappings onto Riemannian manifolds are determined, up to an affine transformation, by the configuration of their geodesics.

1. The following can be checked.

Theorem 7.9 (Sinyukov, see [118, 170, 810]) *If there exists a geodesic mapping of a Riemannian manifold $\mathbb{V}_n$ $(n > 2)$ onto a semisymmetric equiaffine manifold $\bar{\mathbb{A}}_n$, then $\mathbb{V}_n$ is either an Einstein manifold or an equidistant manifold, in which the equidistant congruence is generated by the vector ψ_i, and the following equation holds:*

$$\psi_{ij} = B\, g_{ij}, \tag{7.30}$$

where g_{ij} is the metric tensor of $\mathbb{V}_n$, and B is a function.

Here $\psi_{ij} = \psi_{i,j} - \psi_i\psi_j$, and ψ_i is a gradient vector that determines the geodesic mapping.

Recall that a manifold with affine connection $\bar{\mathbb{A}}_n$ is called *semisymmetric* if the second covariant derivative[88)] of its curvature tensor $\overline{R}$, $\overline{R}^h_{ijk\,|\,lm}$, satisfies

$$\overline{R}^h_{ijk\,|\,lm} = \overline{R}^h_{ijk\,|\,ml}. \tag{7.31}$$

Let us introduce a geometric object

$$\Lambda^h_{ijklm}(\overline{R}) = -\,\overline{R}^\alpha_{ijk}\overline{R}^h_{\alpha lm} + \overline{R}^h_{\alpha jk}\overline{R}^\alpha_{ilm} + \overline{R}^h_{i\alpha k}\overline{R}^\alpha_{jlm} + \overline{R}^h_{ij\alpha}\overline{R}^\alpha_{klm}\,.$$

Using the Ricci identity, we check that (7.31) is equivalent to

$$\Lambda^h_{ijklm}(\overline{R}) = 0. \tag{7.32}$$

These equations, having algebraic character, were studied in connection with the developement of the theory of symmetric spaces and their generalization (É. Cartan [323, 324], Kovalev [553], Nomizu [734], P.A. Shirokov [167] etc.); Sinyukov [810] was the first who started to investigate them in connection with geodesic mappings.

A Riemannian space $\mathbb{V}_n$ with a metric tensor g_{ij} is called *equidistant* if there exists a non-vanishing one-form φ in $\mathbb{V}_n$, $\varphi_i \neq 0$, the covariant derivative of which is proportional to the metric tensor, i.e. satisfying

$$\varphi_{i,j} = \varrho\, g_{ij}. \tag{7.33}$$

[88)] Here covariant differentiation with respect to the connection in $\bar{\mathbb{A}}_n$ is denoted by "|".

Such a 1-form φ_i is necessarily exact (=gradient), therefore it determines in $\mathbb{V}_n$ a normal congruence called *equidistant*; see Sec. 2.3, p. 140.

Proof. of the Theorem. Suppose that a Riemannian manifold $\mathbb{V}_n$ admits a geodesic mapping $f\colon \mathbb{V}_n \to \bar{\mathbb{A}}_n$ $(n > 2)$ onto a semisymmetric equiaffine manifold $\bar{\mathbb{A}}_n$ given by the covector ψ_i. In a common coordinate system with respect to f, let us consider the relationship of the corresponding curvature tensors. Under our assumptions, $\psi_{ij} = \psi_{ji}$, and

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ijk} \tag{7.34}$$

where

$$P^h_{ijk} = \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}. \tag{7.35}$$

Due to particular properties of the difference tensor P^h_{ijk}, we check

$$\Lambda^h_{ijklm}(P) = 0.$$

By (7.32) we get

$$\begin{aligned}\Lambda^h_{ijklm}(R) - {} & R^\alpha_{ijk} P^h_{\alpha lm} + R^h_{\alpha jk} P^\alpha_{ilm} + R^h_{i\alpha k} P^\alpha_{jlm} + R^h_{ij\alpha} P^\alpha_{klm} \\ & - P^\alpha_{ijk} R^h_{\alpha lm} + P^h_{\alpha jk} R^\alpha_{ilm} + P^h_{i\alpha k} R^\alpha_{jlm} + P^h_{ij\alpha} R^\alpha_{klm} = 0.\end{aligned}$$

Now accounting (7.35), lowering the index h in $\mathbb{V}_n$ and using algebraic properties[89] of R in type $(0,4)$ we find

$$\begin{aligned}\Lambda^h_{ijklm}(R) - {} & g_{h[m}\psi_{l]\alpha} R^\alpha_{ijk} - \psi_{i[l} R_{m]hjk} + \psi_{j[l} R_{m]khl} \\ & - \psi_{k[l} R_{m]lhi} + g_{hk}\psi_{\alpha(i} R^\alpha_{j)lm} - g_{hj}\psi_{\alpha(i} R^\alpha_{k)lm} = 0,\end{aligned}$$

where $[\ ,\ ]$ is alternation and $(\ ,\ )$ symmetrization. The curvature R_{hijk} of $\mathbb{V}_n$ is skew-symmetric in h, i, therefore $\Lambda^h_{ijklm}(R)$ is is skew-symmetric in h, i as well, since

$$\Lambda^h_{ijklm}(R) = R^h_{ijk,[lm]}\,,$$

and

$$\Lambda_{hijklm} = g_{h\alpha}\Lambda^\alpha_{ijklm} = R_{hijk,[lm]}.$$

Consequently, if we symmetrize the above equations in h, i and then contract the resulting equations with $g^{k\ell}$ in k as well as in ℓ, we obtain

$$\begin{aligned}& -g_{hm}\psi_{\alpha\beta} R^{\alpha}{}_{ij}{}^{\beta} - g_{im}\psi_{\alpha\beta} R^{\alpha}{}_{hj}{}^{\beta} + g_{hj}(\psi_{\alpha\beta} R^{\alpha}{}_{im}{}^{\beta} + \psi_{i\alpha} R^\alpha_m) \\ & -\psi_{m(i} R_{h)j} + g_{ij}\,(\psi_{\alpha\beta} R^{\alpha}{}_{hm}{}^{\beta} + \psi_{h\alpha} R^\alpha_m) - \psi_{\alpha m} R^\alpha_{(ih)j} + \psi_{\alpha j} R^\alpha_{(ih)m} = 0,\end{aligned}$$

where R_{ij} are components of the Ricci tensor of $\mathbb{V}_n$; we rised indices by means of g^{ij}. Symmetrization of these formulas in m, j yields

$$\begin{aligned}& \psi_{im} R_{hj} + \psi_{ij} R_{hm} + \psi_{hm} R_{ij} + \psi_{hj} R_{im} \\ & -g_{im}\,\psi_{h\alpha} R^\alpha_j - g_{ij}\,\psi_{h\alpha} R^\alpha_m - g_{hm}\,\psi_{i\alpha} R^\alpha_j - g_{hj}\,\psi_{i\alpha} R^\alpha_m = 0.\end{aligned} \tag{7.36}$$

[89] $R_{hijk} = R_{jkhi}$

Further, contracting with g^{ij} in i and j, we get

$$-(n+1)\,\psi_{h\alpha}R^{\alpha}_{m} - g_{hm}\,\psi_{\alpha\beta}R^{\alpha\beta} + \psi_{m\alpha}R^{\alpha}_{h} \\ +\Delta\psi R_{hm} + \psi_{hm}\,\varrho = 0,$$

where $\Delta\psi = \psi_{\alpha\beta}g^{\alpha\beta}$, and ϱ is the scalar curvature of $\mathbb{V}_n$. Since the tensors g_{hm}, R_{hm} and ψ_{hm} are symmetric, it follows

$$\psi_{h\alpha}R^{\alpha}_{m} = \psi_{m\alpha}R^{\alpha}_{h}.$$

again, contraction with g^{hm} shows that $\psi_{\alpha\beta}R^{\alpha\beta} = \varrho\,\Delta\psi$, which reads

$$n\,\psi_{h\alpha}R^{\alpha}_{m} = \Delta\psi R_{hm} + \varrho\,\psi_{hm} - \frac{\varrho}{n}\Delta\psi\,g_{hm}. \tag{7.37}$$

Now consider the tensor

$$Q_{ijhm} = g_{ij}\,\psi_{h\alpha}R^{\alpha}_{m} + g_{hm}\,\psi_{i\alpha}R^{\alpha}_{j} - R_{ij}\psi_{hm} - R_{hm}\psi_{ij}.$$

By (7.37), the following holds:

$$Q_{ijhm} = Q_{jihm} = Q_{hmij}.$$

From (7.36) we get

$$Q_{ijhm} + Q_{hjim} = 0.$$

Hence

$$Q_{ijhm} = -Q_{hjim} = -Q_{jhlm} = Q_{ihjm} = Q_{hijm} = -Q_{jihm} = -Q_{ijhm},$$

or equivalently, $Q_{ijhm} = 0$. Let us introduce

$$\text{a)}\quad E_{ij} = R_{ij} - \frac{\varrho}{n}\,g_{ij} \qquad \text{and} \qquad \text{b)}\quad \mathcal{E}_{ij} = \psi_{ij} - \frac{\Delta\psi}{n}\,g_{ij}. \tag{7.38}$$

Then the above equations yield

$$\mathcal{E}_{ij}E_{hm} + \mathcal{E}_{hm}E_{ij} = 0.$$

It is almost obvious that the above conditions can be satisfied if and only if in $\mathbb{V}_n$, at least one of the following conditions holds:

$$\text{a)} \qquad E_{ij} = 0 \qquad \text{or} \qquad \text{b)} \qquad \mathcal{E}_{ij} = 0.$$

If (a) is satisfied then we deduce from (7.38a) that $\mathbb{V}_n$ is an Einstein space. On the other hand, accounting (7.38b) we get

$$\psi_{i,j} - \psi_i\psi_j = \frac{\Delta\psi}{n}g_{ij}.$$

If we put $\varphi_i = \exp(-\psi)\,\psi_i$ we can see that (7.33) holds, in other words, ψ_i defines an equidistant congruence in $\mathbb{V}_n$. □

2. Let us prove a stronger theorem to improve the result mentioned above.

Theorem 7.10 (Mikeš [643]) *If there exists a geodesic mapping of a Riemannian manifold $\mathbb{V}_n$ $(n > 2)$ with nonconstant curvature onto a semisymmetric equiaffine manifold $\bar{\mathbb{A}}_n$, then $\mathbb{V}_n$ is an equidistant manifold in which (7.30) holds with constant B, and the following condition is satisfied in $\mathbb{V}_n$*

$$\begin{aligned} R_{hijk,[lm]} = B\,(&R_{lijk}g_{mh} + R_{hljk}g_{ml} + R_{hilk}g_{mj} + R_{hiji}g_{mk} \\ &- R_{mijk}g_{lh} - R_{hmjk}g_{li} - R_{himk}g_{lj} - R_{hijm}g_{lk}), \end{aligned} \tag{7.39}$$

where R_{hijk} are components of the Riemannian tensor of $\mathbb{V}_n$.

A manifold satisfying (7.39) is said to be a *generalized semisymmetric manifold*. This concept was introduced by Mikeš in 1976 [117, 623, 640, 642, 643]; let us note that in many papers these manifolds are called *pseudosymmetric* (Deszcz [362]–[366], [372]–[376]).

Note that the Theorem 7.10 had been proved earlier, in 1976, by Mikeš [623], see [118, 149, 170], and later on, in 1978, by Venzi [917] for the case of geodesic mappings of Riemannian manifolds onto semisymmetric Riemannian manifolds. In references, usually the paper [917] by Venzi is cited, see e.g. [920]. Let us mention the results on geodesic mappings of recurrent manifolds obtained by Sinyukov [118, 170, 810]. Further, the author of [917] probably did not realize that projectively recurrent Riemannian manifolds are always recurrent, which has been proved by Glodek [436], and independently by Mikeš [623].

Let us first prove the following

Lemma 7.4 *Given a geodesic correspondence between $\mathbb{V}_n$ and $\bar{\mathbb{A}}_n$, suppose there exist tensor fields $T^h_{i\,l_1\cdots l_m}$, $M^h_{i\,l_1\cdots l_p}$ and $S^{\alpha_1\cdots\alpha_p}_{l_1\cdots l_m sr}$ such that*

$$T_{(hi)l_1\cdots l_m} = M_{(hi)l_1\cdots l_p} = 0; \tag{7.40}$$

$$T^h_{i\,l_1\cdots l_m|[sr]} = M^h_{\alpha_1\cdots\alpha_p} S^{\alpha_1\cdots\alpha_p}_{l_1\cdots l_m sr}\,. \tag{7.41}$$

Then if the condition (7.30) fails, we have $T^h_{i\,l_1\cdots l_m} = 0$.

Here " $|$ " is the covariant derivative in $\bar{\mathbb{A}}_n$, and

$$T_{hi\,l_1\cdots l_m} = g_{h\alpha}T^\alpha_{i\,l_1\cdots l_m}, \qquad M_{hi\,l_1\cdots l_p} = g_{h\alpha}M^\alpha_{i\,l_1\cdots l_p}.$$

Proof. Via the Ricci identity we rearrange (7.41) and replace the Riemannian tensor components $\overline{R}^h_{ijk}$ by means of (6.26). Lowering the index h and symmetrizing the expression in h and i, via (7.40) we obtain

$$\psi_{s(h}T_{i)rl_1\cdots l_m} - \psi_{r(h}T_{i)sl_1\cdots l_m} - g_{s(h}T^\alpha_{i)l_1\cdots l_m}\psi_{\alpha r} + g_{r(h}T^\alpha_{i)l_1\cdots l_m}\psi_{\alpha s} = 0. \tag{7.42}$$

Contracting (7.42) with g^{is} (components of the matrix inverse to g_{ij}), we have

$$T^\alpha_{hl_1\cdots l_m}\psi_{\alpha r} = B\,T_{hrl_1\cdots l_m} + \frac{1}{n}\psi_{\alpha(h}T^\alpha_{r)l_1\cdots l_m}$$

where $B = \frac{1}{n}\psi_{\alpha\beta}g^{\alpha\beta}$. Symmetrizing this in h and r, we find out that $\psi_{\alpha(h}T^{\alpha}_{r)l_1\cdots l_m} = 0$, hence now the last formula takes the form:

$$T^{\alpha}_{hl_1\cdots l_m}\psi_{\alpha r} = B\,T_{hrl_1\cdots l_m}.$$

Then, setting $S_{ij} = \psi_{ij} - B\,g_{ij}$, we see that (7.42) reads:

$$S_{s(h}T_{i)rl_1\cdots l_m} - S_{r(h}T_{i)sl_1\cdots l_m} = 0.$$

Alternating this expression in i and r, and then symmetrizing it in h and s, we obtain that $S_{sh}T_{irl_1\cdots l_m} = 0$. The Lemma has been proved. □

The Lemma 7.4 implies

Corollary 7.1 *Suppose that an equiaffine manifold $\bar{\mathbb{A}}_n$ $(n > 2)$ admits a geodesic mapping onto an Einstein manifold $\mathbb{V}_n$ of nonconstant curvature. If $\overline{W}^h_{ijk|[lm]} = \overline{W}^h_{i\alpha\beta}S^{\alpha\beta}_{jklm}$ where S is a tensor, then $\mathbb{V}_n$ is equidistant and the conditions (7.30) hold.*

Indeed, suppose that the assumptions of Lemma 7.4 hold for the projective curvature tensor of an Einstein manifold $\mathbb{V}_n$. Hence (7.30) is satisfied since, in the case when $W^h_{ijk} = 0$, the manifold $\mathbb{V}_n$ has constant curvature.

Now let us come back to the proof of Theorem 7.10. Assume that the contrary would hold. Suppose that $\mathbb{V}_n$ of nonconstant curvature admits a geodesic mapping onto an equiaffine semisymmetric manifold $\bar{\mathbb{A}}_n$, and at the same time the condition (7.30) fails. Then, due to Sinyukov's theorem 7.9, $\mathbb{V}_n$ is an Einstein manifold. However, in $\bar{\mathbb{A}}_n$ the assumption of Corollary 7.1 holds if $S = 0$. Hence $\mathbb{V}_n$ is a manifold of constant curvature; this gives a contradiction. Therefore, (7.30) give relations which imply that B is constant. Since $\bar{\mathbb{A}}_n$ is semisymmetric, the equations $\overline{R}^h_{ijk|[lm]} = 0$ hold, which take the form (7.39) in $\mathbb{V}_n$, due to the Ricci identities (6.26) and (7.30). So the Theorem 7.10 has been proved completely. □

3. Let us obtain conditions for an equiaffine semisymmetric manifold $\mathbb{A}_n$ to admit geodesic mappings onto a Riemannian manifold $\overline{\mathbb{V}}_n$.

Theorem 7.11 (Mikeš [643])*A semisymmetric equiaffine manifold $\mathbb{A}_n$ which is not projectively flat admits a nontrivial geodesic mapping onto a manifold $\mathbb{V}_n$ $(n > 2)$ if and only if there exists a solution to the Cauchy type system of linear partial differential equations (written in terms of covariant derivatives)*

$$\text{a)}\quad a^{ij}{}_{,k} = \lambda^i\delta^i_k + \lambda^j\delta^j_k; \quad \text{b)}\quad \lambda^h_{,i} = \mu\,\delta^h_i \tag{7.43}$$

with respect to the following unknowns: the non-degenerate symmetric tensor a^{ij}, the nonzero vector λ^i, and the constant μ.

Note that the Theorem 7.11 had been proved earlier, in 1976, by Mikeš [623], see also 1979 the monograph by Sinyukov [170], and [118,149], and later on, in 1978,

by Venzi [917] for the case of geodesic mappings of Riemannian manifolds onto semisymmetric Riemannian manifolds. In references, usually only the paper [917] by Venzi is cited, see e.g. [362–364, 366, 372–376].

Proof. It was shown in Theorem 7.5 that conditions (7.43a) (≡ (7.12)) are necessary and sufficient for an equiaffine manifold $\mathbb{A}_n$ to admit a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$, and, in this case, the conditions (7.13) and (7.14) hold.

Taking the covariant derivative of (7.14) with respect to x^j and using (7.30) and (7.43a), we verify that (7.43b) holds. It follows from the integrability conditions that the invariant μ is constant. This completes the proof. □

4. Suppose that $\mathbb{A}_n$ is an equiaffine manifold (not necessarily semisymmetric). The integrability conditions for the equations (7.43) and their differential prolongations take the form:

$$(0)\ \ \lambda^\alpha R^h_{\alpha jk} = 0; \quad \cdots \quad (\mathrm{m})\ \lambda^\alpha R^h_{\alpha jk,l_1\cdots l_m} + \mu \overset{m}{T}{}^h_{\alpha jk,l_1\cdots l_m} = 0; \tag{7.44}$$

$$(0)\ a^{\alpha(i} R^{h)}_{\alpha jk} = 0; \quad \cdots \quad (\mathrm{m})\ a^{\alpha(i} R^{h)}_{\alpha jk,l_1\cdots l_m} + \lambda^{(i} \overset{m}{T}{}^{h)}_{\alpha jk,l_1\cdots l_m} = 0, \tag{7.45}$$

where

$$\overset{m}{T}{}^h_{\alpha jk,l_1\cdots l_m} = \sum_{s=1}^{m} R^h_{l_s jk,l_1\cdots l_{s-1}l_{s+1}\cdots l_m}. \tag{7.46}$$

The conditions (7.44) and (7.45) constitute a system of linear algebraic equations in a^{ij}, λ^i, and μ. If, at a point $x_\circ \in \mathbb{A}_n$, there exists a solution to this system such that $|a^{ij}| \neq 0$; $a^{[ij]} = 0$; $\lambda^i \neq 0$, then one can construct a solution to (7.43), and thus find a nontrivial geodesic mapping of the semisymmetric manifold $\mathbb{A}_n$ onto a Riemannian manifold.

7.3.2 Geodesic mappings of generalized recurrent manifolds

Now let us find classes of equiaffine manifolds which do not admit nontrivial geodesic mappings.

Theorem 7.12 (Mikeš [643]) *An equiaffine, not projectively flat manifold $\mathbb{A}_n$ ($n > 2$) does not admit nontrivial geodesic mappings satisfying* (7.43) *onto a Riemannian manifold if $\mathbb{A}_n$ obeys one of the following conditions:*

$$R^h_{ijk,l} = R^h_{i\alpha\beta} S^{\alpha\beta}_{jkl}; \tag{7.47}$$

$$Q^{ph}_{(ir)jk,l} = Q^{\gamma\delta}_{(ir)\alpha\beta} S^{\alpha\beta ph}_{\gamma\delta jkl} \tag{7.48}$$

$$R^h_{ijk,lm} = 0; \tag{7.49}$$

$$R^h_{ijk,l_1\cdots l_m} = R^h_{ijk} S_{l_1\cdots l_m} \neq 0; \tag{7.50}$$

where S are certain tensors, and $Q^{ph}_{ijkl} = \delta^{(p}_l R^{h)}_{ijk}$.

Particularly, from Theorems 7.11 and 7.12 it follows that an equiaffine semisymmetric manifold $\mathbb{A}_n$ $(n > 2)$, satisfying one of the above conditions, either does not admit nontrivial geodesic mappings onto Riemannian manifold, or is projectively flat. Conditions (7.47)–(7.50) characterize generalized recurrent manifolds; conditions (7.49) determine 2-symmetric manifolds – they are semisymmetric; and the conditions (7.50) determine m-recurrent manifolds (see [81]).

Notice that the conditions (7.48) imply relations (7.39). Let us prove this fact. Suppose that $\mathbb{A}_n$ $(n > 2)$ admits a nontrivial geodesic mapping onto $\overline{\mathbb{V}}_n$ and the condition (7.43) holds. Contracting (7.40) with a^{ir} and taking into account (7.45), we get $\lambda^{(p}R^{h)}_{ijk} = 0$. Since $\lambda^p \neq 0$, we obtain that $R^h_{ijk} = 0$ which contradicts the assumption that $\mathbb{A}_n$ is not projectively flat. □

Theorem 7.12 considerably improves the results obtained by Sinyukov in [170, 810], by Mikeš in [117, 226, 227, 623, 626], Venzi (see [917]), Prvanović (see [753]), Sobchuk [837, 838], and others.

In works by Fomin [405, 406] and Mikeš [118], geodesic mappings are studied for manifolds with more general conditions of recurrency.

Under a *generalized recurrent* manifold with affine connection we understand $\mathbb{A}_n$ in which

$$R^h_{ijk,l} = \varphi_l R^h_{ijk} + \nu_j R^h_{ikl} - \nu_k R^h_{ijl} + \delta^h_i A_{jkl} + \delta^h_j B_{ikl} - \delta^h_k B_{ijl} + \delta^h_l C_{ijk}, \quad (7.51)$$

holds, where φ, ν, A, B, C are some tensors.

Let us note that these conditions are satisfied by *symmetric*, *recurrent*, and also *projective symmetric* and *projective recurrent* spaces, which are characterized by

$$\begin{array}{ll} \nabla R = 0, & R^h_{ijk,l} = 0; \\ \nabla R = \varphi \otimes R, & R^h_{ijk,l} = \varphi_l R^h_{ijk}; \\ \nabla W = 0, & W^h_{ijk,l} = 0; \\ \nabla W = \varphi \otimes W, & W^h_{ijk,l} = \varphi_l W^h_{ijk}, \end{array} \quad (7.52)$$

where R is the curvature, W is the Weyl projective curvature, and φ is a non vanishing linear form.

Theorem 7.13 *Riemannian manifolds $\overline{\mathbb{V}}_n$ $(n > 4)$ with nonconstant curvature do not admit nontrivial geodesic mappings onto manifolds with affine connection $\mathbb{A}_n$ which satisfy (7.51).*

For infinite-dimensional spaces this result was partially obtained by Fomin [405]. If $\mathbb{A}_n$ is equiaffine and $A_{jkl} = 0$ holds then the theorem is valid also for $n = 3$, 4, see [117]. For equiaffine symmetric manifolds and semisymmetric recurrent manifolds, this theorem was proved by Sinykov in [170, 810]:

Theorem 7.14 *Riemannian manifolds $\overline{\mathbb{V}}_n$ $(n > 2)$ with nonconstant curvature do not admit non trivial geodesic mappings onto symmetric equiaffine manifolds (or onto semisymmetric recurrent equiaffine manifolds).*

Let us prove the more general [479]:

Theorem 7.15 *Non projective flat Weyl manifolds $\overline{\mathbb{W}}_n$ $(n > 8)$ do not admit nontrivial geodesic mappings onto a manifold with affine connection $\mathbb{A}_n$ which satisfies (7.51).*

Let us recall that Weyl manifolds are generalization of Riemannian manifolds.

A manifold $\mathbb{A}_n$ with affine connection is called *Weyl manifold* $\mathbb{W}_n$ if there is a regular form g on $\mathbb{A}_n$ for which the following holds:

$$\nabla g = w \otimes g, \tag{7.53}$$

where w is a linear form.

In local notation: $g_{ij,k} = w_k g_{ij}$. If w is a gradient (or locally a gradient) then $\mathbb{W}_n$ is conformal (= conformally equivalent) to some Riemannian manifold. Definition of conformal mapping see Chapter 4.

By the inspiration of Yıldırım, Arsan [954] and Thomas [893] we get

Theorem 7.16 *A manifold $\mathbb{A}_n$ admits a geodesic mapping onto a Weyl manifold $\overline{\mathbb{W}}_n$ with metric $\overline{g}$ if and only if the following equations are satisfied:*

$$\overline{g}_{ij,k} = 2\varphi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}, \tag{7.54}$$

or equivalently, for any vector fields $X, Y, Z \in \mathcal{X}(\mathbb{A}_n)$:

$$\nabla_Z \overline{g}(X,Y) = 2\varphi(Z)\overline{g}(X,Y) + \psi(X)\overline{g}(Y,Z) + \psi(Y)\overline{g}(X,Z). \tag{7.55}$$

Here ", " is the covariant derivative relative the connection ∇ on $\mathbb{A}_n$, ψ, φ are one-forms, and ψ_i, φ_i denote their components. Moreover, the Weyl form of $\overline{\mathbb{W}}_n$ has the following shape:

$$\overline{w} = \varphi - \psi. \tag{7.56}$$

The *proof* is analogous to the proof of Theorem 7.1, so we restrict ourselves to its formulation only. □

Let us *prove* Theorem 7.15. Let $\mathbb{A}_n$ be such that (7.51) holds, and let $\mathbb{A}_n$ admit a geodesic mapping onto a Weyl manifold $\overline{\mathbb{W}}_n$, in which the metric form $\overline{g}$ satisfies the condition $\overline{\nabla}\,\overline{g} = \overline{w} \otimes \overline{g}$. The corresponding integrability conditions take the form

$$\overline{g}_{i\alpha}\overline{R}^{\alpha}_{jkl} + \overline{g}_{j\alpha}\overline{R}^{\alpha}_{ikl} = \overline{w}_{[l|k]}\,\overline{g}_{ij}, \tag{7.57}$$

where $\overline{R}^{h}_{ijk}$ is the curvature tensor of $\overline{\mathbb{W}}_n$, " | " is the covariant derivative in $\overline{\mathbb{W}}_n$. Let us rise the indices i and j by means of the dual tensor to the metric, i.e. by $\overline{g}^{ij}$:

$$\overline{g}^{i\alpha}\overline{R}^{j}_{\alpha kl} + \overline{g}^{j\alpha}\overline{R}^{i}_{\alpha kl} = \overline{w}_{[l|k]}\,\overline{g}^{ij}. \tag{7.58}$$

Using (6.17), let us express the curvature tensor of $\overline{\mathbb{W}}_n$ from the formula (7.58). We get

$$\overline{g}^{i\alpha}R^{j}_{\alpha kl} + \overline{g}^{j\alpha}R^{i}_{\alpha kl} = b_{kl}\,\overline{g}^{ij} + \delta^i_k a^j_l + \delta^j_k a^i_l - \delta^i_l a^j_k - \delta^j_l a^i_k, \tag{7.59}$$

where b_{kl} and a^i_k are components of tensors. Let us note that the equations (7.54) are equivalent to

$$\overline{g}^{ij}_{,k} = -2\varphi_k \overline{g}^{ij} - \psi_\alpha \overline{g}^{\alpha i}\delta^j_k - \psi_\alpha \overline{g}^{\alpha j}\delta^i_k\,. \tag{7.60}$$

Taking the covariant derivative of (7.59) with respect to x^m and using (7.60) we obtain

$$\Lambda^{(i}R^{j)}_{mkl} + \overline{g}^{\alpha(i}R^{j)}_{\alpha kl,m} = b_{klm}\,\overline{g}^{ij} + \delta^i_k a^j_{lm} + \delta^j_k a^i_{lm} - \delta^i_l a^j_{km} - \delta^j_l a^i_{km} + \delta^i_m c^j_{kl} - \delta^j_m c^i_{kl},$$

where $\Lambda^i = \overline{g}^{i\alpha}\psi_\alpha$, $b_{klm}, a^i_{km}, c^i_{kl}$ are certain tensors. From the last formula we express the derivation of the curvature tensor by means of the formulas (7.51). After some calculations we get

$$\Lambda^{(i}R^{j)}_{mkl} = B_{klm}\,\overline{g}^{ij} + \delta^i_k A^j_{lm} + \delta^j_k A^i_{lm} - \delta^i_l A^j_{km} - \delta^j_l A^i_{km} + \delta^i_m C^j_{kl} - \delta^j_m C^i_{kl}, \tag{7.61}$$

where $A^i_{km}, B_{klm}, C^i_{kl}$ are certain tensors.

From (7.61) it follows that if $n > 8$, then $B_{klm} = 0$.

It was proved that for $n > 2$ when $W \neq 0$ there exists a coordinate system x in which $R^1_{223} \neq 0$, see Theorem 6.9. Step by step, we insert into (7.61):

$$\begin{aligned}
&i = 1, \ldots, n,\ j = 1,\ m = k = 2,\ l = 3;\\
&i = j = k = 1,\ l = 3,\ m = 2;\\
&i = j = m = 1,\ l = 3,\ k = 2;\\
&i = j = k = 1,\ l = m = 2
\end{aligned}$$

and we can see that $\Lambda^i = 0$ ($\Leftrightarrow \psi_i = 0$) holds.

By a detailed analysis, we can check that if $\mathbb{A}_n$ is equiaffine and $\overline{\mathbb{W}}_n$ is Riemannian then $B_{klm} = 0$. And if this is the case, we can restrict ourselves to $n > 2$. □

Since we do not suppose that $\mathbb{A}_n$ is semisymmetric, the last results can be considered as a generalization of the results from the previous subsection above.

8 GEODESIC MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

8.1 General results on geodesic mappings between $\mathbb{V}_n$

8.1.1 Levi-Civita and Sinyukov equations of geodesic mappings

Consider two Riemannian manifolds $\mathbb{V}_n = (M_n, g)$ and $\overline{\mathbb{V}}_n = (\overline{M}_n, \overline{g})$, with the corresponding Riemannian connections ∇ and $\overline{\nabla}$, respectively. Suppose there exists a geodesic map $f : \mathbb{V}_n \to \overline{\mathbb{V}}_n$. Let us identify $\overline{M}$ with M via f, so that the deformation tensor $P = \overline{\nabla} - \nabla$ of f can be defined. Then (4.2), (6.5) and (6.8) hold.

Since Riemannian spaces are manifolds with affine connection (with the canonical Levi-Civita connection constructed from the metric of arbitrary signature), and moreover they are equiaffine manifolds, they have all characteristic features and properties formulated in chapters 4 and 5 above. Of course, some of the results can be specified or be made more precise, and new results can be derived.

By a geodesic mapping $\mathbb{V}_n \to \overline{\mathbb{V}}_n$, the Levi-Civita equations (6.8) and (7.1) hold, i.e.

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \psi_i \delta^k_j + \psi_j \delta^k_i, \tag{8.1}$$

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}. \tag{8.2}$$

In this case the one-form $\psi(X)$, with local components ψ_i, is globally a gradient (= exact form), that is,

$$\psi(X) = \nabla_X \Psi \qquad (\text{i.e. } \psi_i = \delta_i \Psi) \tag{8.3}$$

for the function

$$\Psi = \frac{1}{2(n+1)} \ln \left| \frac{\overline{\mathcal{G}}}{\mathcal{G}} \right|, \tag{8.4}$$

where $\mathcal{G} = \det(g_{ij})$ and $\overline{\mathcal{G}} = \det(\overline{g}_{ij})$. In other words, $\psi(X) = \nabla_X \Psi$. The above formula follows by (6.10) and by the Voss-Weyl formula (3.11):

$$\psi_i = \frac{1}{n+1} (\overline{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}) = \frac{1}{n+1} \left(\partial_i \ln \sqrt{|\overline{\mathcal{G}}|} - \partial_i \ln \sqrt{|\mathcal{G}|} \right) = \frac{1}{2(n+1)} \partial_i \ln \left| \frac{\overline{\mathcal{G}}}{\mathcal{G}} \right|.$$

Further, the equations of Mikeš-Berezovski (7.12): $a^{ij}_{,k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k$ hold, see Theorem 7.5. If we set

$$a_{ij} = a^{\alpha\beta} g_{\alpha i} g_{\beta j} \quad \text{and} \quad \lambda_i = g_{i\alpha} \lambda^\alpha, \tag{8.5}$$

we obtain by (7.12) the equations by Sinyukov [817], see [170, p. 121]:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{8.6}$$

The equations (8.6) have been completed by Sinyukov [170, pp. 132-134] to a system of differential equations of Cauchy type in covariant derivatives. In principle, this system has the form (7.16) for the unknown tensors a_{ij}, λ_i and μ:

$$\begin{aligned} &(a) \qquad a_{ij,k} = \lambda_i\, g_{jk} + \lambda_j\, g_{ik};\\ &(b) \qquad n\,\lambda_{i,j} = \mu\, g_{ij} - a_{i\alpha}R^{\alpha}_{j} - a_{\alpha\beta}R^{\alpha}{}_{ij}{}^{\beta};\\ &(c) \quad (n-1)\,\mu_{,i} = 2(n+1)\lambda_{\alpha}R^{\alpha}_{i} + a_{\alpha\beta}(2R^{\alpha}{}_{i,}{}^{\beta} - R^{\alpha\beta}{}_{,i}). \end{aligned} \tag{8.7}$$

We remark, that λ_i is a globally gradient-like vector, moreover

$$\lambda_i = \tfrac{1}{2}\,\nabla_i(a_{\alpha\beta}g^{a\beta}) \quad \text{and} \quad \mu = \lambda_{,\alpha\beta}g^{\alpha\beta}, \tag{8.8}$$

and from (8.7b) follows [170, p. 138]:

$$a_{i\alpha}R^{\alpha}_{j} = a_{j\alpha}R^{\alpha}_{i}. \tag{8.9}$$

Among others, for not projectively Euclidean spaces (i.e. spaces with non-constant curvature), the formula (7.11) holds, i.e. the vector λ_i can be expressed in the form $\lambda_i = a_{\alpha\beta}(x)\,G_i^{\alpha\beta}(x)$, where $G_i^{\alpha\beta}(x)$ are determined by objects of the metric tensor g of $\mathbb{V}_n$. In such a case, the equations (8.5) form a closed linear system of Cauchy type with respect to the unknown functions $a_{ij}(x)$, and the following holds.

Let us note that it was during the analysis of these equations, when Sinyukov introduced the concept of degree of mobility of $\mathbb{V}_n$ relative to geodesic mappings, see p. 277.

8.1.2 Sinyukov Γ-transformations of geodesic mappings

In [170, p. 125] Sinyukov has proved that if (M, g) admits geodesic mapping onto $(M, \overline{g})$, then (M, a) admits geodesic mapping onto $(M, \tilde{a} = \exp(2\Psi)g)$ with the same 1-form ψ. This construction $\{(M, g), (M, \overline{g})\} \mapsto \{(M, \tilde{a}), (M, a)\}$ is called Γ-*transformation*.

For better understanding of manifolds and mappings among them, we introduce the following diagram

$$\begin{array}{ccc} (M, g) & \overset{\psi}{\longrightarrow} & (M, \overline{g}) \\ \downarrow \textit{ conf. map.} & & \\ (M, \tilde{a} = \exp(2\psi)g) & \overset{\psi}{\longleftarrow} & (M, a). \end{array} \tag{8.10}$$

This process can indefinitely being continued. In this way we obtain an infinite sequence of Riemannian manifolds admitting geodesic mappings.

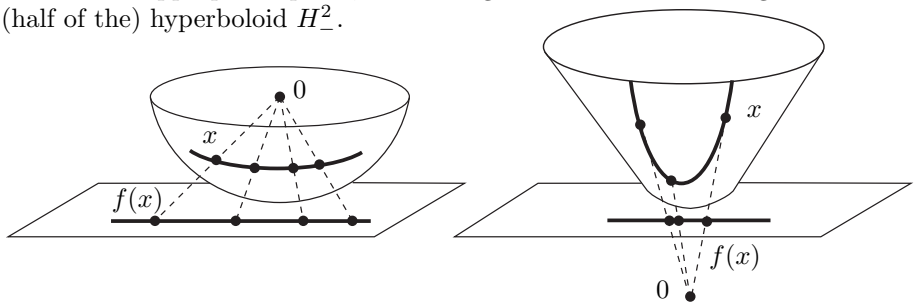
This facts may be directly derived by formulae (6.8), (7.1) and (8.6).

8.2 Classical examples of geodesic mappings

8.2.1 Lagrange and Beltrami projections

The first examples of geodesically equivalent metrics are due to Lagrange [103]. He observed that the radial projection $f(x,y,z)=(-\frac{x}{y},-\frac{y}{z},-1)$ takes geodesics of the half-sphere $S^2=\{(x,y,z)\in\mathbb{R}^3\colon\ x^2+y^2+y^2=1,\ z<0\}$ to the geodesics of the plane $E^2=\{(x,y,z)\in\mathbb{R}^3\colon\ z=-1\}$, see the left-hand side of the following Figure, since geodesics of both metrics are intersection of the 2-plane containing the point (0, 0, 0) with the surface. This projection is called a *gnomonic projection* or *gnomonic map*.

Later, Beltrami [292] generalized the example for the metrics of constant negative curvature, and for the pseudo-Riemannian metrics of constant curvature. In the example of Lagrange, he replaced the half sphere by the half of one of the hyperboloids $H^2_{\pm}=\{(x,y,z)\in\mathbb{R}^3\colon\ x^2+y^2-z^2=\pm1\}$, with the restriction of the Lorenz metrics $dx^2+dy^2-dz^2$ to it. Then, the geodesics of the metric are also intersections of the 2-planes containing the point (0, 0, 0) with the surface, and, therefore, the stereographic projection sends it to the straight lines of the appropriate plane, see the right-hand side of the Figure with the (half of the) hyperboloid H^2_-.



Though the examples of the Lagrange and Beltrami are two-dimensional, one can easily generalize them for every dimension, see [539].

We describe their natural multi-dimensional generalization of gnomonic projection. Consider the sphere

$$S^n=\{(x^1,x^2,\dots,x^{n+1})\in\mathbb{R}^{n+1}\colon\ {x^1}^2+{x^2}^2+\ \cdots\ +{x^{n+1}}^2=1\}$$

with the metric g which is the restriction of the Euclidean metric to the sphere. Next, consider the mapping $a\colon\ S^n\to S^n$ given by $a\colon\ v\mapsto\frac{A(v)}{\|A(v)\|}$, where A is an arbitrary non-degenerate linear transformation of $\mathbb{R}^{n+1}$.

8.2.2 Dimension two

In 1869, U. Dini [379] proved:

Theorem 8.1 *There is a geodesic mapping between two nonisometric surfaces $\mathbb{V}_2$ and $\overline{\mathbb{V}}_2$ if and only if their metrics have the Liouville form:*

$$ds^2=(U-V)(du^2+dv^2)\quad\text{and}\quad d\bar{s}^2=\left(\frac{1}{U}-\frac{1}{V}\right)\left(\frac{du^2}{U}+\frac{dv^2}{V}\right).\qquad(8.11)$$

where $U(u)$ and $V(v)$ are positive functions.

Note that the form (8.11) was found by Darboux. Invariant criterion of Liouville surfaces was obtained by Shulikovski [808].

If the first of the given metrics is positive definite, then geodesics of the corresponding metrics have the expression above, whereas $V(v) \cdot U(u) \neq 0$. Moreover, P.A. Shirokov [167], pp. 383-388, proved the following theorem:

Theorem 8.2 *The pseudo-Riemannian manifold $\mathbb{V}_2$ admits non trivial geodesic mappings if and only if the metric ds^2 of $\mathbb{V}_2$ has one of the following three forms:*

1. *the above Liouville form (8.11), here "+" are substituted by "−",*
2. *$ds^2 = \sigma\,(du^2 - dv^2)$, where σ is a harmonic function,*
3. *$ds^2 = H^2\,du^2 - dv^2$, where H is a function satisfying the differential equation*
$$\frac{\partial H}{\partial u} - H\,\frac{\partial H}{\partial v} = 0.$$

8.2.3 Levi-Civita metrics

We recall some of the classical results of Levi-Civita [107] (without proofs):

Theorem 8.3 *Assume proper Riemannian metrics g and $\overline{g}$ that are strictly nonproportional at the point p. Then in a neighborhood of the point, there exists a coordinate system $(x^1, x^2, \ldots, x^n)$ such that the metric forms ds^2 and $d\overline{s}^2$ are as follows:*

$$ds^2 = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} |X_i(x^i) - X_j(x^j)| \;\cdot\; {dx^i}^2\,, \tag{8.12}$$

$$d\overline{s}^2 = \frac{1}{\prod_{\alpha=1}^{n} X_\alpha(x^\alpha)} \cdot \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} |X_i(x^i) - X_j(x^j)| \;\cdot\; \frac{{dx^i}^2}{X_i(x^i)}\,, \tag{8.13}$$

where $X_i(x^i) > 0$, $i = 1, 2, \ldots, n$, are differentiable functions of the parameters x^i.

The above considered metrics are called *Levi-Civita metrics*.

If we take away in (8.12) and (8.13) the symbol of modulus then a geodesic mapping exists between spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$, whereas ds^2 and $d\overline{s}^2$ may be indefinite and their signatures may be different.

Let all functions $X_i(x_i) \in C^1$ be defined on a line $\mathbb{R}^1$ or a circle $\mathbb{S}^1$ (in this case the function $X_i(x_i)$ is periodic) and $M = \mathbb{R}^1(\mathbb{S}^1) \times \mathbb{R}^1(\mathbb{S}^1) \times \cdots \times \mathbb{R}^1(\mathbb{S}^1)$. If it holds $X_i(x^i) - X_j(x^j) \neq 0$ for all $x^i, x^j \in \mathbb{R}$ and for all $i, j = 1, 2, \ldots, n$, $i \neq j$, then a geodesic mapping $\mathbb{V}_n = (M, g)$ onto $\overline{\mathbb{V}}_n = (M, \overline{g})$ exists globally. For instance, this is the case if $0 < |X_1(x_1)| < |X_2(x_2)| < \cdots < |X_n(x_n)|$.

Therefore we can costruct interesting topological Levi-Civita metrics.

Hence the compact space $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ admit a non-trivial geodesic mapping.

For pseudo-Riemannian spaces, the problem of obtaining metrics which admit geodesic mappings was solved by A.Z. Petrov [90] [139] for $\mathbb{V}_3$, V.I. Golikov (see [139]) for Lorentzian spaces $\mathbb{V}_4$, G.I. Kruchkovich [561] for Lorentzian spaces $\mathbb{V}_n$ and A.V. Aminova for arbitrary $\mathbb{V}_n$. A detailed description is given in the review [9].

8.3 Geodesic mappings and equidistant spaces

In this part we study geodesic mappings of equidistant spaces in a canonical coordinate system, see [465].

Consider a *special mapping* f between equidistant spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$, which have metrics of the form (3.123):

$$\mathrm{d}s^2 = a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2, \tag{8.14}$$

$a, b \in C^1$ are non-zero functions, and $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ is a metric form of a certain Riemannian space $\tilde{\mathbb{V}}_{n-1}$, the equidistant space $\overline{\mathbb{V}}_n$ has the analogous metric

$$\mathrm{d}\bar{s}^2 = A(x^1)\,(\mathrm{d}x^1)^2 + B(x^1)\,\mathrm{d}\hat{s}^2, \tag{8.15}$$

where $A, B \in C^1$ are non-zero functions, and $\mathrm{d}\hat{s}^2 = \hat{g}_{ab}(x^2, \dots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ is the metric form of a certain Riemannian space $\hat{\mathbb{V}}_{n-1}$. Here and in what follows, the indices $a, b, c, \dots$ take the values from 2 to n.

Under this map, the geodesic lines $\ell = \{(t, \overset{\circ}{x}{}^2, \dots, \overset{\circ}{x}{}^n),\ t \in \mathbb{R}$ of $\mathbb{V}_n$ are mapped into the geodesics of $\overline{\mathbb{V}}_n$ and the orthogonal surfaces on this geodesic congruence of $\mathbb{V}_n$ are also mapped into the orthogonal surfaces on the corresponding geodesic congruence of $\overline{\mathbb{V}}_n$.

The deformation tensor $P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ of the mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ has in this case the form

$$\begin{gathered} P^1_{11} = \frac{1}{2}\left(\frac{A'}{A} - \frac{a'}{a}\right); \quad P^1_{1a} = P^1_{a1} = P^c_{11} = 0; \quad P^c_{ab} = \hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab}; \\ P^c_{1a} = P^c_{a1} = \frac{1}{2}\left(\frac{B'}{B} - \frac{b'}{b}\right)\delta^c_a; \quad P^1_{ab} = -\frac{1}{2}\left(\frac{B'}{A}\hat{g}_{ab} - \frac{b'}{a}\tilde{g}_{ab}\right), \end{gathered} \tag{8.16}$$

where $\tilde{\Gamma}^c_{ab}$ and $\hat{\Gamma}^c_{ab}$ are the Christoffel symbols of $\tilde{\mathbb{V}}_{n-1}$ and $\hat{\mathbb{V}}_{n-1}$.

Rewriting the necessary and sufficient condition (8.1) of the geodesic mappings of $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ in terms of the deformation tensor in the form $P^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i$ we obtain

[90] Alexey Zinovievich Petrov (1910-1972), an outstanding Soviet scientist in the field of general relativity and gravitation, the founder of the first and only one in the USSR Department of Relativity and Gravitation of Kazan State University (1960).

$$
\begin{aligned}
P^1_{11} &= \frac{1}{2}\left(\frac{A'}{A} - \frac{a'}{a}\right) && = \psi_1\delta^1_1 + \psi_1\delta^1_1 = 2\psi_1; && \Rightarrow \frac{A'}{A} - \frac{a'}{a} = 4\psi_1\\
P^c_{1a} &= \frac{1}{2}\left(\frac{B'}{B} - \frac{b'}{b}\right)\delta^c_a && = \psi_a\delta^c_1 + \psi_1\delta^c_a = \psi_1\delta^c_a; && \Rightarrow \frac{B'}{B} - \frac{b'}{b} = 2\psi_1\\
P^1_{ab} &= -\frac{1}{2}\left(\frac{B'}{A}\hat{g}_{ab} - \frac{b'}{a}\tilde{g}_{ab}\right) && = \psi_a\delta^1_b + \psi_b\delta^1_a = 0; && \Rightarrow \frac{B'}{A} - \frac{b'}{a} = 0\\
P^c_{ab} &= \hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab} && = \psi_a\delta^c_b + \psi_b\delta^c_a = 0; && \Rightarrow \psi_a = 0.
\end{aligned}
$$

By analysis of these equations we obtain the following theorem.

Theorem 8.4 *The special mapping f between equidistant spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ is non-trivially geodesic if and only if $\hat{\mathbb{V}}_{n-1}$ is homothetic to $\tilde{\mathbb{V}}_{n-1}$, and the metric of $\overline{\mathbb{V}}_n$ reads*

$$\mathrm{d}\bar{s}^2 = \frac{p\,a(x^1)}{(1+q\,b(x^1))^2}\left(\mathrm{d}x^1\right)^2 + \frac{p\,b(x^1)}{1+q\,b(x^1)}\,\mathrm{d}\tilde{s}^2,$$

where p, q are some constants such that $p \neq 0$, $1+q\,b(x^1) \neq 0$, and $q\,b'(x^1) \not\equiv 0$. From this it follows

$$\Psi = -\frac{1}{2}\,\ln\left|1+q\,b(x^1)\right|.$$

Let us note that this theorem is valid provided $a(x^1), b(x^1) \in C^1$. It means that if we choose e.g. $a(x^1), b(x^1) \in C^1$ and $a(x^1), b(x^1) \notin C^2$, the Levi-Civita formulas (8.1), (8.2) hold, and moreover the Mikeš-Berezovsky (7.12), and the Sinyukov's formula (8.6) are satisfied. Under these conditions, neither the Riemannian, nor the Ricci, nor the Weyl projective curvature tensors can be constructed, nor the systems of equations (7.5) or (7.16). The Theorems 7.2 and 7.6 cannot hold.

First time, geodesic mappings between equidistant spaces studied Sinyukov [813] in case of for the metric (3.124). For the metrics of type (3.122), the calculation in [170, pp. 97-98], is incorrect, among others the formula (69) is wrong. These subject was also treated by Gorbatyi [440], Deszcz [376] and Venzi [920].

An equidistant space $\mathbb{V}_n$ with the metric (3.124) admits geodesic mappings onto the Riemannian space $\overline{\mathbb{V}}_n$, whose metric form is

$$d\bar{s}^2 = \frac{p}{f\cdot(1+qf)^2}dx^{1\,2} + \frac{p\,f}{1+qf}d\tilde{s}^2, \tag{8.17}$$

where p, q are some constants such that $1+qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is trivial; here x are coordinates common for $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ [813].

The next part deals with manifolds $\mathbb{V}_n(B)$. We claim that equidistant spaces $\mathbb{V}_n$ are spaces $\mathbb{V}_n(B)$. Moreover, a space $\mathbb{V}_n$ with the Brinkmann metric (3.124) is $\mathbb{V}_n(\mathrm{B})$, $B = \mathrm{const}$, if and only if $f = Bx^{1\,2} + a\,x^1 + b$, where B, a, b are constants.

8.4 Manifods $\mathbb{V}_n(B)$

8.4.1 Geodesic mappings of $\mathbb{V}_n(B)$ spaces

It will be mentioned later that manifolds admitting geodesic mappings play an important role in the theory of geodesic mappings. Let us deal with the following definition (Mikeš, see [117, 118, 226, 227, 542]), which anables us to make our formulations more precise.

Definition 8.1 A Riemannian manifold $\mathbb{V}_n$ will be called a $\mathbb{V}_n(B)$ manifold if it admits a nonhomothetic geodesic mapping with

$$\begin{aligned} &\text{(a)}\ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \\ &\text{(b)}\ \lambda_{i,j} = \mu\, g_{ij} + B\, a_{ij}, \end{aligned} \tag{8.18}$$

where μ and B are some functions.

Note that if a manifold $\mathbb{V}_n(B)$ admits any geodesic mapping then the corresponding conditions (8.18) are satisfied, just with the same B, see Lemma 9.7, p. 331.

Formulas (8.18b) are equivalent to

$$\psi_{ij} = \overline{B}\,\overline{g}_{ij} - B\, g_{ij}, \tag{8.19}$$

where $\overline{B}$ is a function; the proof of this fact follows by covariant differentiation of the formula (8.18b), see [117, 118, 226, 227].

These conditions are especially fulfilled for geodesic mappings of manifolds of constant curvature (Beltrami [292, 293], Eisenhart [50]; see p. 318), Einstein manifolds (Mikeš [118, 121, 627], Hinterleitner, Mikeš [482, 483]; see p. 320) and manifolds $V(K)$ (these were introduced by Solodovnikov [843, 845]).

Manifolds $\mathbb{V}_n(B), B = \text{const}$, with positive definite metric are $V(K)$, $K = -B$. The theory of these spaces was treated by A.S. Solodovnikov [177, 178, 843–846] and G.I. Kruchkovich [98, 561], see [179].

Note. Solodovnikov has proved the equivalence of (8.18b) and (8.19) in a special coordinate system of spaces $V(K)$. These formulas was proved by Mikeš in [226] for conditions $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ belong to C^2, which is presented in [539] without any reference. By Lemma 3.4 we elementary obtain that $B \in C^1$ if only if $\overline{B} \in C^1$ in the case $\mathbb{V}_n, \overline{\mathbb{V}}_n \in C^2$, moreover in this case $\Psi \in C^3, \psi_i \in C^2$.

Clearly, the following lemma follows from the "symmetry" of the formulae (8.19).

Lemma 8.1 *If a manifold $\mathbb{V}_n(B)$ admits a nontrivial geodesic mapping onto $\overline{\mathbb{V}}_n$, then $\overline{\mathbb{V}}_n$ is a $\overline{\mathbb{V}}_n(\overline{B})$ manifold where $\overline{B}$ is some function, i.e. the class of spaces $\mathbb{V}_n(B)$ is closed with respect to geodesic mappings; moreover*

$$B = const \ \Leftrightarrow\ \overline{B} = const.$$

The last fact for $\mathbb{V}_n, \overline{\mathbb{V}}_n \in C^3$ follows from the results of Gorbatyi [440] that $\mathbb{V}_n(B)$, $B \neq \text{const}$, admits nontrivial geodesic mapping only onto $\overline{\mathbb{V}}_n(\overline{B})$,

$\overline{B} \neq$ const , and in these manifolds there exists a concircular field which is not special. Absolutely analogically this follows when $\mathbb{V}_n, \overline{\mathbb{V}}_n \in C^2$ and $B \in C^1$.

Manifolds $\mathbb{V}_n$ admitting nontrivial geodesic mappings and having concircular fields ξ are $\mathbb{V}_n(B)$ manifolds. In this case, if ξ is convergent, then $B = 0$; if ξ is a special concircular field, then $B =$ const , and $B \neq$ const in the remaining cases. On the other hand, $\mathbb{V}_n(B)$, $B \neq$ const and $\mathbb{V}_n(0)$ always admit a concircular vector field.

Under geodesic mappings of $\mathbb{V}_n(B)$ onto $\overline{\mathbb{V}}_n(\overline{B})$, the tensors Z^h_{ijk} and Z_{ij} are invariant:

$$\overline{Z}^h_{ijk} = Z^h_{ijk} \quad \text{and} \quad \overline{Z}_{ij} = Z_{ij}, \tag{8.20}$$

where

$$\begin{aligned} &(a)\ Z^h_{ijk} = R^h_{ijk} - B\,(\delta^h_k g_{ij} - \delta^h_j g_{ik}),\\ &(b)\ \ Z_{ij} = Z^\alpha_{ij\alpha} \equiv R_{ij} - B(n-1)g_{ij}. \end{aligned} \tag{8.21}$$

The above immediately follows from (6.26), (6.27) and (8.19).

The integrability conditions of (8.18a) can be written in the form

$$a_{i\alpha} Z^\alpha_{jkl} + a_{j\alpha} Z^\alpha_{ikl} = 0. \tag{8.22}$$

1) The basic equations of the manifolds $\mathbb{V}_n(B)$, $B =$ const , admitting geodesic mappings have the following form:

$$(a)\ a_{ij,k} = \lambda_i\, g_{jk} + \lambda_j\, g_{ik};\ (b)\ \lambda_{i,j} = \mu g_{ij} + B\, a_{ij};\ (c)\ \mu_{,i} = 2B\,\lambda_i. \tag{8.23}$$

It follows from the analysis of the integrability conditions (8.18). The conditions of integrability of equations (8.23b) take the following form:

$$\lambda_\alpha Z^\alpha_{ijk} = 0. \tag{8.24}$$

2) For manifolds $\mathbb{V}_n(0)$, equations (8.23) may be simplified to the form

$$(a)\quad a_{ij,k} = \lambda_i\, g_{jk} + \lambda_j\, g_{ik}; \qquad (b)\quad \lambda_{i,j} = \mu g_{ij}, \tag{8.25}$$

where $\mu =$ const .

Integrability conditions and their differential prolongations may be written as: (7.47) and (7.48). The conditions may be considered as a system of linear algebraic equations in the variables a_{ij}, λ_i and μ.

It follows from (8.25b) that in $\mathbb{V}_n(0)$ there exist convergent vector fields. As we have shown above, Riemannian manifolds $\mathbb{V}_n(0)$ may be considered as Solodovnikov manifolds. Consequently, Theorem 8.6 implies that manifolds $\mathbb{V}_n(0)$ admitting nontrivial geodesic mappings are equidistant manifolds of basic type [630, 631].

3) For manifolds $\mathbb{V}_n(B)$, $B \neq$ const , equations (8.23) may be simplified to the form [440]:

$$\text{(a)}\quad a_{ij} = \alpha\, g_{ij} + \beta\, \lambda_i\lambda_j, \qquad \text{(b)}\quad \lambda_{i,j} = \gamma\, g_{ij} + \delta\, \lambda_i\lambda_j, \tag{8.26}$$

where $\alpha, \beta, \gamma, \delta$ are functions of Λ, and $\lambda_i = \partial_i \Lambda$.

8.4.2 Properties of the spaces $\mathbb{V}_n(B)$

Theorem 8.5 *Any geodesic mapping of a Riemannian space $\mathbb{V}_n(B)$, $B \neq 0$, is either nontrivial or homothetic.*

Proof. Let us assume that $\mathbb{V}_n(B)$, $B \neq 0$, admits an affine (i.e. a trivial geodesic) mapping. Then, for $\lambda_i = 0$, there exists a solution of (8.18). From (8.18(b)) it follows that $\mu g_{ij} + B\, a_{ij} = 0$. If $B \neq 0$, we get that $a_{ij} = -\frac{\mu}{B}\, g_{ij}$. Hence, by ([170, p. 121]), it follows that any geodesic map is homothetic. □

Theorem 8.6 *If in a Riemannian space $\mathbb{V}_n(B)$ there exists a nonzero isotropic vector among the vectors λ_i satisfying (8.18), then $B = 0$.*

Proof. Assume that in $\mathbb{V}_n(B)$, $B \neq 0$, the tensor a_{ij}, the covector λ_i ($\neq 0$), and the invariant μ give a solution to the equation system of a geodesic map, which has the form (8.18) in $\mathbb{V}_n(B)$. Let us assume that λ_i is an isotropic vector, i.e.,

$$\lambda_\alpha \lambda_\beta g^{\alpha\beta} = 0. \tag{8.27}$$

a) First let us consider the case $B \neq$ const. From [440] it follows that the tensor a_{ij} and the vector λ_i satisfy (8.26). Differentiating (8.27) and substituting (8.26b), we get that $\gamma = 0$. Then, differentiating (8.26a), after substitution (8.18) and (8.26) we have $\lambda_i g_{jk} + \lambda_j g_{ik} = \lambda_k(\alpha' g_{ij} + (\beta' + 2\delta)\lambda_i\lambda_j)$. Hence it follows that $\lambda_i = 0$. Otherwise $\operatorname{rank}(g_{ij}) \geq 2$. Thus the case a) is proved.

b) It remains to consider the case $B =$ const $\neq 0$. Then (8.18) holds true, and μ satisfies

$$\mu_{,i} = 2B\lambda_i. \tag{8.28}$$

We differentiate (8.27). Then we substitute (8.18b) and get

$$\mu\lambda_i + B\, a_{i\alpha}\lambda^\alpha = 0. \tag{8.29}$$

Then we differentiate (8.29), and, taking into account (8.18), (8.27) and (8.28), we have

$$3B\lambda_i\lambda_j + \mu^2 g_{ij} + 2\mu B a_{ij} + B^2 a_{i\alpha}a^\alpha_j = 0. \tag{8.30}$$

Let us differentiate (8.30) with respect to x^k, then, by (8.18), (8.28), we have

$$4\lambda_k(B\mu\, g_{ij} + B^2 a_{ij}) + \lambda_i c_{jk} + \lambda_j c_{ik} = 0, \tag{8.31}$$

where c_{ij} is a symmetric tensor. Since $\lambda_i \neq 0$, there exists ε^i such that $\varepsilon^i\lambda_i = 1$. Contracting (8.31) with $\varepsilon^i\varepsilon^j$, we obtain that $c_{\alpha k}\varepsilon^\alpha = c\,\lambda_k$, where c is an invariant. Then, contracting (8.31) with ε^i, we get $c_{jk} = c_j\lambda_k$, where c_j is a vector. The tensor c_{jk} is symmetric, therefore we have $c_{jk} = \alpha\,\lambda_j\lambda_k$, where α is an invariant. Then, from (8.31) it follows

$$\mu\, B g_{ij} + B^2 a_{ij} + \beta\,\lambda_i\lambda_j = 0, \tag{8.32}$$

where β is an invariant.

By differentiating (8.32) with respect to x^k we get

$$2B^2\lambda_k g_{ij} + \lambda_i d_{jk} + \lambda_j d_{ik} = 0,$$

where d_{ij} is a tensor.

From the last equality, in case $B \neq 0$, it follows that $\operatorname{rank}(g_{ij}) \geq 2$. Hence $B = 0$. □

In a similar way one can prove the following theorem.

Theorem 8.7 *Assume that in a space $\mathbb{V}_n(B)$ among the nonzero vectors λ_i which satisfy (8.18), there are mutually orthogonal vectors. Then $B = 0$.*

The contraposition of Theorem 8.6 is the following Lemma:

Lemma 8.2 *In a space $\mathbb{V}_n(B)$, $B \neq 0$, the vector λ_i is non isotropic.*

Lemma 8.3 *In a space $\mathbb{V}_n(B)$, $B \neq 0$, the vector ψ_i is non isotropic.*

Proof. Assuming that the vector ψ is isotropic (null), we have $\psi_i\psi_i g^{ij} = 0$, where ψ^i are components of the vector Ψ. By covariant differentiation, in virtue of (8.19), we obtain $B\psi_i = \overline{B}\psi^\alpha \overline{g}_{\alpha i}$ whence $\overline{B}\psi^i = B\psi_\alpha \overline{g}^{\alpha i}$. Hence, taking into account (7.13) and (7.14), we infer that λ_i and ψ_i are linearly dependent and the thesis follows from Lemma 8.2. □

8.4.3 Projective transformations and manifolds $\mathbb{V}_n(B)$

It is known that in Riemannian manifolds $\mathbb{V}_n$ the following holds.

Theorem 8.8 (L.P. Eisenhart [50]) *An infinitesimal operator $X = \xi^\alpha(x)\partial_\alpha$ determines a one-parameter Lie group of projective transformations (and infinitesimal projective transformation) of a Riemannian manifold $\mathbb{V}_n$ if and only if it satisfies the following condition (generalized Killing equation)*

$$h_{ij,k} = 2\, g_{ij}\psi_k + g_{ik}\psi_j + g_{jk}\psi_i \quad \textit{and} \quad h_{ij} = \mathcal{L}_\xi g_{ij} \equiv \xi_{(i,j)} \tag{8.33}$$

where $\mathcal{L}_\xi$ is the Lie derivative in the direction ξ, $\xi_i = g_{i\alpha}\xi^\alpha$ and ψ_i is a covector.

Proof. Let us compare formulas (4.90) and (6.52). It follows from Theorem 4.11, p. 200, that the formula (8.8) is true. □

Contracting (8.33) with g^{ij} we obtain, that $\psi_k = \frac{1}{n+1}\left(\xi_{\alpha,\beta}g^{\alpha\beta}\right)_{,k} = \psi_{,k}$. Evidently if $\psi_i = 0$ then X determines an affine transformation. (or trivial projective transformation).

Putting

$$a_{ij} = \xi_{(i,j)} - 2\,\psi\, g_{ij} \tag{8.34}$$

we obtain the equations

$$a_{ij,k} = g_{ij}\psi_k + g_{ik}\psi_j\,. \tag{8.35}$$

This equation is the Sinyukov equation (8.6) of geodesic mappings.

Clearly, (8.34) and (8.35) are necessary and sufficient conditions for the existence of projective transformations.

Analysing (8.33) and (8.34) we have that a nontrivial projective transformation generates a nontrivial geodesic mapping. The converse of this proposition is not true.

Hereafter we shall presume that the structure of the tensor P admits its vanishing (see Theorem 4.13, p. 200). In other words $a_{ij} = \text{const}\ g_{ij}$ is a partial solution of equation (8.33). In this case, we can prove the following statement [633, 635].

Theorem 8.9 *There is the inequality*

$$r_{pt} \le r_{gm} + r_{hom} - 1 \tag{8.36}$$

where r_{pt} and r_{hom} are orders of projective and homothetic groups and r_{gm} is the degree of geodesic mapping on manifold $\mathbb{V}_n$.

For manifolds $\mathbb{V}_n(B)$ the following theorem is true.

Theorem 8.10 *Suppose a manifold $\mathbb{V}_n(B)$, $B = \text{const}$, admitting projective transformations with projective vector λ_i. If $B \neq 0$, it is non trivial, and $r_{pt} = r_{gm} + r_{hom} - 1$. If $B = 0$, it is affine.*

Proof. On a manifold $\mathbb{V}_n(B)$ the equations (8.23) are true. Differentiating (8.23b) we get

$$\lambda_{i,jk} = 2B\, g_{ij}\lambda_k + B\, g_{jk}\lambda_i + B\, g_{jk}\lambda_j. \tag{8.37}$$

Because λ_i is gradient-like we may write:

$$\lambda_{(i,j)k} = 4B\, g_{ij}\lambda_k + 2B\, g_{jk}\lambda_i + 2B\, g_{jk}\lambda_j.$$

Comparing it with (8.33) we have that λ_i is a projective vector, which generates a projective transformation. This transformation is nontrivial for $B \neq 0$ and it is affine for $B = 0$. □

Finally we consider the results found above, the $\mathbb{V}_n$ which admit a group of motions (or homothetic motions) of high order and at the same time have high degree of mobility with respect to geodesic maps, have a projective group of high order.

As is known [139, 170], for flat $\mathbb{V}_n$: $r_{pt} = n(n+2)$, $r_{hom} = n(n+1)/2 + 1$, $r_{gm} = (n+1)(n+2)/2$, so $r_{hom} + r_{gm} = r_{pt} + 2$; for nonflat manifolds of constant curvature $\mathbb{V}_n$: $r_{pt} = n(n+2)$, $r_{hom} = n(n+1)/2$, $r_{gm} = (n+1)(n+2)/2$. In this case the right side of (8.36) becomes a strict equality.

For spaces $\mathbb{V}_n$ of nonconstant curvature, the degree of mobility with respect to geodesic maps is bounded by the following inequality [118, 220, 221, 542]: $r_{gm} \le N^2_{gm} = n(n-3)/2 + 2$. It is proved that N^2_{gm} is achieved only in $\mathbb{V}_n$ which admit exactly $n-2$ concircular vector fields (i.e., of vectors λ_i, satisfying the conditions $\lambda_{i,j} = \varrho\, g_{ij}$, where ϱ is a function). In these spaces, as is easily seen, $r_{pt} \ge n(n-4) + 4$. In more special $\mathbb{V}_n$, admitting exactly $n-2$ absolutely parallel vector fields, the order $r_{pt} \ge n(n-3) + 2$.

As is known [45, 46], the subprojective spaces of Kagan are spaces of the second lacunarity with respect to motions and homothety transformations.

These spaces are conformally Euclidean ones for which, as Kiosak [220] proved, $r_{gm} \leq 2$; this follows from Moldobaev's result [228]. Hence, conformally Euclidean, and in particular subprojective spaces admit a complete group of projective transformations of relatively low order. Hence, one should seek spaces of the second lacunarity with respect to the group of projective transformations among spaces of the third lacunarity with respect to the group of motions. For $\mathbb{V}_n$, different from conformally Euclidean ones, Egorov [45] proved that $r_{mot} \leq N^3_{mot} = n(n-3)/2 + 6$ and that the order N^3_{mot} and is achieved by special $\mathbb{V}_n$ and only them; the metric of these $\mathbb{V}_n$ in some system of coordinates x has the structure (5.54), p. 256.

Considering the estimates cited above and (8.36), we get for $\mathbb{V}_n$ of nonconstant curvature $r_{pt} \leq n(n-3)+8$. It is easy to see that $\mathbb{V}_n$ with the metric (5.54) admit groups of projective transformations of order $r_{pt} = n(n-3)+8$. Obviously, the converse holds, i.e. these $\mathbb{V}_n$ and only these admit groups of projective transformations of the order indicated above.

Thus, one has [633]:

Theorem 8.11 *There do not exist Riemannian spaces $\mathbb{V}_n$ admitting complete groups of projective transformations of orders $n(n-3)+8 < r_{pt} < n(n+2)$. Riemannian spaces with metric (5.54) admit groups of projective transformations of order $n(n-3)+8$, and they are the only ones which do.*

In [170] the notion of the mobility degree r_{gm} with respect to geodesic maps was introduced, see Section 6.1.3, p. 277. The following can be

Theorem 8.12 (Kiosak, Mikeš [118, 541]) *Each Riemannian space $\mathbb{V}_n$, whose mobility degree with respect to geodesic maps is greater than two, is a space $\mathbb{V}_n(B)$ where B is constant.*

8.4.4 Geodesically complete manifolds $\mathbb{V}_n(B)$

The assumption that the metric is complete is very important. Conformal, projective and holomorphically projective mappings and transformations have been studied by many authors, see [539, 568, 722, 889].

The following theorem follows from results of M. Obata [722] and S. Tanno [889] on the solvability of equations (8.37), which simultaneously characterize the manifolds $\mathbb{V}_n(B)$, (see Ph.D. thesis Kiosak [220]):

Theorem 8.13 *A complete manifold $\mathbb{V}_n(B)$, $B = \text{const} < 0$, with positive definite metric, is isometric to the standard sphere $\mathbb{S}^n$.*

The paper [636] (see [121]) contains an explicit construction of the projective transformation of standard sphere $\mathbb{S}_n$.

The gnomonic projection of a half sphere onto the Euclidean space $\mathbb{E}_n$ is an example of a global geodesic mapping from a non complete manifold onto a complete manifold.

Inspired by [539] we prove the more general theorem [122]:

Theorem 8.14 *Let the manifold $\mathbb{V}_n(B)$, $B = \text{const}$, admit a geodesic mapping f onto the complete manifold $\overline{\mathbb{V}}_n \in C^1$.*

1. *If $\mathbb{V}_n(B)$ is pseudo-Riemannian then f is affine.*
2. *If $B \geq 0$ then f is affine.*

Proof. Let us suppose that $f\colon\ \mathbb{V}_n(B) \to \overline{\mathbb{V}}_n$ is a geodesic mapping and the Levi-Civita equation (6.8) holds: $\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i\psi_j + \delta^h_j\psi_i$ with $\psi_i = \partial_i\Psi$. Let γ be a geodesic on $\mathbb{V}_n(B)$ and $\overline{\mathbb{V}}_n = \overline{\mathbb{V}}_n(\overline{B})$, $\overline{B} = \text{const}$, with natural parameter t and τ, respectively. Assume $\dot\tau = d\tau(t)/dt > 0$ for the parameter transformation $\tau = \tau(t)$, τ is a surjection onto $\mathbb{R}$. Then the following holds:

$$\psi_\alpha\dot\gamma^\alpha = \frac{1}{2}\,\frac{d}{dt}\,(\ln\dot\tau) \quad \text{where} \quad \dot\gamma^h = \frac{d}{dt}\,\gamma. \tag{8.38}$$

Then along the geodesic γ, $\Psi(t) = \Psi(\gamma(t))$ may be written in the form

$$\Psi(t) = \frac{1}{2}\,\ln(\dot\tau(t)) + c_o, \quad c_o = \text{const}\,. \tag{8.39}$$

Because g and $\mathrm{e}^{-4\Psi}\,\overline{g}$ are first integrals of geodesics (see (10.31)) then the following holds

$$g_{ij}\dot\gamma^i\dot\gamma^j = \varepsilon = \pm 1, 0 \quad \text{and} \quad \overline{g}_{ij}\dot\gamma^i\dot\gamma^j = c\,\mathrm{e}^{4\Psi(t)}, \quad c = \text{const}\,. \tag{8.40}$$

On the manifold $\mathbb{V}_n(B)$ the equation (8.19) is satisfied, i.e. in expanded form, it may be expressed by

$$\psi_{i,j} = \psi_i\psi_j + \overline{B}\,\overline{g}_{ij} - B\,g_{ij}\,. \tag{8.41}$$

For a geodesic γ, after contraction with $\dot\gamma^i\dot\gamma^j$, we get

$$\ddot\Psi = (\dot\Psi)^2 + b\,\mathrm{e}^{4\,\Psi} - a, \tag{8.42}$$

where $a = \varepsilon\,B$ and $b = c\,\overline{B}$. From the equation (8.42) follows:

$$2q\,\ddot q = \dot q^2 - 4\,b + 4\,a\,q^2 \quad \text{where} \quad q = 1/\dot\tau = \mathrm{e}^{-2\Psi(t)}. \tag{8.43}$$

This equation has the general solution:

$$q(t) = \begin{cases} c_1 + c_2\,(t + c_3)^2, & \text{for } a = 0; \\ c_1 + c_2\,\exp(2\sqrt{a}\cdot t) \;+\; c_3\,\exp(-2\sqrt{a}\cdot t), & \text{for } a > 0; \\ c_1 + c_2\,\sin(2\sqrt{-a}\cdot t) + c_3\,\cos(2\sqrt{-a}\cdot t), & \text{for } a < 0, \end{cases} \tag{8.44}$$

where c_1, c_2, c_3 are constants (which are mutually dependent).

For $a \geq 0$ it is clear to see that the non linear function $\tau(t) = \int_{t_o}^t 1/q(s)\,ds$ is not surjective onto $\mathbb{R}$. Hence Ψ is constant, for all geodesics with $a \geq 0$.

Further if $\Psi = \text{const}$ on an interval of a geodesic, then the solution (8.44) implies that $\Psi = \text{const}$ on such geodesic as a whole.

1. Assume a pseudo-Riemannian manifold $\mathbb{V}_n(B)$, $B = \text{const}$, and a point x_o at $\mathbb{V}_n(B)$. We construct some geodesics $\gamma \ni x_o$ with $a \equiv \varepsilon B \geq 0$. In a "conical domain" containing such geodesics, $\Psi = \text{const} = \Psi(x_o)$ everywhere.

Therefore, we have $\Psi = \text{const} = \Psi(x_o)$ along all geodesics passing through this domain. As a consequence, $\Psi = \text{const}$ on the entire manifold, since any two points can be connected by sequences of the geodesics constructed above.

2. If $\mathbb{V}_n(B)$, $B = \text{const} \geq 0$, is properly Riemanian, γ is a non isotropic geodesic with arc length parameter t (i.e. $(g(\dot\gamma, \dot\gamma) = \varepsilon = 1)$, we get $\varepsilon B \geq 0$. Therefore $\Psi(t) = \text{const}$ along arbitrary geodesics. As a consequence, $\Psi = \text{const}$ on the entire manifold, since any two points can be connected by sequences of geodesics. □

Evidently, from Theorems 8.13 and 8.14 follows:

Theorem 8.15 *Let a complete manifold $\mathbb{V}_n(B)$, $B = \text{const}$, admit a non-affine geodesic mapping f onto the complete manifold $\overline{\mathbb{V}}_n$. Then this manifold is isometric to the standard sphere $\mathbb{S}^n$.*

Because a projective transformation of $\mathbb{V}_n(B)$ is a geodesic mapping from $\mathbb{V}_n(B)$ onto itself, we have:

Theorem 8.16 *Let a complete manifold $\mathbb{V}_n(B)$, $B = \text{const}$, admit a non-affine projective transformation. Then these manifolds are isometric to the standard sphere $\mathbb{S}^n$.*

From the above Theorems follows that a compact manifolds $\mathbb{V}_n(B)$, $B = \text{const}$, admitting nontrivial geodesic mappings is isometric to the standard sphere $\mathbb{S}^n$.

The above mentioned Theorems 8.13, 8.14, 8.15 and 8.16 apply to geodesic mappings of many special complete manifolds, e.g. Einstein, Kähler, semisymmetric, Ricci semisymmetric, pseudosymmetric, Ricci pseudosymmetric, etc., see following Chapter, p. 338

8.5 Geodesics mapping and its field of symmetric linear endomorphisms

It is well known that a geodesic mapping $f\colon (M,g)\to(\overline{M},\overline{g})$ is defined through the tensor field A_f of type (1,1). In the present section we give a geometric interpretation of eigenvalue function of the tensor field A_f and consider a global aspect of the geometry of the geodesic mapping $f\colon (M,g)\to(\overline{M},\overline{g})$ with respect to the trace of A_f.

E. Stepanova, J. Mikeš and I. Tsyganok [873] obtain the following results of geodesic mappings between Riemannian manifolds. In the first subsection we give a geometric interpretation of eigenvalues of the tensor field A_f. In the second subsection we consider a global aspect of the geometry of the geodesic mapping $f\colon (M,g)\to(\overline{M},\overline{g})$ with respect to the trace of A_f.

8.5.1 Geodesic mappings in terms of linear algebra

1. The necessary and sufficient condition for which an n-dimensional ($n\geq 2$) Riemannian manifold $\mathbb{V}_n=(M,g)$ to admit a geodesic mapping onto another n-dimensional Riemannian manifold $\overline{\mathbb{V}}_n=(\overline{M},\overline{g})$ has the following form of the differential equation (see in coordinate form (8.6), [170, pp. 120-121], [122, p. 167]):

$$(\nabla_Z a)(X,Y)=\lambda(X)g(Y,Z)+\lambda(Y)g(X,Z), \tag{8.45}$$

where ∇_Z denotes covariant derivative with respect to a smooth vector field Z on $\mathbb{V}_n$, a is a smooth covariant regular symmetric tensor field of order 2 on $\mathbb{V}_n$ and λ is a gradient-like form for which $\Lambda:=1/2\,\mathrm{grad}(\mathrm{trace}_g a)$. In addition, we note that if $\lambda=0$ then the geodesic mapping $f\colon \mathbb{V}_n\to\overline{\mathbb{V}}_n$ is affine.

Remark (see [170, pp. 120-121], [122, pp. 167-168]). Let $f\colon \mathbb{V}_n\to\overline{\mathbb{V}}_n$ be a geodesic mapping then in terms of local coordinate systems $x^1,\dots,x^n$ of $\mathbb{V}_n$ and $\overline{x}^1,\dots,\overline{x}^n$ of $\overline{\mathbb{V}}_n$ we can suppose that f is represented by the following equations $\overline{x}^1=x^1,\dots,\overline{x}^n=x^n$ for the corresponding points x and $\overline{x}=f(x)$. Also, let g_{ij} and $\overline{g}_{ij}$ be local components of the Riemannian metric tensors g and $\overline{g}$ for $i,j,k,l=1,\dots,n$. Denote by $\overline{g}^{ij}$ the elements of the inverse matrix $\|\overline{g}_{ij}\|^{-1}$, then the tensor a associated with the geodesic mapping $f\colon \mathbb{V}_n\to\overline{\mathbb{V}}_n$ is defined by $a_{ij}=e^{2\Psi}\overline{g}^{kl}g_{ik}g_{jl}$ for some smooth scalar function φ such that

$$\Psi=\frac{1}{2(n+1)}\ \ln|\det\overline{g}\,/\det g\,|\ .$$

Further we define the tensor field A_f, by setting $g(A_fX,Y)=a(X,Y)$ for arbitrary smooth tangent vector fields X and Y. In this case A_f is a self-adjoin section of End TM, i.e. $g(A_fX,Y)=g(A_fY,X)$. Moreover, with respect to the above definition the equation (8.45) can be rewritten as

$$(\nabla_Y A_f)X=g(X,Y)\xi+\lambda(X)Y, \tag{8.46}$$

for arbitrary smooth vector fields X,Y and $\lambda(X):=g(\xi,X)$.

2. Let $\varrho_1(x), \dots, \varrho_n(x)$ be the eigenvalues (some of which may coincide) of the symmetric endomorphism A_f of the tangent space T_xM at each $x \in M$. We can take an orthonormal basis $e_1, \dots, e_n$ of T_xM such that $A_f e_i = \varrho_i(x)$, $1 \geq i \geq n$. Next, for an arbitrary $\varrho_i(x)$ we denote corresponding eigenspace by $D_{\varrho_i(x)} \subset T_xM$. It is well known that the geometric multiplicity of an eigenvalue $\varrho_i(x)$ is the dimension of the eigenspace $D_{\varrho_i(x)}$ associated to $\varrho_i(x)$, i.e. number of linearly independent eigenvectors with the eigenvalue $\varrho_i(x)$.

We shall denote by M_f a connected component of the open dense subset of M, which consists of points at which the number of all distinct eigenvalues $\varrho_1(x), \dots, \varrho_r(x)$ of A_f is locally constant and the geometric multiplicities $n_1(x), \dots, n_r(x)$ of the eigenvalues $\varrho_1(x), \dots, \varrho_r(x)$ that are also locally constant. In this case, the eigenvalues of A_f form mutually distinct smooth eigenvalue functions as ϱ_α and, the assignment $x \in M_f \to D_{\varrho_\alpha(x)} \subset T_xM_f$ for all $x \in M_f$ defines a smooth eigenspace distribution D_{ϱ_α} of A_f. Then the following theorem is true.

Theorem 8.17 *Let $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ be a geodesic mapping with its field of linear symmetric endomorphisms A_f and ϱ_α is an eigenvalue function of A_f, defined in a connected component of $M_f \subset M$. If the geometric multiplicity n_α of ϱ_α is at least two, then the eigenspace distribution D_{ϱ_α} is integrable and each maximal integral manifold of D_{ϱ_α} is an umbilical submanifold of $\mathbb{V}_n$ and ϱ_α is constant.*

Proof. Let $\varrho_\alpha = \varrho_\alpha(x)$ be an arbitrary eigenvalue functions of A_f, defined in a connected component of $M_f \subset M$. If the geometric multiplicity n_α of ϱ_α is at least two, then $\dim D_{\varrho_\alpha} = n_\alpha > 1$ for eigenspace distribution D_{ϱ_α} which corresponds to ϱ_α. In this case we can define the second fundamental form Q_{ϱ_α} and the integrability tensor F_{ϱ_α} of D_{ϱ_α} by the identities [153, p. 148]

$$g(F_{\varrho_\alpha}(X_\alpha, Y_\alpha), W) = 1/2\ g((\nabla_{Y_\alpha} X_\alpha - \nabla_{X_\alpha} Y_\alpha), W),$$
$$g(Q_{\varrho_\alpha}(X_\alpha, Y_\alpha), W) = 1/2\ g((\nabla_{Y_\alpha} X_\alpha + \nabla_{X_\alpha} Y_\alpha), W),$$

for arbitrary $X_\alpha, Y_\alpha \in C^\infty D_{\varrho_\alpha}$ and $W \in C^\infty D^\perp_{\varrho_\alpha}$. Here $D^\perp_{\varrho_\alpha}$ is the orthogonal complement of $D_{\varrho_\alpha(x)}$ in T_xM_f, i.e. $T_xM_f = D_{\varrho_\alpha(x)} \oplus D^\perp_{\varrho_\alpha(x)}$ at each point $x \in M_f$.

Next, we find the form of the second fundamental form Q_{ϱ_α} and the integrability tensor F_{ϱ_α} of D_{ϱ_α} in our case.

First, from the identity $A_f Y_\alpha = \varrho_\alpha Y_\alpha$, by covariant differentiation alog D_{ϱ_α} we obtain

$$\varrho_\alpha \nabla_{X_\alpha} Y_\alpha + A_f(\nabla_{X_\alpha} Y_\alpha) = g(X_\alpha, Y_\alpha)\xi + \lambda(Y_\alpha) X_\alpha - X_\alpha(\varrho_\alpha) Y_\alpha \qquad (8.47)$$

for arbitrary $X_\alpha, Y_\alpha \in C^\infty D_{\varrho_\alpha}$.

Second, let ϱ_β be another arbitrary eigenvalue function of A_f defined in a connected component of $M_f \subset M$ and D_{ϱ_β} be the eigenspace distribution which corresponds to ϱ_β. Now if we multiply both sides of (8.47) by $W_\beta \in C^\infty D_{\varrho_\beta}$ we obtain the identity

$$(\varrho_\alpha - \varrho_\beta) g(\nabla_{X_\alpha} Y_\alpha, W_\beta) = g(X_\alpha, Y_\alpha)\lambda(W_\beta), \qquad (8.48)$$

where $g(A_f\nabla_{X_\alpha}Y_\alpha, W_\beta) = g(\nabla_{X_\alpha}Y_\alpha, A_fW_\beta) = \varrho_\beta g(\nabla_{X_\alpha}Y_\alpha, W_\beta)$. At the same time, from (8.48) it is implied that

$$g(Q_{\varrho_\alpha}(X_\alpha, Y_\alpha), W_\beta) = \frac{1}{\varrho_\alpha - \varrho_\beta}\ g(X_\alpha, Y_\alpha)\,\lambda(W_\beta),$$
$$g(F_{\varrho_\alpha}(X_\alpha, Y_\alpha), W_\beta) = 0.$$

In this case [153, p. 151] the distribution D_{ϱ_α} is integrable and each its maximal integral manifold N is umbilical.

For an arbitrary $X_\alpha \in C^\infty D_{\varrho_\alpha}$ we can find, locally, a non-zero $Y_\alpha \in C^\infty D_{\varrho_\alpha}$ such as $g(X_\alpha, Y_\alpha) = 0$ and $g(Y_\alpha, Y_\alpha) = 1$. In this case, applying (8.47) we obtain

$$\begin{aligned} X_\alpha(\varrho_\alpha) &= \varrho_\alpha\ g(\nabla_{X_\alpha}Y_\alpha, Y_\alpha) + g(A_f(\nabla_{X_\alpha}Y_\alpha), Y_\alpha) \\ &= 2\ \varrho_\alpha\ g(\nabla_{X_\alpha}Y_\alpha, Y_\alpha) = \varrho_\alpha\ X_\alpha g(Y_\alpha, Y_\alpha) = 0. \end{aligned}$$

From the above equation we obtain $X_\alpha(\varrho_\alpha) = 0$. This implies that ϱ_α is constant along D_{ϱ_α}. On the other hand, from the identity $A_fY_\alpha = \varrho_\alpha Y_\alpha$, by covariant differentiation along D_{ϱ_β} we obtain

$$\lambda(Y_\alpha)W_\beta + A_f(\nabla_{W_\beta}Y_\alpha) = W_\beta(\varrho_\alpha)Y_\alpha + \varrho_\alpha\nabla_{W_\beta}Y_\alpha \tag{8.49}$$

for an arbitrary $Y_\alpha \in C^\infty D_{\varrho_\alpha}$ and $W_\beta \in C^\infty D_{\varrho_\beta}$. Next, we multiply both sides of (8.49) by an arbitrary $X_\alpha \in C^\infty D_{\varrho_\alpha}$ and obtain the identity

$$\varrho_\alpha\ g(\nabla_{W_\alpha}Y_\alpha, X_\alpha) = W_\beta(\varrho_\alpha)\ g(X_\alpha, Y_\alpha) + \varrho_\alpha\ g(\nabla_{W_\alpha}Y_\alpha, X_\alpha), \tag{8.50}$$

where $g(A_f\nabla_{W_\beta}Y_\alpha, X_\alpha) = g(\nabla_{W_\beta}Y_\alpha, A_fX_\alpha) = \varrho_\alpha\ g(\nabla_{W_\beta}Y_\alpha, X_\alpha)$. From (8.50) we obtain $W_\beta(\varrho_\alpha) = 0$. This implies that ϱ_α is constant along D_{ϱ_β}. In conclusion, we can note that $Z(\varrho_\alpha) = 0$ for every smooth vector field on M_f which completes the proof.

Corollary 8.1 *Let $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ be a geodesic mapping with its field of linear symmetric endomorphisms A_f. If geometric multiplicity of every eigenvalue function of A_f, defined in a connected component of $M_f \subset M$, is at least two, then* $\operatorname{trace}_g A_f$ *is constant along M_f and the mapping f is affine.*

Proof. Let the linear symmetric endomorphisms A_f has r distinct eigen-values $\varrho_1(x), \ldots, \varrho_r(x)$ at each point of a connected component of $M_f \subset M$. If geometric multiplicity of every eigenvalue function ϱ_α for an arbitrary $\alpha = 1, \ldots, r$ is at least two, then according to the above theorem, ϱ_α is a constant. In this $\operatorname{trace}_g A_f = \sum_{\alpha=1}^{r} n_\alpha\varrho_\alpha$ is constant along M_f. This implies that the geodesic mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ is affine. □

Remark. The corollary is due to the classic paper of Levi-Civita [107] and was proved in [538] under the assumption that every Jordan-Block of A_f is at the most 3-dimensional at each point $x \in M$.

3. It is well known, that for an arbitrary symmetric tensor field a of order 2 on $\mathbb{V}_n$ there is an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xM at each point $x \in M$ such as $a(e_i, e_j) = \varrho_i(x)\,\delta_{ij}$ for the Kronecker delta δ_{ij}, i.e. this basis diagonalizes the tensor a. At the same time an orthonormal basis $\{e_1, \dots, e_n\}$ of T_xM at an arbitrary point $x \in M$ is said to diagonalize the curvature tensor R of $\mathbb{V}_n$ if $R(e_i, e_j)e_k = 0$ whenever i, j, k mutually distinct (see [599]). The following theorem is true.

Theorem 8.18 *Let $\mathbb{V}_n$ be an n-dimensional Riemannian manifold which admits a geodesic mapping f onto another n-dimensional Riemannian manifold $\overline{\mathbb{V}}_n$. If its field of linear symmetric endomorphisms A_f has n distinct eigenvalues at all points of a dense subset of $\mathbb{V}_n$, then, for any $x \in M$, the curvature tensor R_x is diagonalised by some orthonormal basis of the tangent space T_xM, diagonalizing A_f at $x \in M$. Moreover, in this case the Pontryagin forms of $\mathbb{V}_n$ and the real Pontryagin classes of $\mathbb{V}_n$ are all zero.*

Proof. Let $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ be a geodesic mapping with its field of linear endomorphisms A_f and let A_f has n distinct eigenvalues $\varrho_1(x), \dots, \varrho_n(x)$ at each point of a connected component of $M_f \subset M$.

Consider the tensor field $B = A_f + 1/2\ (\mathrm{trace}A_f)\ \mathrm{Id}_{TM}$. On the one hand, the tensor field B has n following distinct eigenvalues

$$\begin{gathered}\mu_1(x) = 1/2\ (2\varrho_1(x) + \varrho_2(x) + \dots + \varrho_n(x)),\\ \dots\\ \mu_n(x) = 1/2\ (\varrho_1(x) + \varrho_2(x) + \dots + 2\varrho_n(x)).\end{gathered}$$

On the other hand, it is easy to show that the tensor field B satisfies the *Codazzi equation* [444, 599]

$$(\nabla_X B)Y = (\nabla_Y B)X$$

for arbitrary $X, Y \in C^\infty TM$. Then the first proposition of our theorem is obvious from Proposition 4 of [368] together with a continuity argument.

Next, following Maillot [153] we call the curvature tensor R a *pure tensor* if it is diagonalized by some orthonormal basis of Tx_M at each point $x \in M$. In addition, we recall if R is pure, then all the *Pontryagin forms* of $\mathbb{V}_n$ and the *real Pontryagin classes* of $\mathbb{V}_n$ are zero [368, Corollary 3], which completes the proof of our theorem.

Remark. The theory of Pontryagin forms and the real Pontryagin classes of $\mathbb{V}_n$ can be found in the monograph [124].

4. We denote by μ_α and μ_β distinct eigenvalues of the tensor $B_x = (A_f)_x + 1/2\ (\mathrm{trace}A_f)_x\ \mathrm{Id}_{T_xM}$ at arbitrary point $x \in M$. Then by [368, Theorem 1] we can conclude that the subspace $D_{\mu_\alpha(x)} \wedge D_{\mu_\beta(x)} \subset \Lambda^2 T_xM$, spanned by all exterior products of elements of $D_{\mu_\alpha(x)}$ and $D_{\mu_\beta(x)}$, is invariant under the curvature operator $\mathcal{R}_x \in \mathrm{End}\,\Lambda^2 T_xM$ which is defined by the identity [138, p. 36]

$$g(\mathcal{R}(X \wedge Y), Z \wedge V) = g(R(X, Y), V, Z)$$

for arbitrary smooth vector fields X, Y, Z, V. This allows us to formulate the following result.

Theorem 8.19 *Let f: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ be a geodesic mapping with its field of linear symmetric endomorphisms A_f, $\mu_\alpha(x)$ and $\mu_\beta(x)$ distinct eigenvalues of the tensor $B_x = (A_f)_x + 1/2\,(\operatorname{trace} A_f)_x\ \mathrm{Id}_{T_xM}$ at arbitrary point $x \in M$. Then the subspace $D_{\mu_\alpha(x)} \wedge D_{\mu_\beta(x)} \subset \Lambda^2 T_xM$, spanned by all exterior products of elements of $D_{\mu_\alpha(x)}$, and $D_{\mu_\beta(x)}$, is invariant under the curvature operator $\mathcal{R}_x \in \operatorname{End} \Lambda^2 T_xM$.*

8.5.2 GM of complete noncompact Riemannian manifolds

It is known [138, pp. 381-382] that the Stokes theorem for an $(n-1)$-form ω on an n-dimensional compact oriented Riemannian manifold $\mathbb{V}_n$ implies the Green theorem for a vector field X on $\mathbb{V}_n$. One could expect that the noncompact analog of the Stokes theorem found by Yau in [953] will imply a noncompact analogue of the Green theorem for a complete Riemannian manifold $\mathbb{V}_n$. From that it follows that on a complete and oriented Riemannian manifold $\mathbb{V}_n$, for a smooth vector field X, the fact that div X has a constant sign on $\mathbb{V}_n$ and $\int\limits_M \|X\| dV_g < \infty$ implies that $\operatorname{div} X = 0$ [319, Prop. 2.1], [320, Prop. 1].

To prove this statement, the authors used the method of determining a vector field X using an $(n-1)$-form ω that was applied in the proof of the Green theorem [319, p. 381], and then, with some specification, in the proof of the Yau theorem.

Based on the noncompact analog of the Green theorem, we prove the following theorem.

Theorem 8.20 *Let $\mathbb{V}_n$ be an n-dimensional complete and oriented Riemannian manifold which admits a geodesic mapping f onto another Riemannian manifold $\overline{\mathbb{V}}_n$. If the trace of the field of linear symmetric endomorphisms A_f corresponding to f satisfies the following properties*

1. *$\Delta(\operatorname{trace} A_f)$ does not change sign on $\mathbb{V}_n$ for the Laplacian Δ;*
2. *$(\operatorname{trace} A_f) \in \mathcal{L}^p(M)$ for $1 < p < \infty$,*
3. *$|d(\operatorname{trace} A_f)| \in \mathcal{L}^1(M)$,*

then f: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ is affine.

Proof. Let A_f be the field of linear symmetric endomorphisms corresponding to the geodesic mapping f: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ of an n-dimensional complete, non-compact, oriented Riemannian manifold $\mathbb{V}_n$ onto another Riemannian manifold $\overline{\mathbb{V}}_n$. Let us suppose that X is the vector field on $\mathbb{V}_n$ which is dual to $d(\operatorname{trace} A_f)$. If $|d(\operatorname{trace} A_f)| \in \mathcal{L}^1(M)$ and $\Delta(\operatorname{trace} A_f)$ doesn't change sign on $\mathbb{V}_n$ then, using the above statement, we can conclude that $\Delta(\operatorname{trace} A_f) = 0$. This implies that $\operatorname{trace} A_f$ is a harmonic function.

On the other hand, the classical Liouville's theorem states that every bounded harmonic function in $\mathbb{R}^n$ is a constant. Yau proved in [953] that every

harmonic function u (positive subharmonic function) satisfying $u \in \mathcal{L}^p(M)$ for $1 < p < \infty$ is constant on a complete Riemannian manifold $\mathbb{V}_n$, in other words, that the $\mathcal{L}^p(M)$-Liouville type result holds. In our case it means that trace A_f is constant and $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ is affine.

9 GEODESIC MAPPINGS OF SPECIAL RIEMANNIAN MANIFOLDS

9.1 Geodesic mappings of spaces of constant curvature

9.1.1 Spaces of constant curvature

To be able to exploit the theory of geodesic maps of Riemannian manifolds in details we must pay attention to special types of Riemannian spaces: namely to spaces of constant curvature, in particular, to flat spaces.

Recall that the Riemannian space $\mathbb{V}_n = (M, g)$ is called *flat* (sometimes also *locally Euclidean*) if for any point, there exists a local map (U, φ), $\varphi = (y^1, \dots, y^n)$ (called a *cartesian map*) with respect to which the metric tensor (3.14) takes the form

$$ds^2 = e_1(dy^1)^2 + e_2(dy^2)^2 + \ldots + e_n(dy^n)^2, \qquad e_i = \pm 1.$$

The space $\mathbb{V}_n$ is flat if and only if the curvature tensor vanishes everywhere, $R = 0$ ($R^h_{ijk} = 0$). If this is the case then also the Ricci tensor vanishes, $\mathrm{Ric} = 0$ ($R_{ij} = 0$), and the Weyl tensor vanishes, $W = 0$ ($W^h_{ijk} = 0$). Euclidean spaces belong, as particular cases, to the class of Riemannian spaces of constant curvature.

Recall that the definition of *spaces of constant curvature* K was introduced in subsection 2.4.4 (Definition 3.3). Such spaces are characterized by conditions for the Riemann (curvature) tensor (3.18): $R^h_{ijk} = K \cdot (\delta^h_j g_{ik} - \delta^h_k g_{ij})$, $K = \mathrm{const}$, or equivalently, if its curvature tensor of type $(0, 4)$ satisfies

$$R_{hijk} = K \cdot (g_{hj} g_{ik} - g_{hk} g_{ij}), \qquad K = \mathrm{const}. \tag{9.1}$$

Obviously, spaces of constant curvature $K = 0$ are flat. It can be checked that two spaces of the same constant curvature, under the condition of the same signature of metrics, are locally isometric (Theorem 4.8, p. 195).

Lemma 9.1 *For spaces of constant curvature, the Ricci tensor is related to the metric tensor by* $Ric = K(n-1)g$*, and the Weyl tensor vanishes,* $W = 0$.

Proof. For the proof of the first assertion, it suffices to contract (3.18) over h, k. Then plugging in components of the Ricci tensor (6.23) we verify that $W^k_{ijk} = 0$ ($W = 0$). □

Vice versa, if $n > 2$ the other implication also holds (note that for any V_2, $W = 0$ holds, too):

Theorem 9.1 *A Riemannian manifold $\mathbb{V}_n$ with $n > 2$ is a space of constant curvature if and only if its Weyl tensor of projective curvature vanishes.*

Proof. To complete the proof, suppose that in $\mathbb{V}_n$ with $n > 2$, $W = 0$ is satisfied. Then according to (6.23), $R^h_{ijk} = \frac{1}{n-1}(\delta^h_j R_{ik} - \delta^h_k R_{ij})$, or equivalently,

$$R_{hijk} = \frac{1}{n-1}\,(g_{hj}R_{ik} - g_{hk}R_{ij}). \tag{9.2}$$

If we use multiplication of (9.2) by the inverse matrix (g^{ij}) followed by summation with respect to i and j, and define the so-called *scalar curvature* of the space (in the terminology of Levi-Civita) by $R = g^{ij}R_{ij}$ we get $R_{hk} = \frac{R}{n}\,g_{hk}$.

Evidently $\mathbb{V}_n$ is an Einstein space and if $n > 2$ then R is necessarily constant. Substituting $R_{hk} = \frac{R}{n}\,g_{hk}$ in (9.2) we have (9.1) with $K = \frac{R}{n(n-1)}$. Now we can easily see that $W^h_{ijk} = 0$ for spaces of constant curvature, and only for them. □

9.1.2 Geodesic mappings of spaces of constant curvature

Let us examine geodesic mappings of a (fixed) space $\mathbb{V}_n$ of constant curvature onto a (general) Riemannian space $\overline{\mathbb{V}}_n$. Since $W = 0$ in $\mathbb{V}_n$ for $n > 2$ the Weyl tensor of $\overline{\mathbb{V}}_n$ must vanish as well, $\overline{W} = 0$. Consequently, by Theorem 9.1, $\overline{\mathbb{V}}_n$ is a space of constant curvature. This turns out to be true even for $n = 2$. The situation is described by the so-called Beltrami' Theorem:

Theorem 9.2 *The only Riemannian manifolds the geodesics of which correspond (under geodesic mappings) to geodesics of spaces of constant curvature, are again spaces of constant curvature.*

The question of existence of geodesic mappings is clarified by the following.

Theorem 9.3 *Given a pair of spaces of constant curvature we can always find a non-trivial geodesic mapping of one onto the other.*

Proof. Assume two spaces $\mathbb{V}_n$ and $\widetilde{\mathbb{V}}_n$ of constant curvature K and $\widetilde{K}$, respectively. Note that if there exists a geodesic mapping of $\mathbb{V}_n$ onto some Riemannian space $\overline{\mathbb{V}}_n$ then there exist a solution of the Levi-Civita equations (8.2) for some metric tensor $\overline{g}$ and some one-form $\psi = \psi_i\,dx^i$.

Let us introduce

$$\psi_{i,j} = \psi_i\psi_j + K\,g_{ij} - \tilde{K}\,\overline{g}_{ij}. \tag{9.3}$$

In a space of constant curvature the system of equations (8.2) and (9.3) is a completely integrable system of Cauchy type of PDEs in the unknown functions g_{ij} and ψ_i. So it has a solution for any initial values $\overline{g}_{ij}(x_o) = g^o_{ij}$, $\psi_j(x_o) = \psi^o_i$.

These initial conditions are chosen so that g^o_{ij} coincide, at some point, with the components of the metric tensor of the a space of constant curvature $\widetilde{\mathbb{V}}_n$, and $\psi^o_i \not\equiv 0$. We find a solution for the system (8.2), (9.3) just under such initial conditions.

Further, let us characterize those spaces $\overline{\mathbb{V}}_n = (M, \overline{g})$ which can serve as images of $\mathbb{V}_n = (M, g)$ under a geodesic map. Evidently, $\overline{\mathbb{V}}_n$ is a space of constant curvature $\tilde{K}$.

According to the construction, the space $\overline{\mathbb{V}}_n$ has the same signature of metric and the same curvature $\widetilde{K}$ as the given space $\widetilde{\mathbb{V}}_n$, hence $\overline{\mathbb{V}}_n$ and $\widetilde{\mathbb{V}}_n$ are isometric (Theorem 4.8, p. 195), that is, they can be identified from the view-point of Riemannian geometry. So we have constructed a non-trivial geodesic mapping of the given space $\mathbb{V}_n$ of constant curvature onto another given space of constant curvature $\widetilde{\mathbb{V}}_n$, which finishes the proof of Theorem 9.3. □

We say that the Riemannian space $\mathbb{V}_n$ is *projectively flat* if there exists a geodesic map of $\mathbb{V}_n$ onto a flat space.

The following interesting properties are consequences of Beltrami' Theorem and the Theorem 9.3.

Theorem 9.4 *A pseudo-Riemannian manifold is projectively flat if and only if it is a space of constant curvature.*

If $\mathbb{V}_n$ is geodesically mapped onto a flat space $\overline{\mathbb{V}}_n$ then one implication (necessity) immediately follows by Theorem 9.2 (since a flat space is a particular case of the space of constant curvature). The other implication (sufficiency) follows by Theorem 9.3 since any two spaces of constant curvature are geodesically mapped onto each other.

A theorem by E. Beltrami in modern formulation states that *a Riemannian space V_n, admitting a geodesic mapping onto a Euclidean space, is a space of constant curvature.* The proofs of this theorem (see [50, 107, 149, 170]) are given under the condition $V_n \in C^2$, i.e. $g_{ij}(x) \in C^2$.

There exists a more general theorem:

Theorem 9.5 (Pogorelov [749]**)** *In Euclidean space, let a Riemannian metric be given by the line element $ds^2 = g_{ij}(x)dx^i dx^j$, $g_{ij}(x) \in C^0$ in cartesian coordinates. Let the geodesic lines of the space with this metric be straight lines (segments of straight lines). Then this space has constant curvature.*

It was proved for the n-dimensional sphere S_n that it admits global nontrivial projective transformations and nontrivial geodesic mappings [636]. By application of a global Γ-*transformation* (its local application is considered in ([170], p. 127]) to two spheres S_n and $\overline{S}_n$ we find that the spheres are in a nontrivial global geodesic correspondence, hence an infinite set of compact orientable proper (classical) Riemannian spaces with nonconstant curvature can be obtained, including some spaces L_n.

Comparing formulae (8.19) and (9.3) we obtain the following lemma.

Lemma 9.2 *The space of constant curvature K is the space $\mathbb{V}_n(B)$, $B = -K$.*

The fact that the spaces of constant curvature have all properties of spaces $\mathbb{V}_n(B)$, $B =$ const, follows from the previous Lemma.

Note that spaces of constant curvature admit geodesic mappings with maximal degree $r_{gm} = (n+1)(n+2)/2$, and projective transformations with maximal order $r_{pt} = n^2 + 2n$.

As these spaces belong to Einstein spaces, some of their further properties will be clear from the following Section.

9.2 Geodesic mappings of Einstein spaces

9.2.1 Einstein spaces are closed under geodesic mappings

It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings [292], see the previous Subsection. It is remarkable that for Einstein spaces, there is a similar property, too. There exists a more general theorem generalizing the theorem of E. Beltrami:

Theorem 9.6 (Mikeš [627]**)** *If an Einstein space $\mathbb{V}_n$ admits a nontrivial geodesic mapping onto a Riemannian space $\overline{\mathbb{V}}_n$, $n > 2$, then $\overline{\mathbb{V}}_n$ is an Einstein space.*

Proof. Let an Einstein space $\mathbb{V}_n$, $n > 2$, admit a nontrivial geodesic mapping onto $\overline{\mathbb{V}}_n$. Recall that $\mathbb{V}_n$ is characterized by the equation (3.19): $Ric = \varrho \cdot g$.

Then, as it is well known, with respect to the map in general, the Levi-Civita equations (8.1) and (8.2) hold in any local coordinate system. On the other hand, (8.1) is equivalent with the existence of a solution in $\mathbb{V}_n$ of the Sinyukov system of equations (8.6) with respect to the tensor a_{ij} $(= a_{ji})$ and the vector λ_i $(\lambda_i \neq 0)$. If (8.6) has a solution, then $\mathbb{V}_n$ admitting a nontrivial geodesic mapping onto $\overline{\mathbb{V}}_n$ is determined by means of relations (8.5).

The integrability conditions for (8.6) read

$$a_{\alpha i}R^{\alpha}_{jkl} + a_{\alpha j}R^{\alpha}_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}. \tag{9.4}$$

Taking (8.6) into account, we differentiate (9.4) with respect to x^m, contract the result with g^{lm}, and then we alternate with respect to i, k. By (3.19), we get $\lambda_\alpha R^{\alpha}_{ijk} = g_{ij}\xi_k - g_{ik}\xi_j$, where ξ_i is some covector, $[j,k]$ denotes the alternation. Contracting the latter with g_{ij} and using (3.19), we see that $\xi_i = B\,\lambda_i$, that is, the formula reads

$$\lambda_\alpha R^{\alpha}_{ijk} = B\,(g_{ij}\lambda_k - g_{ik}\lambda_j). \tag{9.5}$$

We contract (9.4) with λ^l. Considering (9.5), we get

$$g_{ki}\Lambda_{j\alpha}\lambda^\alpha + g_{kj}\Lambda_{i\alpha}\lambda^\alpha - \lambda_i\Lambda_{jk} - \lambda_j\Lambda_{ik} = 0 \tag{9.6}$$

where $\Lambda_{ij} = \lambda_{i,j} - Ba_{ij}$. It is easy to show that $\lambda^\alpha\Lambda_{\alpha i} = \mu\lambda_i$, where μ is some function. Since $\lambda_i \neq 0$, we find from (9.6) that

$$\lambda_{i,j} = \mu\, g_{ij} + B\, a_{ij}. \tag{9.7}$$

Differentiating (8.5) and accounting (8.5), (8.6) and (9.7), it is easy to get the following equation: $\psi_{ij} = \overline{B}\,\overline{g}_{ij} - B\,g_{ij}$, where $\overline{B}$ is some function. Then from (6.27), by virtue of the last relation, and considering (3.19), we get that $\overline{R}_{ij} = (n-1)\,\overline{B}\,\overline{g}_{ij}$. Hence $\overline{\mathbb{V}}_n$ is an Einstein space. The theorem is proved. □

Remark In 1978 this Theorem was proved in the case $\mathbb{V}_n, \overline{\mathbb{V}}_n \in C^3$, see [226, 625, 627]. Based on Theorem 7.8 we can assume that $\overline{\mathbb{V}}_n \in C^1$. Moreover, see Sec. 3.2.3, p. 115, the Einstein space $\mathbb{V}_n \in C^\omega$ and from Theorem 7.8 it follows that $\overline{\mathbb{V}}_n \in C^\omega$. Note, in [538] was studied the case $\mathbb{V}_n \in C^3$ and $\overline{\mathbb{V}}_n \in C^2$. In [482] last case is simplier.

9.2.2 Einstein spaces admit projective transformations

From (8.18b) and (9.7) it follows that the Einstein spaces $\mathcal{E}_n$ admitting nontrivial geodesic mappings are exactly the spaces $\mathbb{V}_n(B)$, $B = -K = \frac{R}{n(n-1)} = \text{const}$, and from Theorem 8.10, p. 307, they do admit projective transformations (when $R \neq 0$, they admit nontrivial projective transformations). Hence the following theorem holds

Theorem 9.7 (Mikeš [627]) *An Einstein space with nowhere-vanishing scalar curvature admits a nontrivial geodesic mapping if and only if it admits a nontrivial projective transformation.*

If an Einstein space with zero scalar curvature admits a nontrivial geodesic mapping, then it admits an affine transformation.

R. Couty [356] proved that under additional requirements, Ricci-flat spaces do not admit nontrivial projective transformations, see also [118].

From Theorems 8.14, 8.15 and 8.16 for Einstein spaces (and also for spaces of constant curvature) we obtained following facts which ar more general than [118, 356, 539].

Theorem 9.8 *Let an Einstein manifold $\mathbb{V}_n$ admit a non-affine geodesic mapping f onto a complete manifold $\overline{\mathbb{V}}_n$. If $\mathbb{V}_n$ is pseudo-Riemannian then f is affine. If the scalar curvature $R \leq 0$ then f is affine.*

In [539] the authors accepted assumptions: Let a complete Einstein manifold $\mathbb{V}_n$ admit a non-affine projective transformation. Then this manifold is isometric to the standard sphere $\mathbb{S}^n$.

9.2.3 Metrics of Einstein manifolds admitting geodesic mappings

Petrov [742], when having studied geodesic mappings of four-dimensional Einstein spaces with metric of Minkowski signature onto spaces of the same type, has proved that either these spaces are of constant curvature, or the geodesic mapping is affine. In connection with this result, the question can arise: do there exist Einstein spaces, beside spaces of constant curvature, admitting nontrivial geodesic mappings?

Geodesic mappings of Einstein spaces were investigated by Formella [407–411, 415], and of Einstein-Finsler spaces by Z. Shen [792]. An affirmative answer to this question follows from the example given below.

The space $\mathbb{V}_n$ constructed in Theorem 3.18, p. 151, is an example of an equidistant Einstein space of the basic class, with non-constant curvature, and, as it is known [170], it admits nontrivial geodesic mappings.

The following two subsections comprehend results on geodesic mappings of Einstein spaces obtained by Formella and Mikeš, see [417].

Let a be a differentiable symmetric bilinear form on $U_a \subseteq M$ satisfying (8.6) and having t different eigenvalues $\overset{1}{\Lambda}, \ldots, \overset{t}{\Lambda}$. From the very definition, at each point $p \in U_a$ they coincide with the eigenvalues of the endomorphism A_p of tangent space $T_p(M)$ corresponding to a, i.e. $g(AX, Y) = a(X, Y)$ for all X, Y.

Let (U, x) be a chart on M such that $U \subseteq U_a$. Suppose that $\overset{\alpha}{\nu}$ is an eigenvector of the matrix a_{ij} corresponding to the eigenvalue $\overset{\alpha}{\Lambda}$, $\alpha = 1, \ldots, l$, i.e. satisfying the condition

$$(a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{\alpha}{\nu}{}^j = 0. \tag{9.8}$$

We define generalized eigenvectors $\overset{r}{u}$ corresponding to $\overset{\alpha}{\Lambda}$ as follows

$$(a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{1}{u}{}^j = \overset{\alpha}{\nu}{}_i, \qquad (a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{r}{u}{}^j = {}^{r-1}\overset{}{u}{}_i, \quad r = 2, 3, \ldots, l. \tag{9.9}$$

Differentiating (9.8) and (9.9) covariantly with respect to x^k and making use of (8.6) we have

$$\begin{aligned}
&\lambda_i \overset{\alpha}{\nu}{}_k + \lambda_j \overset{\alpha}{\nu}{}^j g_{ik} - \overset{\alpha}{\Lambda}_k \overset{\alpha}{\nu}{}_i + (a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{\alpha}{\nu}{}^j_{,k} = 0,\\
&\lambda_i \overset{1}{u}{}_k + \lambda_j \overset{1}{u}{}^j g_{ik} - \overset{\alpha}{\Lambda}_k \overset{1}{u}{}_i + (a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{1}{u}{}^j_{,k} = \overset{\alpha}{\nu}{}_{i,k},\\
&\lambda_i \overset{r}{u}{}_k + \lambda_j \overset{r}{u}{}^j g_{ik} - \overset{\alpha}{\Lambda}_k \overset{r}{u}{}_i + (a_{ij} - \overset{\alpha}{\Lambda} g_{ij})\overset{r}{u}{}^j_{,k} = {}^{r-1}\overset{}{\nu}{}_{i,k}
\end{aligned} \tag{9.10}$$

(the comma denotes covariant differentiation on $\mathbb{V}_n = (M, g)$). From the above relations one can obtain ([409])

$$\begin{aligned}
\lambda_i \overset{\alpha}{\nu}{}^i \overset{l}{u}{}_k + \lambda_i \overset{1}{u}{}^i \overset{l-1}{u}{}_k + \lambda_i \overset{2}{u}{}^i \overset{l-2}{u}{}_k + \cdots + \lambda_i \overset{l-2}{u}{}^i \overset{2}{u}{}_k \\
+ \lambda_i \overset{l-1}{u}{}^i \overset{1}{u}{}_k + \lambda_i \overset{l}{u}{}^i \overset{\alpha}{\nu}{}_k - \tfrac{l+1}{2} \overset{\alpha}{\Lambda}_k u_i \overset{\alpha}{\nu}{}^i = 0.
\end{aligned} \tag{9.11}$$

If a manifold $\mathbb{V}_n$ admits non-trivial geodesic mappings determined by $\overset{\alpha}{\Lambda} \neq \text{const}$, then, in virtue of (9.11), we have $\overset{\alpha}{\nu}{}^i \overset{l}{u}{}_i \neq 0$. On the other hand, the relation (9.9) results in $\overset{\alpha}{\nu}{}^i \overset{r}{u}{}_i = 0$ for $r = 1, \ldots, l-1$. Contracting the second equation from $l+1$ relations (9.10) with $\overset{l-2}{u}{}^i$ and the l-th equation with $\overset{\alpha}{\nu}{}^i$, like (9.11), we obtain

$$\begin{aligned}
\lambda_i \overset{\alpha}{\nu}{}^i \overset{l-1}{u}{}_k + \lambda_i \overset{1}{u}{}^i \overset{l-2}{u}{}_k + \lambda_i \overset{2}{u}{}^i \overset{l-3}{u}{}_k + \cdots + \lambda_i \overset{l-2}{u}{}^i \overset{1}{u}{}_k \\
+ \lambda_i \overset{l-1}{u}{}^i \overset{\alpha}{\nu}{}_k - \tfrac{l}{2} \overset{\alpha}{\Lambda}_k \overset{l-1}{u}{}_i \overset{\alpha}{\nu}{}^i = 0.
\end{aligned} \tag{9.12}$$

Hence, since the vectors $\overset{\alpha}{\nu}$, $\overset{r}{u}$ are linearly independent and condition $\overset{l-1}{u}{}_i \overset{\alpha}{\nu}{}^i = 0$ holds, we have $\lambda_i \overset{\alpha}{\nu}{}^i = 0$ and $\lambda_i \overset{r}{u}{}^i$ for $r = 1, \ldots, l-1$. Because $2\lambda = \sum_{\alpha=1}^{t} \tau_\alpha \overset{\alpha}{\Lambda}$,

where τ_α denotes algebraic multiplicity of $\overset{\alpha}{\Lambda}$ and $\overset{\alpha}{\Lambda} = \overset{\alpha}{\Lambda}(x^{n_\alpha+\tau_\alpha})$, $n_1 = 0$, $n_\beta = \tau_1 + \tau_2 + \cdots + \tau_{\beta-1}$, $\beta = 2, \ldots, t$ ([255, 409]), therefore from (9.11) we get

$$\overset{\alpha}{\nu}_{i\alpha} = \overset{\alpha}{A}\lambda_{i\alpha}, \quad \overset{\alpha}{\nu}_j = 0 \text{ for } j \neq i_\alpha, \tag{9.13}$$

where $i_\alpha = n_\alpha + 1, \ldots, n_\alpha + \tau_\alpha$, $\overset{\alpha}{A}$ are functions on U.

Now we shall prove

Theorem 9.9 *On an Einstein manifold $\mathbb{V}_n$ with $B \neq 0$ all eigenvectors of the matrix $a_{ij}(p)$, $p \in U_\alpha$, are non-isotropic (non-null).*

Proof. Suppose that on an Einstein manifold with $B \neq 0$ the eigenvectors $\overset{\alpha}{\nu}$ corresponding to the eigenvalue $\overset{\alpha}{\Lambda}$ of the matrix $a_{ij}(p)$ are isotropic. Differentiating (9.13) covariantly with respect to x^k and then contracting the resulting equation with $\overset{\alpha}{\nu}{}^i$ and applying the relation $\lambda_{j,i_\alpha} = 0$ if $j \neq j_\alpha$, we obtain $\lambda_{t,k}\overset{\alpha}{\nu}{}^t$. Therefore, form (8.18b) and (9.8), we have

$$B\overset{\alpha}{\Lambda} + \mu = 0. \tag{9.14}$$

Moreover differentiating (7.14) covariantly with respect to x^k, in view of (8.6), (6.27) and (8.18b), we have $\mu = -\overline{B}\exp(2\psi) + \lambda^t\psi_t$, $\lambda^t = \lambda_s g^{st}$.

Differentiating the above equation once more, we easily get $\mu_k = 2B\lambda_k$, whence $\mu = B\,(\sum_{\alpha=1}^{t}\tau_\alpha\overset{\alpha}{\Lambda}) + C$, where $C = \text{const}$. Consequently, from (9.14) it follows either $B = 0$ (and $C = 0$) or the manifold (M, g) admits trivial geodesic mappings, which contradicts the assumption. This completes the proof. □

From the above Theorem it follows

Lemma 9.3 *If an Einstein manifold with non-zero scalar curvature admits geodesic mappings, then the matrix $a_{ij}(p), p \in U_\alpha$ of the form a is nondefective, i.e. for each eigenvalue its geometric multiplicity is equal to the algebraic one.*

9.2.4 Local structure Theorem

Assume that on a manifold (M, g), the form a is nondefective, and suppose that it has k different eigenvalues with multiplicity greater than one. In the case under consideration, from the papers [255, 409] and the results of the previous section, it follows

Lemma 9.4 *Let a be a form satisfying the condition (8.6) and suppose that for each $p \in U_a \subseteq M$, its matrix $a_{ij}(p)$ is nondefective, then in some neighbourhood of p there exists a coordinate system such that the metric tensors g and $\overline{g}$ take the form*

$$g = \sum_{\mu=1}^{k} e_\mu \prod_{\substack{\beta=1 \\ \beta\neq\mu}}^{t} (f_\beta - f_\mu)^{\tau_\beta} (dx^\mu)^2 + \sum_{\varrho=k+1}^{t} \prod_{\substack{\beta=1 \\ \beta\neq\mu}}^{t} (f_\varrho - f_\beta)^{\tau_\beta} \overset{\varrho}{g}_{i_\varrho j_\varrho} dx^{i_\varrho} dx^{j_\varrho},$$

$$\overline{g} = \prod_{\beta=1}^{t} (f_\beta)^{-\tau_\beta} \left\{ \sum_{\mu=1}^{k} e_\mu (f_\mu)^{-1} \prod_{\beta=1,\beta\neq\mu}^{t} (f_\beta - f_\mu)^{\tau_\beta} (dx^\mu)^2 + \sum_{\varrho=k+1}^{t} (f_\varrho)^{-1} \prod_{\beta=1,\beta\neq\mu}^{t} (f_\varrho - f_\beta)^{\tau_\beta} \overset{\varrho}{g}_{i_\varrho j_\varrho} dx^{i_\varrho} dx^{j_\varrho} \right\}, \tag{9.15}$$

where $f_\mu = f_\mu(x^\mu)$, $f_\varrho = \text{const} \neq 0$; $e_\mu = \pm 1$; $\mu = 1, \ldots, k$, $\varrho = k+1, \ldots, t$; $\tau_1 = \cdots = \tau_k = 1$, $\tau_\varrho > 1$; $i_\varrho, j_\varrho = n_\varrho + 1, n_\varrho + 2, \ldots, n_\varrho + \tau_\varrho$, $n_1 = 0$, $n_\gamma = \tau_1 + \tau_2 + \cdots + \tau_{\gamma-1}$, $\gamma = 2, 3, \ldots, t$ and $\overset{\varrho}{g}_{i_\varrho j_\varrho}(x^{n_\varrho+1}, \ldots, x^{n_\varrho+\tau_\varrho})$ are metric tensors on τ_ϱ-dimensional submanifolds $\overset{\varrho}{\mathbb{V}}$.

From (9.15) it follows that M is locally warped reducible. Assume now that M is locally a warped product manifold. Let $p \in M$ and U_1 be a neighbourhood of p such that $U_1 = \mathbb{V}_0 \times_{\sigma_1} \mathbb{V}_1 \times \cdots \times_{\sigma_m} \mathbb{V}_m$, $\sigma_\varrho \in \mathcal{F}(\mathbb{V}_0)$ and $\mathbb{V}_0 \times_{\sigma_1} R^1 \times \cdots \times_{\sigma_m} R^1$ is the space of constant curvature K. Such decomposition of U_1 is said to be a K-decomposition whereas the Riemannian manifold $\mathbb{V}_n = (M, g)$ is said to be of type $\mathbb{V}(K)$ ([178, 845]).

As it is known, $\mathbb{V}_n = (M, g)$, $n \geq 3$, is of constant curvature if and only if the Weyl projective curvature tensor W vanishes identically on M. A manifold $\mathbb{V}_n$ is an Einstein one if and only if $g(W(X,Y)U, V) + g(W(X,Y)V, U) = 0$.

If an Einstein manifold $\mathbb{V}_n$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n = (M, \overline{g})$, then the condition

$$\overline{g}(W(X,Y)U, V) + \overline{g}(W(X,Y)V, U) = 0, \quad X, Y, U, V \in \chi(M), \tag{9.16}$$

is satisfied.

Considering (9.16) in such a coordinate system in which metrics g and $\tilde{g}$ take the forms (9.15), in virtue of [117, 178] and the considerations made above, we easily obtain

Theorem 9.10 *An Einstein manifold $\mathbb{V}_n$ admits geodesic mappings onto an Einstein manifold $\overline{\mathbb{V}}_n$ if and only if:*

1. *the manifolds $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ are of types $\mathbb{V}(K)$, $K = -B$, and $\mathbb{V}(\overline{K})$, $\overline{K} = -\overline{B}$, respectively,*

2. *for each $p \in M$ there exist neighbourhoods $U_1 = \mathbb{V}_0 \times_{\sigma_1} \mathbb{V}_1 \times \cdots \times_{\sigma_m} \mathbb{V}_m$ and $U_2 = \overline{\mathbb{V}}_0 \times_{\overline{\sigma}_1} \mathbb{V}_1 \times \cdots \times_{\overline{\sigma}_m} \mathbb{V}_m$, $\overline{\sigma}_\varrho \in \mathcal{F}(\mathbb{V}_0)$, such that $\mathbb{V}_0$ and $\overline{\mathbb{V}}_0$ are geodesically corresponding k-dimensional spaces of the constant curvatures K and $\overline{K}$, respectively,*

3. *$(\mathbb{V}_\varrho, \overset{\varrho}{g})$, $\varrho = 1, 2, \ldots, m$, are n_ϱ-dimensional Einstein spaces with scalar curvature $R_\varrho = n_\varrho(n_\varrho - 1)K_\varrho$, where*

$$K_\varrho = -\frac{\sigma_{\varrho,a}\sigma_{\varrho,b}\overset{o}{g}{}^{ab}}{4\sigma_\varrho} + B\,\sigma_\varrho = -\frac{\overline{\sigma}_{\varrho,a}\overline{\sigma}_{\varrho,b}\overset{o}{\overline{g}}{}^{ab}}{4\overline{\sigma}_\varrho} + \overline{B}\,\overline{\sigma}_\varrho = \text{const}\,,$$

$\sigma_{\varrho,a} = X_a(\sigma_\varrho)$, $\overset{o}{g}{}^{ab} = (\overset{o}{g}{}^{ab})^{-1}$, $a, b = 1, \ldots, k$, $\overset{o}{g}$ and $\overset{o}{\overline{g}}$ being metric tensors on $\mathbb{V}_0$ and $\overline{\mathbb{V}}_0$ respectively,

4. *$\overset{o}{g}{}^{ab}\,\sigma_{\varrho,b} = \overset{o}{\overline{g}}{}^{ab}\,\overline{\sigma}_{\varrho,b}$.*

If an Einstein manifold $\mathbb{V}_n$ admits a geodesic mapping onto an Einstein manifold $\overline{\mathbb{V}}_n$ then the functions f_α from (9.15) must satisfy the system (8.19). So, in virtue of Theorem 9.10 and the results of [178] (pp. 156-162), by an elementary calculation we obtain

Theorem 9.11 *If an Einstein manifold $\mathbb{V}_n$ with $Ric = (n-1)B\,g, B \neq 0$, admits a geodesic mapping onto an Einstein manifold $\overline{\mathbb{V}}_n$ with $\overline{Ric} = (n-1)\overline{B}\,\overline{g}$ and $\psi_i \neq 0$ at a point $p \in M$, then in some neighborhood of p, there exists a coordinate system such that the metric forms g and $\overline{g}$ take one of the following forms*

$$g = \frac{1}{4(c_1 - x^1)(c_2 - Bx^1)}\,(dx^1)^2 + (c_1 - x^1)\,h_{i_1 j_1}dx^{i_1}dx^{j_1},$$
$$\overline{g} = \frac{1}{(x^1)^2(c_1)^{n-1}}\,g_{11}\,(dx^1)^2 + \frac{1}{x^1(c_1)^n}\,g_{i_1 j_1}dx^{i_1}dx^{j_1}, \tag{9.17}$$

where $c_1, c_2 = \text{const}$, $h = h(x^2, \ldots, x^n)$ is the $(n-1)$-dimensional Einstein metric such that the Ricci tensor $\overset{1}{R}_{i_1 j_1} = (n-2)(c_1 B - c_2)h_{i_1 j_1}$, $\overline{B} = (c_1)^n c_2$, $i_1, j_1 = 2, \ldots, n$,

$$\begin{aligned} g = {} & \frac{1}{4(c_1 - x^1)(c_2 - Bx^1)}(dx^1)^2 + (c_1 - x^1)(c_1 - c^2)^{\tau_2}\overset{1}{h}_{i_1 j_1}dx^{i_1}dx^{j_1} \\ & + (c_2 - x^1)(c_2 - c^1)^{\tau_1}\overset{2}{h}_{i_2 j_2}dx^{i_2}dx^{j_2}, \\ \overline{g} = {} & \frac{1}{(x^1)^2(c_1)^{\tau_1}(c_2)^{\tau_2}}g_{11}(dx^1)^2 + \frac{1}{x^1(c_1)^{\tau_1+1}(c_2)^{\tau_2}}g_{i_1 j_1}dx^{i_1}dx^{j_1} \\ & + \frac{1}{x^1(c_1)^{\tau_1}(c_2)^{\tau_2+1}}g_{i_2 j_2}dx^{i_2}dx^{j_2}, \end{aligned} \tag{9.18}$$

where $c_1, c_2 = \text{const}$, $\overset{1}{h} = \overset{1}{h}(x^2, \ldots, x^{\tau_1+1})$ is a τ_1-dimensional Einstein metric such that $\overset{1}{R}_{i_1 j_1} = (\tau_1 - 1)B(c_1 - c_2)^{\tau_2+1}\overset{1}{h}_{i_1 j_1}$, $\overset{2}{h} = \overset{2}{h}(x^{\tau_1+2}, \ldots, n)$ is the

τ_2-dimensional Einstein metric such that $\overset{2}{R}_{i_2j_2} = (\tau_2-1)B(c_2-c_1)^{\tau_1+1}\overset{2}{h}_{i_2j_2}$, $i_1, j_1 = 2, \dots, \tau_1+1$; $i_2, j_2 = \tau_1+2, \dots, n$; $1+\tau_1+\tau_2 = n$, $\overline{B} = B(c_1)^{\tau_1+1}(c_2)^{\tau_2+1}$,

$$g = \sum_{\mu=1}^{k}\prod_{\nu=1,\nu\neq\mu}^{k}\frac{(x^\nu - x^\mu)}{Q(x^\mu)}(dx^\mu)^2 + \sum_{\varrho=k+1}^{t}\prod_{\mu=1}^{k}(f_\varrho - x^\mu)A_\varrho \overset{\varrho}{g}_{i_\varrho j_\varrho}dx^{i_\varrho}dx^{j_\varrho},$$

$$\overline{g} = \frac{1}{x^1\dots x^k}\prod_{\varrho=k+1}^{t}(f_\varrho) - \tau_\varrho\left\{\sum_{\mu=1}^{k}(x^\mu)^{-1}\prod_{\nu1,\nu\neq\mu}^{k}\frac{(x^\nu - x^\mu)}{Q(x^\mu)}(dx^\mu)^2 + \sum_{\varrho=k+1}^{t}(f_\varrho)^{-1}\prod_{\mu=1}^{k}(f_\varrho - x^\mu)A_\varrho\overset{\varrho}{g}_{i_\varrho j_\varrho}dx^{i_\varrho}dx^{j_\varrho}\right\}, \tag{9.19}$$

where $A_\varrho = \prod_{\omega=k+1,\omega\neq\varrho}^{t}(f_\varrho - f_\omega)^{\tau\omega}$, $f_\varrho, f_\omega = \text{const} \neq 0$, $\varrho, \omega = k+1, \dots, t$; $Q(z)$ *is a polynomial of the form* $Q(z) = (-1)^{k+1}4Bz^{k+1} + Cz^k + A_{k-1}z^{k-1} + \dots + A_1z + 4C_1$, $C_1 = \overline{B}\prod_{\varrho=k+1}^{t}(f_\varrho)^{-\tau_\varrho}$, $C, A_1, \dots, A_{k-1} = \text{const}$, *for each* ϱ f_ϱ *are a root of the polynomial* $Q(z)$, $k > 1$, $t \leq 2k+1$, $\overset{\varrho}{g}$ *are metric tensors of* τ_ϱ*-dimensional Einstein manifolds with* $\overset{\varrho}{R}_{i_\varrho j_\varrho} = (\tau_\varrho - 1)K_\varrho \overset{\varrho}{g}_{i_\varrho j_\varrho}$ *and* $K_\varrho = \frac{1}{4}(-1)^{k+1}A_\varrho Q'(f_\varrho)$, $i_\varrho, j_\varrho = n_\varrho+1, \dots, n_\varrho+\tau_\varrho$, $n_1 = 0$, $n_\gamma = \tau_1+\tau_2+\dots+\tau_{\gamma-1}$, $\gamma = 2, !\dots, t, \tau_\varrho > 1$. *The function* ψ *is defined by* $\exp(-2\psi) = x^1\cdots x^k\prod_{\varrho=k+1}^{t}(f_\varrho)^{\tau_\varrho}$.

Remark. In the case $B = 0$ the local structure of Einstein manifolds admitting geodesic mappings can be easily obtained on the ground of [139].

9.2.5 Geodesic mappings of four-dimensional Einstein spaces

Investigations of geodesic mappings of Lorentzian 4-dimensional Einstein spaces were initiated in 1961 by A.Z. Petrov [742], see [139]. A space $\mathbb{V}_n$ is called *Lorentzian* if it has a metric with Minkowski signature. The following holds:

Theorem 9.12 (Golikov and Petrov, see [139]) *Lorentzian four-dimensional Einstein spaces with nonconstant curvature do not admit nontrivial geodesic mappings onto Lorentzian Riemannian spaces.*

These investigations are completed by the following:

Theorem 9.13 *Four-dimensional Einstein spaces with nonconstant curvature do not admit nontrivial geodesic mappings onto Riemannian spaces* $\overline{\mathbb{V}}_n \in C^1$.

From these results is follows that the spaces $\mathcal{E}_4$ with nonconstant curvature are characterized, among Riemannian spaces, by the position of their geodesic curves.

Proof. To obtain a contradiction, assume that a four-dimensional Einstein space $\mathbb{V}_4$ with nonconstant curvature admits a nontrivial geodesic mapping onto a Riemannian space $\overline{\mathbb{V}}_4$.

Then, by the well-known results, see Theorem 9.6, $\overline{\mathbb{V}}_4$ is necessarily an Einstein space, and $\mathbb{V}_4$ is a space $\mathbb{V}_4(B)$, where $B = -\frac{R}{12}$. In this case the basic equations (7.1) of geodesic mappings are written as

$$\text{(a) } a_{ij,k} = \lambda_i\, g_{jk} + \lambda_j\, g_{ik}; \quad \text{(b) } \lambda_{i,j} = \mu\, g_{ij} - \frac{R}{12}\, a_{ij}; \quad \text{(c) } \mu_{,i} = -\frac{R}{6}\, \lambda_i. \tag{9.20}$$

The integrability conditions for (9.20b) are $\lambda_\alpha Y^\alpha_{ijk} = 0$, where the tensor Y^h_{ijk} is the Yano tensor of concircular curvature, see (5.38), p. 248. Since the space $\mathbb{V}_4$ has nonconstant curvature, its tensor of concircular curvature is not equal to zero. By [163], from $\lambda_\alpha Y^\alpha_{ijk} = 0$ it follows that for $n = 4$, the vector λ^h is isotropic. Then by analysis of $\lambda_\alpha Y^\alpha_{ijk} = 0$ we obtain that $B = 0$. For details, see [670]. Hence the scalar curvature R is equal to zero, therefore the space $\mathbb{V}_4$ is Ricci flat, i.e. $R_{ij} = 0$.

Since the vector λ_i is isotropic, μ is zero. Hence the vector λ_i is covariantly constant.

The analysis of Einstein manifolds $\mathbb{V}_4$ admitting an isotropic covariantly constant vector λ^h is contained in the monograph [139] by Petrov and the paper [561] by Kruchkovich, see also [455]. It follows from this, that there exists no non-trivial geodesic mappings. □

Remark In the 1982 Theorem 9.13 was proved by Mikeš and Kiosak [118, 668, 670] for $\mathbb{V}_n, \overline{\mathbb{V}}_n \in C^3$. See Remark on the page 320.

9.2.6 Petrov's conjecture on geodesic mappings of Einstein spaces

Petrov extended the methods developed originally for the investigation of geodesic mappings of four-dimensional Lorentzian-Einstein spaces to Einstein spaces of higher dimensions $n > 4$, and conjectured that *the Lorentzian-Einstein spaces $\mathcal{E}_n$ $(n > 4)$ which do not have constant curvature, do not admit nontrivial geodesic mappings onto Lorentzian-Einstein spaces* ([139], pp. 355, 461).

Let us construct a counterexample to Petrov's conjecture, see [665, 670].

Let $\mathcal{E}_n$ $(n > 4)$ be an equidistant Einstein space of nonconstant curvature with the Brinkmann metric (3.124), see Theorem 3.18, p. 151. It is known that the space $\mathcal{E}_n$ with a coordinate system (3.124) admits a geodesic mapping onto an Einstein space $\overline{\mathcal{E}}_n$ with the metric (8.17). If $qf' \neq 0$, the mapping is nontrivial (we work in common coordinates (x^i) relative to the mapping). The signatures of the metrics of $\mathcal{E}_n$ and $\overline{\mathcal{E}}_n$ are distinct provided $1 + qf < 0$ holds, otherwise they coincide, see [466].

One can easily see that under an appropriate choice of the constant q, it is possible to construct an example of a nontrivial geodesic mapping between Einstein spaces with Minkowski signature which have nonconstant curvatures and whose dimensions are greater than four. This provides a counterexample to the reduced Petrov conjecture.

9.3 Geodesic mappings of pseudo-symmetric manifolds

9.3.1 T-pseudo-symmetric manifolds

For any Riemannian manifold $\mathbb{V}_n$ let us define the following tensors (8.21):

$$Z^h_{ijk} \overset{\text{def}}{=} R^h_{ijk} - B(\delta^h_k g_{ij} - \delta^h_j g_{ik}) \ \text{ and } \ Z_{ij} \overset{\text{def}}{=} Z^\alpha_{ij\alpha}, \tag{9.21}$$

where B is a function on $\mathbb{V}_n$.

Let T be an arbitrary tensor of type (p, q). Using a tensor Z we define the operation $\langle lm\rangle$ on $\mathbb{V}_n$ in the following way:

$$T^{h_1\cdots h_p}_{i_1\cdots i_q\langle lm\rangle} = \sum_{r=1}^{q} T^{h_1\cdots h_p}_{i_1\cdots i_{r-1}\alpha\, i_{r+1}\cdots i_q} Z^\alpha_{i_r lm} - \sum_{r=1}^{p} T^{h_1\cdots h_{r-1}\alpha\, h_{r+1}\cdots h_p}_{i_1\cdots i_q} Z^{h_r}_{\alpha lm}. \tag{9.22}$$

When T is a function on $\mathbb{V}_n$ then $T_{\langle lm\rangle} = 0$, by definition. If $B \equiv 0$, then using the Ricci identity we obtain for any tensor T:

$$T^{\cdots}_{\cdots\langle lm\rangle} \equiv T^{\cdots}_{\cdots,[lm]}.$$

If U and V are tensors, then the operation $\langle lm\rangle$ has the following property:

$$(U \pm V)_{\langle lm\rangle} = U_{\langle lm\rangle} \pm V_{\langle lm\rangle} \ \text{ and } \ (UV)_{\langle lm\rangle} = U_{\langle lm\rangle} V \pm U\, V_{\langle lm\rangle}. \tag{9.23}$$

For arbitraty tensors U and V (which are covariant), we have by definition

$$\begin{aligned} \sum_{s=1}^{q} (U_{h_1\cdots h_p\langle i_s\alpha\rangle}\, V_{i_1\cdots i_{s-1}\beta i_{s+1}\cdots i_q}\, g^{\alpha\beta}) = \\ = \sum_{s=1}^{p} (V_{i_1\cdots i_q\langle h_s\alpha\rangle}\, U_{h_1\cdots h_{s-1}\beta h_{s+1}\cdots h_p}\, g^{\alpha\beta}). \end{aligned} \tag{9.24}$$

The formulas above follow from the definition of $\langle lm\rangle$ and properties of the tensor Z, immediately.

Without troubles we may derive that the following relations are true for the Kronecker tensor, the Riemannian tensor, the metric tensor and the inverse tensor of the last one:

$$g_{ij\langle lm\rangle} = 0;\ g^{ij}{}_{\langle lm\rangle} = 0;\ \delta^h_{i\langle lm\rangle} = 0;$$

$$R_{hijk\langle lm\rangle} + R_{jklm\langle hi\rangle} + R_{lmhi\langle jk\rangle} = 0. \tag{9.25}$$

In the case when $B = 0$ we obtain from (9.25) the well known Walker identities (3.9d), p. 111.

Let us introduce the following class of the Riemannian manifolds, pseudosymmetric Riemannian and Ricci-semisymmetric Riemannian manifolds $\mathbb{V}_n$. Let T be some tensor on the Riemannian manifold $\mathbb{V}_n$. Again we recall the following definition:

Definition 9.1 A Riemannian manifold $\mathbb{V}_n$ will be called a *T-pseudosymmetric Riemannian manifold* a $(TPs_n(B))$ if there holds:

$$T_{\langle lm\rangle} = 0. \tag{9.26}$$

In the case $B = 0$, it is suitable to call $\mathbb{V}_n$ *T-semisymmetric* and denote it by TPs_n.

Considering a particular choice of the tensor T we will use the following nomenclature for the corresponding manifolds TPs_n:

Tensor T	TPs_n	Nomenclature of TPs_n
R	Ps_n	semisymmetric $\mathbb{V}_n$
Ric	$RicPs_n$	Ricci semisymmetric $\mathbb{V}_n$
W	WPs_n	projective semisymmetric $\mathbb{V}_n$
C	CPs_n	conformal semisymmetric $\mathbb{V}_n$

Recalling that $\mathcal{S}_n$ is a space with constant curvature (p. 114), $\mathcal{E}_n$ is an Einstein manifold and C_n is a conformal Euclidean manifold we obtain:

The following inclusions hold true:

$$\begin{array}{ccccccc} \mathcal{S}_n & \subset & Ps_n(B) & \equiv & WPs_n(B) & \subset & CPs_n(B) \\ \cap & & \cap & & & & \cup \\ \mathcal{E}_n & \subset & RicPs_n(B) & & & & C_n. \end{array}$$

Let us investigate certain properties of the manifolds $Ps_n(B)$ and $RicPs_n(B)$ which we are mostly interested in.

We may easily see that if $Ps_n(B)$ and $RicPs_n(B)$ are not manifolds with constant curvature then the invariant B is determined uniquely.

Considering the identity (9.25) we see that Riemannian manifolds with $R_{hijk\langle lm\rangle} = A_{hijk}C_{lm}$ are pseudosymmetric manifolds.

As we have mentioned, research of the semisymmetric and Ricci-semisymmetric manifolds brings many results. P.I. Kovalev [553] and others applied Lie algebras to find properties of Riemannian tensor of semisymmetric manifolds. This research was based on the expression of the relation (7.29) in the form

$$R^h_{\alpha jk}R^\alpha_{ilm} + R^h_{i\alpha k}R^\alpha_{jlm} + R^h_{ij\alpha}R^\alpha_{klm} - R^\alpha_{ijk}R^h_{\alpha lm} = 0. \tag{9.27}$$

Clearly, it is also possible to use this algebraic methods in pseudosymmetric Riemannian manifolds $Ps_n(B)$ because (7.39) may be equivalently written in the form

$$Z^h_{\alpha jk}Z^\alpha_{ilm} + Z^h_{i\alpha k}Z^\alpha_{jlm} + Z^h_{ij\alpha}Z^\alpha_{klm} - Z^\alpha_{ijk}Z^h_{\alpha lm} = 0. \tag{9.28}$$

Analogously, the classes $RicPs_n$ of Ricci-semisymmetric and $RicPs_n(B)$ of pseudosymmetric manifolds may be investigated together. These manifolds are characterized by corresponding identities:

$$\text{(a)}\ \ R_{\alpha j}R^\alpha_{ilm} + R_{i\alpha}R^\alpha_{jlm} = 0; \qquad \text{(b)}\ \ Z_{\alpha j}Z^\alpha_{ilm} + Z_{i\alpha}Z^\alpha_{jlm} = 0. \tag{9.29}$$

Let us take into consideration basic properties of T-pseudosymmetric manifolds. Naturally, expressing a tensor T by a tensor composition of functions, the Kronecker symbol, the metric tensor, and the dual tensor to the metric tensor, and considering (9.23), (9.24) and (9.25) we have that (9.26) is fulfilled automaticaly.

In what follows, we will treat the manifolds $TPs_n(B)$ as *principal* $(TPs_n^*(B))$, if a tensor T (which appears in the definition of this manifolds) cannot be expressed by a tensor composition of functions, the Kronecker symbols, the metric tensor, and the inverse of the metric tensor.

Let a tensor $\tilde{T}$ be a contraction of a tensor T with Kronecker tensors, the metric tensor, or inverse of the metric tensor. Then $T_{\langle lm\rangle} = 0$ implies that $\tilde{T}_{\langle lm\rangle} = 0$. Thus we have

$$TPs_n(B) \subset \tilde{T}Ps_n(B). \tag{9.30}$$

Analogously, if a tensor $\tilde{T}$ is a sum of a tensor T and a tensor composition of functions, Kronecker tensors, the metric tensor, and is dual, then we get $T_{\langle lm\rangle} = 0 \Longleftrightarrow \tilde{T}_{\langle lm\rangle} = 0$, i.e.

$$TPs_n(B) \equiv \tilde{T}Ps_n(B). \tag{9.31}$$

In the following part, an important role will be played by T-pseudosymmetric manifolds $TPs_n^*(B)$ such that the contraction of the tensor T with Kronecker tensors, the metric tensor, or inverse of the metric tensor, may be expressed by a tensor composition of these tensors.

It can be proved that this decomposition may be always choosen in the form when the contraction of the sum mentioned above, as well as of a tensor T, by Kronecker tensors, the metric tensor or the dual tensor are zero.

In Section 2.4, there are contained theorems showing that this decomposition is uniquely determined.

9.3.2 Geodesic mappings of c_i - and c_{ij} - pseudosymmetric $\mathbb{V}_n$

First, let us consider geodesic mappings of T-pseudosymmetric manifolds where the valence of T is one or two.

Lemma 9.5 *Let a tensor c_{ij} on $\mathbb{V}_n$ be given and let $c_{ij} \neq \varrho\, g_{ij}$ where ϱ is a function, and*

$$c_{ij\langle lm\rangle} = 0. \tag{9.32}$$

If $\mathbb{V}_n$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n$, then $\mathbb{V}_n$ is $\mathbb{V}_n(B)$.

Proof. Let the assumptions of the Lemma be satisfied. Since $\mathbb{V}_n$ admits a geodesic mapping, we obtain the equations (8.6) $\equiv$ (8.18a). The integrability conditions of the equations (8.18a) take the form:

$$a_{\alpha(i}R^{\alpha}_{j)kl} = \lambda_{l(i}g_{j)k} - \lambda_{k(i}g_{j)l}, \tag{9.33}$$

where $\lambda_{ij} \overset{\text{def}}{=} \lambda_{i,j}$. These conditions may be written as

$$a_{ij\langle lm\rangle} = \Lambda_{l(i}g_{i)k} - \Lambda_{k(i}g_{i)l}, \tag{9.34}$$

where

$$\Lambda_{ij} \overset{\text{def}}{=} \lambda_{i,j} - Ba_{ij} - \mu g_{ij} \quad \text{and} \quad \mu \overset{\text{def}}{=} \tfrac{1}{n}(\lambda_{\alpha\beta} - Ba_{\alpha\beta})g^{\alpha\beta}.$$

Putting $C_{ij} \overset{\text{def}}{=} c_{ij} - \frac{1}{n}c_{\alpha\beta}g^{\alpha\beta}g_{ij}$ we clearly get the following conditions

$$\text{(a)}\ \ C_{ij\langle lm\rangle} = 0; \quad \text{(b)}\ \ C_{\alpha\beta}g^{\alpha\beta} = 0; \quad \text{(c)}\ \ C_{ij} \neq cg_{ij}. \tag{9.35}$$

Then, putting in (9.24) $T \equiv a_{ij}$ and $U \equiv C_{ij}$ we obtain

$$a_{ij\langle k\alpha\rangle}C^{\alpha}_{\cdot l} + a_{ij\langle l\alpha\rangle}C_{k\cdot}^{\ \alpha} = 0.$$

Due to (9.34) we have:

$$C^{\alpha}_{\cdot l}\Lambda_{\alpha(i}g_{j)k} - \Lambda_{k(i}C_{j)l} + C_{k\cdot}^{\ \alpha}\Lambda_{\alpha(i}g_{j)l} - C_{k(i}\Lambda_{j)l} = 0. \tag{9.36}$$

Contracting (9.36) with g^{jk} and contracting (9.36) with g^{jl} we get:

$$\begin{aligned} &\text{(a)} \quad nC^{\alpha}_{\cdot l}\Lambda_{\alpha i} + C_{l\cdot}^{\ \alpha}\Lambda_{\alpha i} - C_{i\cdot}^{\ \alpha}\Lambda_{\alpha l} + C^{\alpha\beta}_{\cdot\cdot}\Lambda_{\alpha\beta}\, g_{il} = 0; \\ &\text{(b)} \quad nC_{k\cdot}^{\ \alpha}\Lambda_{\alpha i} + C^{\alpha}_{\cdot k}\Lambda_{\alpha i} - C^{\alpha}_{\cdot i}\Lambda_{\alpha k} + C^{\alpha\beta}_{\cdot\cdot}\Lambda_{\alpha\beta}\, g_{ik} = 0. \end{aligned} \tag{9.37}$$

Now, contracting (9.37) with g^{il} we may write $C^{\alpha\beta}_{\cdot\cdot}\Lambda_{\alpha\beta} = 0$. Then (9.37) implies $C^{\alpha}_{\cdot l}\Lambda_{\alpha i} = C_{l\cdot}^{\ \alpha}\Lambda_{\alpha i} = 0$. Therefore formulas (9.36) may be written in a simplified form:

$$\Lambda_{k(i}C_{j)l} + C_{k(i}\Lambda_{j)l} = 0. \tag{9.38}$$

Let us suppose $\Lambda_{ij} \neq 0$. Then there exists a vector ε^i with $\varepsilon^\alpha\varepsilon^\beta\Lambda_{\alpha\beta} = \varepsilon = \pm 1$. Contracting (9.38) with $\varepsilon^i\varepsilon^j\varepsilon^k\varepsilon^l$ we derive $\varepsilon^\alpha\varepsilon^\beta C_{\alpha\beta} = 0$, and then, contracting (9.38) with $\varepsilon^i\varepsilon^j\varepsilon^k$ and with $\varepsilon^i\varepsilon^j\varepsilon^l$, respectively, we obtain $\varepsilon^\alpha C_{\alpha l} = 0$ and $\varepsilon^\alpha C_{k\alpha} = 0$.

Therefore, contraction (9.38) with $\varepsilon^i\varepsilon^k$ yields $C_{ij} = 0$. This contradicts (9.35c). Thus $\Lambda_{ij} = 0$. Considering the definition of this tensor we see that last one is equivalent to (8.18b). Then $\mathbb{V}_n$ is a space $\mathbb{V}_n(B)$. □

Lemma 9.6 *Let a nonzero vector c_i on $\mathbb{V}_n$ with $c_{i\langle lm\rangle} = 0$ be given. If $\mathbb{V}_n$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n$, then $\mathbb{V}_n$ is a space $\mathbb{V}_n(B)$.*

Proof. Putting $c_{ij} \overset{\text{def}}{=} c_ic_j$ we clearly see $c_{ij\langle lm\rangle} = 0$ and $c_{ij} \neq cg_{ij}$. Now, Lemma 9.6 follows from the Lemma 9.5. □

Lemmas 9.5 and 9.6 may be formulated as follows:

Theorem 9.14 *If a principal pseudosymmetric manifold $c_iPs^*_n(B)$ or $c_{ij}Ps^*_n(B)$ admits geodesic mappings, then this manifold is a space $\mathbb{V}_n(B)$.*

Lemma 9.7 *The Definition 8.1 of a manifold $\mathbb{V}_n(B)$ is correct: if a manifold $\mathbb{V}_n(B)$ admits any geodesic mapping, then the conditions (8.18) are satisfied for just for this particular B.*

Proof. Let a manifold $\mathbb{V}_n(B)$ admit a non trivial geodesic mapping onto $\overline{\mathbb{V}}_n$ for which a solution $\tilde{a}_{ij}$ ($\neq \varrho g_{ij}$) and $\tilde{\lambda}_i$ satisfy the equation (8.18). We rewrite the integrability conditions for the equation $\tilde{a}_{ij,k} = \tilde{\lambda}_i\, g_{jk} + \tilde{\lambda}_j\, g_{ik}$ in the form $\tilde{a}_{ij\langle lm\rangle} = 0$, i.e. any manifold $\mathbb{V}_n(B)$ is a $\tilde{a}_{ij}Ps_n(B)$ manifold. Using Lemma 9.5 we see that any solution (8.5) has the form (8.18). □

Note that as the formula (8.22) may be written in the form $a_{ij\langle kl\rangle} = 0$, any manifold $\mathbb{V}_n(B)$ is $a_{ij}Ps_n^*(B)$. Moreover, a manifold $\mathbb{V}_n(B)$ with $B = \text{const}$ is $\lambda_i Ps_n^*(B)$, because $\lambda_{i\langle kl\rangle} = 0$.

9.3.3 Geodesic mappings of T-pseudosymmetric manifolds

Lemmas 9.5 and 9.6 have been formulated for tensor valence 1 or 2. Let us prove an analogous Lemma for arbitrary tensor valences.

Lemma 9.8 *Let T be a tensor of the type (p, q) on $\mathbb{V}_n$ which cannot be written as tensor composition of functions, Kronecker tensors, the metric tensor, and the dual tensor to the metric tensor. Let $T_{\langle lm\rangle} = 0$ be fulfilled. If $\mathbb{V}_n$ ($n \leq 2(p+q)-1$) admits a geodesic mapping, then it is $\mathbb{V}_n(B)$.*

Proof. Let us suppose that on $\mathbb{V}_n$ such a tensor T be given and $\mathbb{V}_n$ admits a geodesic mapping. Consequently, besides the equations (8.6) $\equiv$ (8.18a), also their integrability conditions which are written as (9.33), are fulfilled in $\mathbb{V}_n$.

Let us investigate a tensor T of the type $(p, 0)$ where $m = p+q$ is its valency. It follows from (9.26) that T has the following property:

$$T_{h_1h_2\cdots h_m\langle lm\rangle} = 0. \tag{9.39}$$

Furthermore, T cannot be expressed by a composition of functions, Kronecker tensors, the metric tensor, and the dual metric tensor.

Lemma 9.8 will be proved by mathematical induction. The validity of the proved lemma for $m = 1, 2$ follows from Lemmas 9.5 and 9.6. Supposing the validity of Lemma 9.8 for $1, 2, \ldots, m-1$ we prove it for arbitrary $m > 2$.

In the opposite case, conditions (8.18b) are not fulfilled for a geodesic mapping. As we already mentioned, relations (9.39) are satisfied for every contraction of T with g^{ij}. Thus, these contractions may be (with respect to our hypothesis) considered as a tensor composition of functions and the metric tensor. For odd m, every such contraction vanish.

Let us suppose that m is even, $n \geq m-1$. Then, using Theorem 9.6, we may construct a tensor $\tilde{T}$ equal to the sum of the tensor T and a tensor sum of functions and the metric, where any contraction of $\tilde{T}$ with g^{ij} is zero. The tensor $\tilde{T}$ satisfies (9.39). In what follows, we may, without loss of generality, assume $\tilde{T}$ instead of T.

Let us contract relation (9.39) with $a_{k\alpha}g^{\alpha m}$ and then symmetrize it in l and k. Considering a contruction of tensor Z^h_{ijk} and conditions (9.33), (9.24) we obtain (after suitable grouping)

$$\sum_{s=1}^{m}\{\Lambda_{h_s(i}T_{|h_1\cdots h_s|j)h_{s+1}\cdots h_m} - g_{h_s(i}\overset{s}{T}_{|h_1\cdots h_s|j)h_{s+1}\cdots h_m}\} = 0, \tag{9.40}$$

where $\Lambda_{ij} \overset{\text{def}}{=} \lambda_{i,j} - Ba_{ij} - \mu g_{ij}$ and $\overset{s}{T}_{h_1\cdots h_m} \overset{\text{def}}{=} \Lambda_{\alpha h_s} g^{\alpha\beta} T_{h_1\cdots h_{s-1}\beta h_{s+1}\cdots h_m}$.

Contracting (9.40) with g^{ih_r}, where $r = \overline{1,m}$, we get

$$n\overset{r}{T}_{h_1\cdots h_{r-1}jh_{r+1}\cdots h_m} = \sum_{s=1(\neq r)}^{m} \{\overset{r}{T}_{h_1\cdots h_{s-1}jh_{s+1}\cdots h_{r-1}h_s h_{r+1}\cdots h_m} -$$
$$-\overset{s}{T}_{h_1\cdots h_{s-1}jh_{s+1}\cdots h_{r-1}h_s h_{r+1}\cdots h_m} - g_{h_s j} g^{\alpha\beta} \overset{s}{T}_{h_1\cdots h_{s-1}\alpha h_{s+1}\cdots h_{r-1}\beta h_{r+1}\cdots h_m}\}.$$

Then, contracting this condition with g^{jh_t}, where $t = \overline{1,m}$ $(t \neq r)$, we have

$$2ng^{\alpha\beta}\overset{r}{T}_{h_1\cdots h_{t-1}\alpha h_{t+1}\cdots h_{r-1}\beta h_{r+1}\cdots h_m} =$$
$$= -\sum_{s=1(\neq r,t)}^{m} \{g^{\alpha\beta}\overset{s}{T}_{h_1\cdots h_{s-1}\alpha h_{s+1}\cdots h_{r-1}\beta h_{r+1}\cdots h_{t-1}h_s h_{t+1}\cdots h_m} +$$
$$+ g^{\alpha\beta}\overset{s}{T}_{h_1\cdots h_{s-1}\alpha h_{s+1}\cdots h_{r-1}h_s h_{r+1}\cdots h_{t-1}\beta h_{t+1}\cdots h_m}\}.$$

Doing all possible substitutions of n indices we may consider this conditions as a system of linear homogenous equations in the variables

$$g^{\alpha\beta}\overset{r}{T}_{h_1\cdots h_{t-1}\alpha h_{t+1}\cdots h_{p-1}\beta h_{p+1}\cdots h_m}.$$

For $n \geq m-1$, this system has only the trivial solution. Hence the previous formulas may be simplified as follows:

$$n\overset{r}{T}_{h_1\cdots h_{r-1}jh_{r+1}\cdots h_m} =$$
$$= \sum_{s=1(\neq r)}^{m} \{\overset{r}{T}_{h_1\cdots h_{s-1}jh_{s+1}\cdots h_{r-1}h_s h_{r+1}\cdots h_m} - \overset{s}{T}_{h_1\cdots h_{s-1}jh_{s+1}\cdots h_{r-1}h_s h_{r+1}\cdots h_m}\}.$$

These relations may be also treated as a system of linear homogenous equations in the variables $\overset{r}{T}_{h_1h_2\cdots h_m}$. This system has, for $n \geq 2m-1$, only the trivial solution $\overset{r}{T}_{h_1h_2\cdots h_m} = 0$. Now, formulas (9.40) have the final expression:

$$\sum_{s=1}^{m} \{\Lambda_{h_s(i}T_{|h_1\cdots h_s|j)h_{s+1}\cdots h_m}\} = 0. \tag{9.41}$$

Since we have supposed $\Lambda_{ij} \neq 0$, a vector ε_i with $\varepsilon^\alpha\varepsilon^\beta\Lambda_{\alpha\beta} = \varepsilon = \pm 1$ may be found. Contracting (9.41) with $\varepsilon^i\varepsilon^j\varepsilon^{h_1}\varepsilon^{h_2}\cdots\varepsilon^{h_m}$ we clearly obtain:

$$\varepsilon^{h_1}\varepsilon^{h_2}\cdots\varepsilon^{h_m}T_{h_1h_2\cdots h_m} = 0.$$

Further, let us contract (9.41) with $\varepsilon^i\varepsilon^j\varepsilon^{h_1}\varepsilon^{h_2}\cdots\varepsilon^{h_{r-1}}\varepsilon^{h_{r+1}}\cdots\varepsilon^{h_m}$, $r = \overline{1,m}$, step by step. We obtain

$$\varepsilon^{h_1}\varepsilon^{h_2}\cdots\varepsilon^{h_{r-1}}\varepsilon^{h_{r+1}}\cdots\varepsilon^{h_m}T_{h_1h_2\cdots h_m} = 0.$$

Continuing this process and declining the number of vectors ε^h for tensor contraction of (9.41) we get $T_{h_1h_2\cdots h_m} = 0$. This contradiction implies that $\Lambda_{ij} = 0$, and conditions (8.18) are fulfilled. □

In terms of T - pseudosymmetric manifolds, Lemma 9.8 may be formulated as follows.

Theorem 9.15 *If a principal T - pseudosymmetric manifold $TPs_n^*(B)$ ($n \geq 2m-1$, where m is the valence of the tensor T) admits nontrivial geodesic mapping, then it is a $\mathbb{V}_n(B)$ manifold.*

Especially, we may write

Lemma 9.9 *Let T on $\mathbb{V}_n$ be a tensor of the type (p, q) which cannot be written as tensor composition of functions, Kronecker tensors, the metric tensor and its dual tensor. Let $T_{,[lm]} = 0$ be fulfilled. If $\mathbb{V}_n$ ($n \geq 2(p+q)-1$) admits geodesic mappings then $\mathbb{V}_n$ is $\mathbb{V}_n(0)$.*

Proof follows from Lemma 9.8 for $B = 0$. The fact $\mu = \text{const}$ follows from results for manifolds $\mathbb{V}_n(B)$, $B = \text{const}$, and $\mathbb{V}_n(0)$ in the Subsection 7.3.1, p. 303.

We obtain:

Theorem 9.16 *If a principal T - semisymmetric manifold TPs_n^* ($n \geq 2m-1$, where m is the tensor valence of T) admits a nontrivial geodesic mapping, then it is a $\mathbb{V}_n(0)$ manifold.*

Let us remark that Theorems 9.15 and 9.16 as well as Lemmas 9.8, 9.9 apriori do not hold in the case $n = m$, since they do not hold for the discriminant tensor of Riemannian manifold $\mathbb{V}_n$. Evidently, they are valid for $n > m$.

If we assume that the tensor T satisfies some additional algebraic properties, then some of the above mentioned requirements (related to dimension of manifold or to tensor valence) may be weaken. For example, we have

Lemma 9.10 *Let a tensor T_{hijk} on $\mathbb{V}_n$ be given. Let the following algebraic conditions*

$$T_{hijk} + T_{hikj} = T_{hijk} + T_{hjik} + T_{hkij} = T_{hijk} - T_{jkhi} = 0, \tag{9.42}$$

and

$$\text{(a)}\ \ T_{hijk\langle lm\rangle} = 0, \quad \text{(b)}\ \ T_{hijk} \neq \varrho(g_{hj}g_{ik} - g_{hk}g_{ij}), \tag{9.43}$$

where ϱ is a function, be satisfied.

If $\mathbb{V}_n$ admits a nontrivial geodesic mapping, then it is $\mathbb{V}_n(B)$.

Proof. This theorem may be proved in an analogous way as Lemma 9.8. Instead of a initial tensor T we choose a tensor which may be written as a sum of the initial tensor and tensor composition of functions and metric tensors such that the contraction of choosen tensor with the metric tensor is zero.

Contracting (9.40) with g^{ih_1} and g^{jh_3}, using (9.42), we obtain that $g^{\alpha\beta}\overset{1}{T}_{\alpha ij\beta} = 0$. Now, contracting (9.40) with g^{ih_1} and the following simplified expression: $4\overset{1}{T}_{hijk} = \overset{1}{T}_{ihjk} + \overset{1}{T}_{jihk} + \overset{1}{T}_{kijh} - \overset{2}{T}_{ihjk} - \overset{3}{T}_{jihk} - \overset{4}{T}_{kijh}$.

Using previous results and the definition of the tensors $\overset{\sigma}{T}$, we may write:

$$\overset{1}{T}_{hijk} = -\overset{1}{T}_{ihjk} + \overset{1}{T}_{jihk} + \overset{1}{T}_{kijh}. \tag{9.44}$$

Cyclically permuting the indices h, i and j we get $\overset{1}{T}_{hijk} + \overset{1}{T}_{ijhk} + \overset{1}{T}_{jhik} = 0$.

Therefore, (9.44) may be written as $4\overset{1}{T}_{hijk} = -\overset{1}{T}_{hjik} + \overset{1}{T}_{kijh}$. Clearly, it follows that $\overset{1}{T}_{hijk} = 0$. Analogously $\overset{1}{T} = \overset{2}{T} = \overset{3}{T} = \overset{4}{T} = 0$, then formulas (9.40) may be written in the form (9.41), which implies that T vanish. Lemma 9.10 is proved.

It follows from this lemma that Theorems 9.15 and 9.16 hold in the case when $n > 2$ and the valence of the tensors fulfilling (9.42) is equal 4. □

9.4 Generalized symmetric, recurrent and semisymmetric $\mathbb{V}_n$

Spaces of constant curvature and Einstein spaces are generalized by the following special Riemannian spaces (see [25, 80, 81, 118, 139, 170, 197, 837])

symmetric spaces (S_n^1)	–	$R^h_{ijk,l} = 0$,
recurrent spaces (K_n^1)	–	$R^h_{ijk,l} = \varphi_l\, R^h_{ijk}$,
Ricci-symmetric spaces $(RicS_n^1)$	–	$R_{ij,l} = 0$,
Ricci-recurrent spaces $(RicK_n^1)$	–	$R_{ij,l} = \varphi_l\, R_{ij}$,
$\mathbb{V}_n$ *with harmonic curvature* (H_n)	–	$R^\alpha_{ijk,\alpha} = 0$ $(\Leftrightarrow\ R_{ij,k} = R_{ik,j})$,
spaces L_n	–	$R_{ij,k} = a_k g_{ij} + b_i g_{jk} + b_j g_{ik}$.

Here "," denotes the covariant derivative with respect to the connection of the space $\mathbb{V}_n$ and φ_l, a_k, b_i are nonvanishing covectors. Generalizations of the spaces S_n^1 and $RicS_n^1$ mentioned above are

semisymmetric space (Ps_n)	–	$R^h_{ijk\langle lm\rangle} = 0$,	$B = 0$,
pseudosymmetric space $(Ps_n(B))$	–	$R^h_{ijk\langle lm\rangle} = 0$,	
Ricci-semisymmetric space $(RicPs_n)$	–	$R_{ij\langle lm\rangle} = 0$,	$B = 0$,
Ricci-pseudosymmetric space $(RicPs_n(B))$	–	$R_{ij\langle lm\rangle} = 0$.	
T-semisymmetric space (TPs_n)	–	$T_{\langle lm\rangle} = 0$,	$B = 0$,
T-pseudosymmetric space $(TPs_n(B))$	–	$T_{\langle lm\rangle} = 0$.	

Informations about semisymmetric and Ricci-semisymmetric manifolds can be found in Sec. 5.3.1. The research on pseudosymmetric and Ricci-pseudosymmetric spaces has been started by Mikeš (see [117, 118, 122, 149, 170, 226, 623, 643]), but he has used the name *generally semisymmetric and Ricci-semisymmetric manifolds*. The name pseudosymmetric and Ricci-pseudosymmetric were introduced by R. Deszcz. These spaces were studied further by Defeverer, Deszcz, Grycak, Hotlos, Prvanović, Verstraelen, Vrancken, etc. [362–364, 366, 372–376].

9.4.1 GM of semisymmetric spaces and their generalizations

As we have mentioned in the part before, spaces $V_n(B)$, spaces of constant curvature, and Einstein spaces form closed classes of geodesic mappings of Riemannian spaces.

From Lemmas 9.5 and 9.10 follows:

Theorem 9.17 (Mikeš [117,118,226,227]) *If a pseudosymmetric space $Ps_n(B)$ (resp. a Ricci pseudosymmetric space $RicPs_n(B)$) admits nontrivial geodesic mappings onto $\overline{\mathbb{V}}_n$, then the space $\overline{\mathbb{V}}_n$ is a space $\mathbb{V}_n(B)$, and $B, \overline{B}$ are constants.*

Theorem 9.18 (Mikeš [117, 118, 226]) *If a pseudosymmetric space $Ps_n(B)$ (resp. a Ricci pseudosymmetric space $RicPs_n(B)$) admits geodesic mappings onto $\overline{\mathbb{V}}_n$, then the space $\overline{\mathbb{V}}_n$ is a pseudosymmetric space $\overline{P}s_n(\overline{B})$ (resp. a Ricci pseudosymmetric space $Ric\overline{P}s_n(\overline{B})$), and $B, \overline{B}$ are constants.*

Proof. It follows from Beltrami theorem [292,293], see p. 318, that manifolds of constant curvature form a class of Riemannian manifolds closed under geodesic mappings. Therefore we suppose that $Ps_n(B)$ is not a manifold of constant curvature.

Due to the Theorem 9.17, we obtain that this manifold is a $\mathbb{V}_n(B)$ manifold. Therefore the conditions (8.20) are satisfied. Now, we see that the conditions $Z_{hijk\langle lm\rangle} = 0$ (characterizing pseudosymmetric manifolds) are invariant under geodesic mappings.

Analogously, using invariancy of the relations (9.29b) with respect to geodesic mappings we find that the Ricci pseudosymmetric manifolds form the class $RicPs_n(B)$. □

It follows from Theorem 9.18 that the *pseudosymmetric and Ricci-semisymmetric manifolds form classes of Riemannian manifolds closed under geodesic mappings.*

The problems mentioned above concerning particular spaces were studied by Defeverer, Deszcz, Grycak, Hotlos [362,364,373,374,376]. More general metrics were found in [117].

It can be shown that a Riemannian space $\mathbb{V}_n$ with the metric (3.124) is a space $Ps_n(B)$ (resp. $RicPs_n(B)$), $B = \text{const}$, if $f = B\,x^{1^2} + 2a\,x^1 + b$, where B, a, b are constants, and $d\tilde{s}^2$ is a metric of the space $\tilde{P}s_{n-1}(\tilde{B})$ (resp. $Ric\tilde{P}s_{n-1}(\tilde{B})$), and $\tilde{B} = b\,B - a^2$.

Theorem 9.19 (Mikeš [623], see [170,226,626], Venzi [917]) *If a semisymmetric Riemannian space Ps_n admits nontrivial geodesic mappings onto $\overline{\mathbb{V}}_n$, then $\overline{\mathbb{V}}_n$ is an equidistant pseudosymmetric space, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

See the more general Theorem 7.10 and 7.11.

Theorem 9.20 (Mikeš [626]) *If a Ricci semisymmetric space $RicPs_n \not\equiv \mathcal{E}_n$ admits nontrivial geodesic mappings onto $\overline{\mathbb{V}}_n$, then this space is an equidistant space, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

The following statements are corollaries of Theorem 9.14.

Corollary 9.1 *If a reducible (or Kähler) space $\mathbb{V}_n$ $(n > 2)$ admits a non trivial geodesic mapping then it is equidistant, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

This follows from the existence of covariantly constant tensors of the second valence in these spaces, which are not proportional to the metric tensor, and from Theorem 9.14.

By Theorem 9.15, and taking into account [227, 687], it is easy to check that the following assertion is true:

Theorem 9.21 *A Riemannian space with nonconstant curvature $\mathbb{V}_n$ and satisfying anyone of the following relations a), b) does not admit a nontrivial geodesic mapping:*

$$\text{a)} \qquad R^h_{ijk,m_1m_2\cdots m_p} = 0, \qquad (n > 2(p+1)), \quad \textit{or}$$

$$\text{b)} \qquad R_{i[j,k]m_1m_2\cdots m_p} = 0, \qquad R_{i[j,k]} \neq 0 \qquad (n > 2p)$$

where R_{ij} is the Ricci tensor, $[j,k]$ denotes the alternation with respect to j and k.

Because in next theorems $\mathbb{V}_n$ there are spaces $\mathbb{V}_n(0)$, from the Theorem 8.14 follows:

Theorem 9.22 *Let on a complete Riemannian space $\mathbb{V}_n \in C^3$ $(n \geq 2)$ one of the following relation be valid at each point:*

1. *$T_{h_1h_2\cdots h_p,[lm]} = 0$ and the tensor T cannot be represented as a sum of products of functions and the metric tensor $(n > 2(p-1))$;*
2. *$R_{ij,[lm][pr]} = 0$.*

If the condition 1) is satisfied at least at one point then $\mathbb{V}_n$ does not admit global nontrivial geodesic mappings onto the complete $\overline{\mathbb{V}}_n \in C^3$.

Theorem 9.23 *Suppose that in a complete Riemannian space $\mathbb{V}_n \in C^k$ $(n > 2)$ one of the following conditions is satisfied:*

$$\text{a)}\ R_{ij,[lm][pr]} = 0;\ R_{ij,[lm]} \neq 0; \qquad (k = 6);$$

$$\text{b)}\ R_{ij,[lm]} = 0; \qquad R^h_{ijk} \neq \varrho\,(\delta^h_j g_{ik} - \delta^h_k g_{ij});\ (k = 4);$$

$$\text{c)}\ R_{ij} = 0; \qquad (k = 3).$$

Then the Riemannian space $\mathbb{V}_n$ does not admit a global nontrivial geodesic mapping onto complete $\mathbb{V}_n \in C^3$.

In the works by Mikeš [118, 121, 634] and Sinyukova [832], a series of results on global geodesic mappings of compact semisymmetric ($R^h_{ijk,[lm]} = 0$) and Ricci-semisymmetric ($R_{ij,[lm]} = 0$) Riemannian manifolds with additional conditions is obtained.

From Theorems 8.14, 8.15 and 8.16, for Ricci pseudosymmetric spaces we obtain the following facts which ar more general than in [118].

Theorem 9.24 *Let a complete Ricci pseudosymmetric manifold $\mathbb{V}_n$ admits a non-affine geodesic mapping f onto a complete manifold $\overline{\mathbb{V}}_n$ or a non-affine projective transformation. Then these manifolds are isometric to the standard sphere.*

Remark 9.1 We remark that the Riemannian manifold of constant curvature, Einstein, Kähler, semisymmetric, Ricci semisymmetric and pseudosymmetric Riemannian spaces are Ricci pseudosymmetric.

The property of spaces $\mathcal{E}_4$, which is the subject of Theorem 9.14, is shared by many Riemannian spaces, which are generalizations of S_n and $\mathcal{E}_n$.

In 1953, H. Takeno and M. Ikeda [886] considered geodesic mappings from the spherically symmetric spaces $\mathbb{V}_4$ and, in 1954 Sinyukov [170] proved that the symmetric and recurrent Riemannian spaces $\mathbb{V}_n$ with nonconstant curvature do not admit nontrivial geodesic mappings.

Kaygorodov [80,81] introduced into consideration the so-called *generally recurrent spaces* D_n^m, defined by the conditions

$$R^h_{ijk,l_1l_2\cdots l_m} = \sum_{s=1}^{m} \overset{s}{\Omega}_{l_s l_{s+1}\cdots l_m} R^h_{ijk,l_1l_2\cdots l_{s-1}},$$

where $\overset{s}{\Omega}$ are some tensors. The spaces with $R^h_{ijk,l_1l_2\ldots l_m} = 0$ are called *m-symmetric spaces* S_n^m, and the spaces where $R^h_{ijk,l_1l_2\ldots l_m} = \Omega_{l_1l_2\ldots l_m} R^h_{ijk}$, $\Omega \not\equiv 0$, are called *m-recurrent spaces* K_n^m. Note that many spaces D_n^m are semisymmetric spaces Ps_n. In particular, S_n^1, S_n^2, $K_n^m \subset Ps_n$.

Mikeš [118, 170, 626] proved that semisymmetric spaces considered below with nonconstant curvature do not admit nontrivial geodesic mappings: (a) K_n^m; (b) S_n^2; (c) D_n^2; (d) D_n^m, where $\Omega^2 \not\equiv 0$.

Sobchuk added to this list semisymmetric spaces S_n^m. He also showed that the spaces of nonconstant curvature S_n^3, $n > 4$ (see [839]), S_n^4, $n > 4$ (see [691]), and S_n^m, $2n > m + 3$ (see [642]), cannot be semisymmetric and do not admit nontrivial geodesic mappings.

This is true also for non-Einstein Ricci-symmetric ($R_{ij,k} = 0$, see [249,626]), Ricci-2-symmetric ($R_{ij,kl} = 0$, see [626]), Ricci-3-symmetric ($R_{ij,klm} = 0$, $n > 4$, see [691]), Ricci-4-symmetric ($R_{ij,klmp} = 0$, $n > 4$, see [839]) and Ricci-m-symmetric ($R_{ij,l_1l_2\ldots l_m} = 0$, $2n > m + 2$, see [642]) spaces.

Remark 9.2 The results listed above were previously formulated only locally, see [118,121,170]. For the present part, $\psi \equiv 0$ holds strictly. The complement of the space, for which the local theorems mentioned above are not valid, is either a space of constant curvature or an Einstein space. We can formulate these theorems even globally.

9.4.2 Geodesic mappings of spaces with harmonic curvature

A Riemannian space $\mathbb{V}_n$ *with harmonic curvature* is defined as a space where $R^\alpha_{ijk,\alpha} = 0$ ($\Leftrightarrow R_{ij,k} = R_{ik,j}$), see [23, pp. 443-447]. In particular, $\mathbb{V}_n$ with $R_{ij,k} = 0$ is Ricci symmetric $RicS_n^1$; in [626], it is proved that $RicS_n^1 \not\equiv \mathcal{E}_n$ admit neither nontrivial projective transformations, nor nontrivial geodesic mappings, see also [249].

Theorem 9.25 (Sobchuk [837]) *If non Einstein spaces $\mathbb{V}_n$ with harmonic curvature admitting nontrivial geodesic mappings, there exist concircular vector fields and special coordinates (3.124), where $d\widetilde{s}^2$ is a metric of some Einstein space with scalar curvature $\widetilde{R}$, and the function $f \not\equiv$ const satisfies the differential equation*

$$(n-1)(4f\,f'' + (n-2)f'^2) + 4e(\widetilde{R} - Rf) = 0,$$

where R is the constant scalar curvature of $\mathbb{V}_n$.

Tanno [889] studied projective transformations of complete Riemannian spaces $\mathbb{V}_n$ with harmonic curvature. His results are generalized by the following theorems:

Theorem 9.26 (Mikeš, Radulović [687]) *Non-Einstein spaces $\mathbb{V}_n$ with harmonic curvature do not admit a nontrivial geodesic mapping onto $\overline{\mathbb{V}}_n$ with harmonic curvature. Non-Einstein spaces $\mathbb{V}_n$ with harmonic curvature do not admit any non-affine projective transformations.*

The Riemannian spaces $\mathbb{V}_n$ with nonconstant curvature R such that

$$R_{ij,k} = \sigma_k g_{ij} + \nu_i g_{jk} + \nu_j g_{ik}, \quad \text{where} \quad \sigma_k \equiv \frac{n}{(n-1)(n+2)} R_{,k};\ \nu_k \equiv \frac{n-2}{2n}\sigma_k,$$

are called the *spaces L_n* (Sinyukov [170]). This condition has intrinsic character.

The tensor a_{ij}, which is a nontrivial solution of the basic geodesic mappings equations (8.6) in S_n , is a metric tensor of the space L_n [170]. A similar circumstance is stated for nontrivial geodesic mappings of Einstein spaces [408, 413]. The general solution of (8.6) in the space L_n has the form

$$a_{ij} = c_1\, g_{ij} + c_2 \left(R_{ij} - \frac{nR}{(n-1)(n+2)} g_{ij} \right),$$

where c_1, c_2 are constants. The same result has been partially proved earlier, in [831, 832], under the assumption $\mathrm{Rang}\big(R_{ij} - \frac{R}{n} g_{ij}\big) > 2$.

The local expression of the metric L_n is given by Formella [413].

The problems of global geodesic mappings of spaces L_n were considered in [637, 831, 832]. In [637], the principal scheme for construction of a compact orientable space L_n admitting global nontrivial geodesic mappings is given.

We have shown the following theorem.

Theorem 9.27 *n-dimensional conformally flat and 3-dimensional Riemannian manifolds which admit nontrivial mappings are manifolds of constant curvature or manifolds L_n.*

9.5 Geodesic mappings of Kähler manifolds

Geodesic mappings of a Kähler manifold $\mathbb{K}_n$ onto a Kähler manifold $\overline{\mathbb{K}}_n$ will be investigated in this part. We present a construction of a Kähler manifold $\mathbb{K}_n$ which admits a non-trivial geodesic mapping onto a Kähler manifold $\overline{\mathbb{K}}_n$.

9.5.1 Introduction

Under Kähler manifolds we mean here a wide class of manifolds defined as follows (Definition 3.8):
A Riemannian manifold is called a *Kähler manifold* $\mathbb{K}_n$ if, together with the metric tensor g, an affinor structure F is defined on $\mathbb{K}_n$ which satisfies the relations

$$F^2 = e\,\mathrm{Id}; \quad g(FX,Y) + g(X,FZ) = 0; \quad \nabla F = 0, \tag{9.45}$$

where $e = \pm 1, 0$. If $e = -1$ then $\mathbb{K}_n$ is said to be an *elliptic Kähler manifold* $\mathbb{K}_n^-$, if $e = +1$ then $\mathbb{K}_n$ is said to be a *hyperbolic Kähler manifold* $\mathbb{K}_n^+$, and if $e = 0$ and $\mathrm{Rank}(F) = m \leq 2$ then $\mathbb{K}_n$ is said to be an *m-parabolic Kähler manifold* $\mathbb{K}_n^{0(m)}$. The manifold $\mathbb{K}_n^{0(n/2)}$ is called a *parabolic Kähler manifold* $\mathbb{K}_n^0$.

Geodesic mappings of Kähler manifolds $\mathbb{K}_n^-$ were investigated particularly by Coburn [353], Yano [198], Westlake [940], Yano, Nagano [952] and others. In these works the authors proved that in the case when the structure of the Kähler manifold $\mathbb{K}_n^-$ is preserved under geodesic mappings then these mappings are trivial (i.e. affine). Koga Mitsuru [550] has found more general conditions for the structure of a Kähler manifold forcing any geodesic mapping to be trivial. Similar questions for geodesic mappings of almost Hermitian manifolds were investigated by Karmazina and Kurbatova [528].

The geodesic mappings from a Kähler manifold $\mathbb{K}_n$ onto a Riemannian manifold $\overline{\mathbb{V}}_n$ were studied by Mikeš (see [118, 626, 630, 631]).

Some of the papers of Mikeš, Starko and Shiha were devoted to geodesic mappings of hyperbolic and parabolic Kähler manifolds which are generalizations of the classical Kähler manifolds (see [118, 693, 802]).

In the sequel, by Kähler manifold we mean both classical (i.e. elliptic) as well as hyperbolic and parabolic Kähler manifold.

In this part, we investigate geodesic mappings from a Kähler manifold $\mathbb{K}_n$ onto a Kähler manifold $\overline{\mathbb{K}}_n$. We present a construction of non-trivial Kähler manifolds $\mathbb{K}_n$ which are geodesically mapped onto Kähler manifolds $\overline{\mathbb{K}}_n$.

9.5.2 GM of $\mathbb{K}_n$ which preserve the structure tensor

According to the results of Westlake [940], Yano and Nagano [952], there are no non-trivial geodesic mappings of elliptic Kähler manifolds onto elliptic Kähler manifolds preserving the structure tensor. We prove the following generalization:

Theorem 9.28 *There are no non-trivial geodesic mappings of Kähler manifolds onto Kähler manifolds preserving the structure tensor.*

Proof. Let us suppose the opposite: let there be a non-trivial geodesic mapping of the Kähler manifold $\mathbb{K}_n$ with the metric g and the structure tensor F onto a Kähler manifold $\overline{\mathbb{K}}_n$with the metric $\overline{g}$ and the structure tensor $\overline{F}$. Moreover, suppose we work in both manifolds in a common coordinate system relative to this mapping. Then the fact that the diffeomorphism preserves the structures is expressed simply by

$$\overline{F}_i^h(x) = F_i^h(x). \tag{9.46}$$

Using covariant differentiation we get

$$\overline{F}_{i|j}^h = \partial_j \overline{F}_i^h + \overline{F}_i^\alpha \overline{\Gamma}_{\alpha j}^h - \overline{F}_\alpha^h \overline{\Gamma}_{ij}^\alpha \tag{9.47}$$

where "$|$" denotes covariant derivative in $\overline{\mathbb{K}}_n$. Excluding $\overline{\Gamma}$'s from (9.47) with help (6.8) we get

$$\overline{F}_{i|j}^h = \partial_j \overline{F}_i^h + \overline{F}_i^\alpha \Gamma_{\alpha j}^h - \overline{F}_\alpha^h \Gamma_{ij}^\alpha + \delta_j^h \psi_\alpha \overline{F}_i^\alpha - \overline{F}_j^h \psi_i.$$

The part $\partial_j \overline{F}_i^h + \overline{F}_i^\alpha \Gamma_{\alpha j}^h - \overline{F}_\alpha^h \Gamma_{ij}^\alpha$ is just $\overline{F}_{i,j}^h$ which is in fact $F_{i,j}^h$ by (9.46). Hence we get

$$\overline{F}_{i|j}^h = F_{i,j}^h + \delta_j^h \psi_\alpha \overline{F}_i^\alpha - \overline{F}_j^h \psi_i.$$

But since the structure tensors are covariantly constant in respective spaces, $F_{i,j}^h = 0$ and $\overline{F}_{i|j}^h = 0$ hold, we can rewrite the above conditions as

$$\delta_j^h \psi_\alpha \overline{F}_i^\alpha - \overline{F}_j^h \psi_i = 0 \tag{9.48}$$

since the mapping is non-trivial the one-form ψ must be non-zero. Hence there is a (contravariant) vector ε such that $\varepsilon^\alpha \psi_\alpha = 1$. Now contracting (9.48) with ε^i and settng $a = \psi_\alpha \varepsilon^\beta F_\beta^\alpha$ we get $F_j^h = a\delta_j^h$, a contradiction with the fact that F is the structure tensor of a Kähler manifold. □

This result for elliptic Kähler manifolds have been generalized by M. Kora [550] for the case when the structures commute under the geodesic mappings, that is, when in a common coordinate system

$$\overline{F}_\alpha^h(x) F_i^\alpha(x) = \pm F_\alpha^h(x) \overline{F}_i^\alpha(x).$$

It is possible to prove a similar property for hyperbolic Kähler manifolds.

9.5.3 Geodesic mappings onto Kähler manifolds

In this section, we determine conditions which are necessary and sufficient for a Riemannian manifold $\mathbb{V}_n$ to admit a nontrivial geodesic mapping onto a Kähler manifold $\overline{\mathbb{K}}_n$ satisfying the formulas (9.45). The following theorem holds:

Theorem 9.29 *The Riemannian manifold $\mathbb{V}_n$ admits a nontrivial geodesic mapping onto a Kähler manifold $\overline{\mathbb{K}}_n$ if and only if, in the common coordinate system x with respect to the mapping, the conditions*

$$\begin{array}{lrcl} a) & \overline{g}_{ij,k} & = & 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik}, \\ b) & \overline{F}_{i,k}^h & = & \overline{F}_k^h \psi_i - \delta_k^h \overline{F}_i^\alpha \psi_\alpha \end{array} \tag{9.49}$$

hold, where $\psi_i \not\equiv 0$ *and the tensors* $\overline{g}_{ij}$ *and* $\overline{F}_i^h$ *satisfy the following conditions:*

$$\overline{g}_{ij} = \overline{g}_{ji}, \quad \det(\overline{g}_{ij}) \neq 0, \quad \overline{F}_\alpha^h \overline{F}_i^\alpha = \overline{e}\delta_i^h, \quad \overline{F}_i^\alpha \overline{g}_{\alpha j} + \overline{F}_j^\alpha \overline{g}_{\alpha i} = 0. \tag{9.50}$$

Then $\overline{g}_{ij}$ *and* $\overline{F}_i^h$ *are the metric tensor and the structure of* $\overline{\mathbb{K}}_n$*, respectively.*

Proof. The Levi-Civita equation (9.49a)≡(8.2) guarantees the existence of geodesic mappings from a Riemannian manifold $\mathbb{V}_n$ onto a Riemannian manifold $\overline{\mathbb{V}}_n$ with the metric tensor $\overline{g}_{ij}$.

The formula (9.49b) implies that the structure $\overline{F}_i^h$ in $\overline{\mathbb{V}}_n$ is covariantly constant. Further, the algebraic conditions (9.50) guarantee that $\overline{g}_{ij}$ and $\overline{F}_i^h$ are the metric tensor and the structure of the same Kähler manifold $\overline{\mathbb{K}}_n$, respectively.

The system (9.49) is a system of partial differential equations with respect to the unknown functions $\overline{g}_{ij}(x)$, $\overline{F}_i^h(x)$ and $\psi_i(x)$ which moreover must satisfy the algebraic conditions (9.50). □

9.5.4 Geodesic mappings between Kähler manifolds

As was said in the introduction, a geodesic mapping between Kähler manifolds $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ which preserves the structure (i.e. in the common coordinate system x with respect to the mapping the conditions $\overline{F}_i^h(x) = F_i^h(x)$ hold, where F_i^h and $\overline{F}_i^h$ are the structures of $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$, respectively) is trivial (i.e. affine).

Since the structures F_i^h and $\overline{F}_i^h$ are covariantly constant in $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$, respectively, we can deduce from the results of [118, 626] that for the tensor ψ_{ij} under a geodesic mapping $\mathbb{K}_n$ onto $\overline{\mathbb{K}}_n$ the relation $\psi_{ij} = 0$ holds, i.e.

$$\psi_{i,j} = \psi_i \psi_j. \tag{9.51}$$

It follows from the relations (6.26) and (9.51) that the Riemannian tensor for a geodesic mapping of $\mathbb{K}_n$ onto $\overline{\mathbb{K}}_n$ is invariant.

We shall construct a Kähler manifold $\mathbb{K}_n$ admitting a nontrivial geodesic mapping; of course, the structure of $\mathbb{K}_n$ is not preserved.

Obviously, the existence of a nontrivial geodesic map between (pseudo-) Euclidean manifolds $\mathbb{E}_n$ and $\overline{\mathbb{E}}_n$ follows from the Beltrami theorem. On the other hand, under some specific conditions on the dimension and the signature of metrics, the manifolds $\mathbb{E}_n$ and $\overline{\mathbb{E}}_n$ are Kähler manifolds $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ in our sense.

For example, $\mathbb{E}_{2m}$ is $\mathbb{K}_{2m}^-$ and $\mathbb{K}_{2m}^+$, too, where

$$g = (I); \quad F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \quad F = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

hold, respectively.

We now construct a nontrivial example of a geodesic mapping between Kähler manifolds.

Let $\mathbb{K}_n$ be a product of Riemannian manifolds with the metric

$$ds^2 = d\hat{s}^2 + d\tilde{\hat{s}}^2, \tag{9}$$

where $d\tilde{s}^2$ is the metric of the Euclidean Kähler manifold $\tilde{\mathbb{K}}_{\tilde{n}}$ with the metric tensor $\tilde{g}_{ab}$ and the structure $\tilde{F}^a_b$, $(a,b,c=1,2,...,\tilde{n})$;

$d\tilde{\tilde{s}}^2$ is the metric of a Kähler manifold $\tilde{\tilde{\mathbb{K}}}_{\tilde{\tilde{n}}}$ with the metric tensor $\tilde{\tilde{g}}_{AB}$ and the structure $\tilde{\tilde{F}}^A_B$, $(A,B,C=\tilde{n}+1,\ ...\ ,\tilde{n}+\tilde{\tilde{n}})$, and such that a noncovariantly constant concircular vector field $\tilde{\tilde{\xi}}^h$ exists on $\mathbb{K}_n$.

This manifold is a Kähler manifold, and

$$g=\begin{pmatrix}\tilde{g} & 0\\ 0 & \tilde{\tilde{g}}\end{pmatrix} \qquad \text{and} \qquad F=\begin{pmatrix}\tilde{F} & 0\\ 0 & \tilde{\tilde{F}}\end{pmatrix}$$

are its metric and structure, respectively.

The manifolds $\tilde{\mathbb{K}}_{\tilde{n}}$ and $\tilde{\tilde{\mathbb{K}}}_{\tilde{\tilde{n}}}$ must be of the same type, i.e. both of them must be either elliptic or hyperbolic or parabolic.

We prove the following result.

Theorem 9.30 *The Kähler manifold $\mathbb{K}_n$, constructed above, admits a nontrivial geodesic mapping onto a Kähler manifold $\overline{\mathbb{K}}_n$.*

Proof. In the manifold $\tilde{\mathbb{K}}_{\tilde{n}}$ we shall investigate the equations (analogous to (9.49)):

$$\begin{aligned}\tilde{q}_{ab\backslash c} &= 2\tilde{\psi}_c\tilde{q}_{ab}+\tilde{\psi}_a\tilde{q}_{bc}+\tilde{\psi}_b\tilde{q}_{ac},\\ \tilde{B}^a_{b\backslash c} &= \tilde{B}^a_c\tilde{\psi}_b-\delta^a_c\tilde{B}^d_b\tilde{\psi}_d,\\ \tilde{\psi}_{a\backslash b} &= \tilde{\psi}_a\tilde{\psi}_b, \quad \tilde{\psi}_a=\tilde{\psi}_{\backslash a}\neq 0,\end{aligned} \tag{9.52}$$

where "$\backslash$" is the covariant derivative in $\tilde{\mathbb{K}}_{\tilde{n}}$; $\tilde{q}_{ab}$, $\tilde{B}^a_b$, $\tilde{\psi}_a$ are some tensors satisfying the algebraic conditions

$$\tilde{B}^a_c\tilde{B}^c_b=e\delta^a_b, \quad \tilde{B}^c_a\tilde{q}_{cb}+\tilde{B}^c_b\tilde{q}_{ca}=0, \quad \tilde{q}_{ab}=\tilde{q}_{ba}, \quad |\tilde{q}_{ab}|\neq 0. \tag{9.53}$$

The solution of the equations (9.52) satisfying (9.53) exists, because the equations (9.52) are completely integrable in the Euclidean manifold $\tilde{\mathbb{K}}_{\tilde{n}}$.

On the other hand, since there exists a noncovariantly constant concircular vector field in $\tilde{\tilde{\mathbb{K}}}_{\tilde{\tilde{n}}}$, we can find a function $\tilde{\tilde{\xi}}$ satisfying the conditions

$$2\tilde{\tilde{\xi}}=\tilde{\tilde{\xi}}^A\tilde{\tilde{\xi}}_A, \quad \tilde{\tilde{\xi}}^A_{\wr B}=\delta^A_B,$$

where $2\tilde{\tilde{\xi}}^A\equiv\tilde{\tilde{\xi}}_B\tilde{\tilde{g}}^{AB}$, $\tilde{\tilde{\xi}}_A\equiv\tilde{\tilde{\xi}}_{\wr A}$, $\|\tilde{\tilde{g}}^{AB}\|=\|\tilde{\tilde{g}}_{AB}\|^{-1}$ and "$\wr$" denotes the covariant derivative of $\tilde{\tilde{K}}_{\tilde{\tilde{n}}}$.

We put

$$\begin{gathered}\overline{g}_{ab}=2\,k\exp(2\overline{\psi})\ \tilde{\tilde{\xi}}\ \overline{\psi}_a\,\overline{\psi}_b+\tilde{q}_{ab},\\ \overline{g}_{aB}=k\exp(2\overline{\psi})\,\tilde{\tilde{\xi}}_B\,\overline{\psi}_a, \quad \overline{g}_{AB}=k\exp(2\overline{\psi})\,\tilde{\tilde{g}}_{AB},\\ \overline{F}^a_b=\tilde{B}^a_b, \quad \overline{F}^a_B=0, \quad \overline{F}^A_B=\tilde{\tilde{F}}^A_B, \quad \overline{F}^A_b=\tilde{\tilde{F}}^A_B\,\tilde{\tilde{\xi}}\,\overline{\psi}_b-\tilde{\tilde{\xi}}^A\,\tilde{B}^c_b\,\tilde{\xi}_c,\end{gathered}$$

where k is a constant such that $|g_{ij}| \neq 0$.

Putting $\psi = \tilde{\psi}$, we can verify the formulas (9.49) and (9.50). Hence the tensors $\overline{g}_{ij}$ and $\overline{F}_i^h$ constructed by Theorem 9.29 are the metric and structure tensors of the Kähler manifold $\overline{\mathbb{K}}_n$, and $\overline{\mathbb{K}}_n$ is the geodesic image of $\mathbb{K}_n$. □

10 GLOBAL GEODESIC MAPPINGS AND DEFORMATIONS

10.1 On the theory of geodesic mappings of Riemannian manifolds "in the large"

Theoretical preliminaries

As before, let (M, g) and $(\overline{M}, \overline{g})$ be Riemannian manifolds and f: $(M, g) \to (\overline{M}, \overline{g})$ a smooth mapping. We say that f has the *constant rank* ϱ if the Jacobian matrix of f has the rank ϱ at each point $x \in M$. With rare exception the study of geodesic mappings of Riemannian manifolds "in the large" was restricted to consideration of geodesic diffeomorphisms. By definition, any such mapping has the maximal rank ϱ which is equal to dimension of manifolds, i.e. $\varrho = \dim M = \dim \overline{M}$.

In this paragraph we will study geodesic mapping of Riemannian manifolds "in large" without the restriction on the equality of their dimensions (see [854]). Moreover, we will assume that $\dim M \neq \dim \overline{M}$. In particular, we will consider the concept of a *geodesic submersion* which was introduced into consideration (see [746, 856, 858]) as a mapping of the maximal rank $\varrho = \dim \overline{M} < \dim M$ that preserves geodesics and the concept of a *geodesic immersion* which was introduced into consideration (see [858]) as a mapping of the maximal rank $\varrho = \dim M < \dim \overline{M}$ that preserves geodesics.

We recall that the differential f_*: $TM \to T\overline{M}$ of the mapping f is a section of the vector bundle $f^*T\overline{M} \otimes TM$, i.e. $f_{*x} \in T_{f(x)}\overline{M} \otimes T_x^*M$ at each point $x \in M$ (see [721]). Let $\tilde{\nabla}$ be the connection in the vector bundle $f^*T\overline{M} \otimes TM$ induced ∇ and $\overline{\nabla}$ from (M, g) and $(\overline{M}, \overline{g})$, respectively. Then we have (see [721])

$$(\tilde{\nabla} f_*)(X, Y) = (\tilde{\nabla}_X f_*)Y = \overline{\nabla}_X f_* Y - f_* \nabla_X Y$$

for any vector fields X and Y on (M, g). It is obvious that $\tilde{\nabla} f_*$ is a symmetric tensor: for each $x \in M$ we have $(\tilde{\nabla} f_*)_x \in T_{f(x)}\overline{M} \otimes S^2 T_x M$.

Next, we denote by $|\dots|$ the norm of any type of tensor, for either the metric g, or the metric $\overline{g}$. For example,

$$|\tilde{\nabla} f_*|^2 = \sum_{i=1,\dots,n} \sum_{j=1,\dots,n} \overline{g}((\tilde{\nabla} f_*)(e_i, e_j), (\tilde{\nabla} f_*)(e_i, e_j))$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame of TM.

Let R^* be the tensor $f^*\overline{R}$ induced in (M,g) from the Riemannian curvature tensor $\overline{R}$ of $(\overline{M},\overline{g})$ by f:

$$R^*(X,Y,Z,W) = \overline{R}(f_*X, f_*Y, f_*Z, f_*W)$$

for any vector fields X,Y,Z and W from TM. Then we can define the first scalar invariant

$$r^* = \sum_{i=1,\dots,n} \sum_{j=1,\dots,n} R^*(e_i,e_j,e_i,e_j).$$

On the other hand, let Ric_* be the tensor f_*Ric induced in $(\overline{M},\overline{g})$ from the Ricci tensor Ric of (M,g) by f. Then we can define the second scalar invariant by the formula $r_* = \mathrm{trace}_{\overline{g}} Ric_*$.

Let $\{e_1,\dots,e_n\}$ be a local orthonormal frame of TM such that $g(e_i,e_j)=\delta_{ij}$ and $g^*(e_i,e_j) = (f^*\overline{g})(e_i,e_j) = \lambda_i\delta_{ij}$ where $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ at each point of (M,g) and δ_{ij} is the Kronecker symbol. Then we obtain

$$r^* = \sum_{i=1,\dots,n} \sum_{j=1,\dots,n} \lambda_i\lambda_j\ \overline{K}(f_*e_i, f_*e_j), \tag{10.1}$$

$$r^* = \sum_{i=1,\dots,n} \lambda_i\ Ric(e_i,e_i), \tag{10.2}$$

for the sectional curvature $\overline{K}$ of $(\overline{M},\overline{g})$. In [854] was proved the following theorem.

Theorem 10.1 *Let f: $(M,g) \to (\overline{M},\overline{g})$ be a smooth mapping between Riemannian manifolds and assume (M,g) to be compact (without boundary) and oriented. Then*

$$r^*(M) = r_*(M) + \int_M |\tilde{\nabla} f_*|^2\, d\nu - \int_M |\mathrm{trace}_g \tilde{\nabla} f_*|^2\, d\nu, \tag{10.3}$$

where $d\nu$ is the volume form of (M,g) and

$$r^*(M) = \int_M r^*\, d\nu, \quad r_*(M) = \int_M r_*\, d\nu.$$

10.1.1 Geodesic mappings between Riemannian manifolds of different dimensions

Let f: $(M,g) \to (\overline{M},\overline{g})$ be a geodesic mapping of constant rank $\varrho \leq \inf\{\dim M, \dim \overline{M}\}$, then due Har'el we have, see [458],

$$\tilde{\nabla} f_* = \theta \otimes f_* + f_* \otimes \theta \tag{10.4}$$

where θ is a smooth 1-form on (M,g). In order for a mapping f: $(M,g) \to (\overline{M},\overline{g})$ to be *affine*, it is necessary and sufficient that $\tilde{\nabla} f_* = 0$. For a proof see [930] and [951].

Now, using (10.3) and (10.4), we find

$$r^*(M) = r_*(M) + \int_M |\theta \otimes f_* - f_* \otimes \theta|^2 \, d\nu. \tag{10.5}$$

It follows from (10.5) that $r^*(M) \geq r_*(M)$. For $r^*(M) \leq r_*(M)$ from (10.5) we shall have $r^*(M) = r_*(M)$ and $\varrho \leq 1$. Then we can formulate the following theorem.

Theorem 10.2 *Let f: $(M, g) \to (\overline{M}, \overline{g})$ be a smooth mapping of constant rank ϱ between Riemannian manifolds and assume (M, g) to be compact (without boundary) and oriented.*

(i) If $r^*(M) < r_*(M)$ then f is not geodesic.

(ii) If $r^*(M) \leq r_*(M)$ and f is geodesic then $r^*(M) = r_*(M)$ and $\varrho \leq 1$.

On the other hand, using (10.3) and (10.4), one can rewrite the integral formula in the following form

$$\int_M \left(\sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} \lambda_i \lambda_j \, \overline{K}(f_* e_i, f_* e_j) - \sum_{i=1,\ldots,n} \lambda_i \; Ric(e_i, e_i) - |\theta \otimes f_* - f_* \otimes \theta|^2 \right) d\nu = 0.$$

Then we obtain particularly the proposition.

Corollary 10.1 *Let f: $(M, g) \to (\overline{M}, \overline{g})$ be a geodesic mapping of constant rank between Riemannian manifolds. Next, we assume that (M, g) is compact (without boundary) and oriented and the sectional curvature $\overline{K}$ of $(\overline{M}, \overline{g})$ is non positive at each point of $f(M)$.*

1. *If $Ric > 0$ then f is a constant mapping.*
2. *If $Ric \geq 0$ and the equality is attained, then $\varrho \leq 1$.*

If $\varrho < \dim M$ then the distributions Ker f_* and the distribution $(\text{Ker } f_*)^\perp$ which is orthogonal complement to Ker ${}_*f$ are integrable. The integral manifolds of these distributions are totally geodesic and umbilical respectively (see [858] and [855]). In this case, for each geodesic γ which is tangent to the distribution Ker f_* we have $f(\gamma)$ is a point in $(\overline{M}, \overline{g})$. Note that this does not contradict the definition of geodesic mapping (see the definition above). This argument and results from [853] and [315] imply

Theorem 10.3 *Let f: $(M, g) \to (\overline{M}, \overline{g})$ be a geodesic mapping of constant rank ϱ between Riemannian manifolds. If (M, g) is one of the following two types:*

1. *a complete manifold with non-negative sectional curvature and* $\dim M > \varrho$,
2. *a compact (without boundary) oriented manifold with non-positive sectional curvature and* $\dim M > \varrho > 1$,

then (M, g) is locally the Riemannian product of the leaves of two integral manifolds of Ker f_* and $(\text{Ker } f_*)^\perp$ respectively.

In particular, for $\varrho = \dim M - 1$ we can obtain the corollary from Theorem 10.3. Namely, the following proposition holds.

Corollary 10.2 *Let $f\colon (M,g) \to (\overline{M},\overline{g})$ be a geodesic mapping of constant rank $\varrho = \dim M - 1$ between Riemannian manifolds. If (M,g) is one of the following two types:*

1. *a complete manifold with non-negative Ricci curvature,*
2. *a compact (without boundary) oriented manifold with non-positive Ricci curvature*

then (M,g) is locally the Riemannian product of the leaves of two integral manifolds of Ker f_* and (Ker $f_*)^{\perp}$ respectively.

Next, we consider two special types of geodesic mapping: geodesic immersions and geodesic submersions.

10.1.2 Geodesic immersions

Let $f\colon (M,g) \to (\overline{M},\overline{g})$ be a geodesic immersion. It follows from (10.4) that for tensor field $g^* = f^*\overline{g}$ we have

$$(\nabla_Z g^*)(X,Y) = 2\theta(Z)g^*(X,Y) + \theta(X)g^*(Y,Z) + \theta(Y)g^*(X,Z)$$

for arbitrary vector fields X, Y and Z on (M,g) . Let $G = \det[g^*]$ for the matrix $[g^*]$ of the tensor g^*, then $X(\ln G) = 2(n+1)\,\theta(X)$.

Hence, $\theta = \operatorname{grad}\varphi$ for $\varphi = \frac{1}{2(n+1)}\ln G$. In this case, the tensor field $A = e^{-4\varphi}\,g^*$ satisfies the equation $(\nabla_X A)(X,X) = 0$ for an arbitrary vector field X on (M,g). We recall that on a compact Riemannian manifold (M,g) with $(\nabla_X A)(X,X) = 0$ for an arbitrary vector field X and with non-positive sectional curvature K we have $\nabla_X A = 0$, see [23, p. 451]. Moreover, the equation $\nabla_X A = 0$ implies that f is affine mapping. Then we have, see [855],

Theorem 10.4 *An arbitrary geodesic immersion $f\colon (M,g) \to (\overline{M},\overline{g})$ of a compact (without boundary) Riemannian manifold (M,g) with non-positive sectional curvature into a Riemannian manifold $(\overline{M},\overline{g})$ is an affine mapping.*

10.1.3 Geodesic submersions

Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, $\pi_a\colon M_1 \times M_2 \to M_a$ be the canonical projection for $a = 1,2$ and $\mu\colon M_1 \times M_2 \to \mathbb{R}$ be a strictly positive smooth function. Then the manifold $M = M_1 \times M_2$ endowed with the metric $g = g_1(\pi_{1*},\pi_{1*}) \oplus \mu\, g_2(\pi_{2*},\pi_{2*})$ is called a *twisted product* of Riemannian manifolds (M_1,g_1) and (M_2,g_2) or twisted manifold and is denoted by $M_1 \times_\mu M_2$. In [856] was proved the following

Theorem 10.5 *A simply connected Riemannian manifold (M,g) admitting a geodesic submersion with geodesically complete fibres is isometric to the twisted product $M_1 \times_\mu M_2$ of Riemannian manifolds (M_1,g_1) and (M_2,g_2) such that the fibres of submersion and their orthogonal complements correspond to the canonical foliations of $M_1 \times M_2$.*

We recall that a totally geodesic submanifold in a complete Riemannian manifold is geodesically complete (see [90, p. 180]) and therefore we arrive at

Corollary 10.3 *A simply connected complete Riemannian manifold* (M, g) *admitting a geodesic submersion is isometric to the twisted product* $M_1 \times_\mu M_2$ *of Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that the fibres of submersion and their orthogonal complements correspond to the canonical foliations of* $M_1 \times M_2$.

In this case, from Theorem 10.3 we can obtain the following two corollaries, see [858] and [855].

Corollary 10.4 *If a simply connected complete Riemannian manifold* (M, g) *admitting a geodesic submersion has a non-negative sectional curvature, then* (M, g) *is isometric to a direct product of Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that the fibres of submersion and their orthogonal complements correspond to the canonical foliations of* $M_1 \times M_2$.

Corollary 10.5 *If a compact (without boundary) oriented manifold Riemannian manifold* (M, g) *admitting a geodesic submersion has a non-positive sectional curvature, then* (M, g) *is isometric to a direct product of Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *such that the fibres of submersion and their orthogonal complements correspond to the canonical foliations of* $M_1 \times M_2$.

10.2 Projective transformations and deformation of surfaces

A transformation of a Riemannian space $\mathbb{V}_n$ which maps geodesics into geodesics is called *projective*. A projective transformation which is not affine is customarily called *nontrivial*.

General problems of the theory of projective transformations have been studied by G. Fubini [422], L.P. Einsenhart [50], A.S. Solodovnikov [176, 843, 844] etc. It has been established that many Riemannian spaces admit neither nontrivial geodesic maps nor nontrivial projective transformations, [117, 118, 170]. The condition of compactness of the space turns out to be especially rigid with respect to the global existence of such transformations, [118].

As far as we know, no examples have been given of compact Riemannian spaces admitting nontrivial global projective transformations.

We are going to construct an example of a nontrivial global projective transformation of an n-dimensional sphere.

10.2.1 Global projective transformation of n-sphere

Let $\mathbb{S}_n \subset \mathbb{E}_{n+1}$, $n \geq 2$, be the n-dimensional n-sphere of radius $R > 0$ (which is a compact orientable n-dimensional Riemannian space of constant curvature),

$$\mathbb{S}_n : \qquad {x^1}^2 + {x^2}^2 + \ldots + {x^{n+1}}^2 = R^2,$$

where $(x^1, x^2, \ldots, x^{n+1})$ are rectangular Cartesian coordinates in the Euclidean space $\mathbb{E}_{n+1}$. Consider the following parametrization of the sphere $\mathbb{S}_n$:

$$x^i = R\, u^i \cos w \quad \text{for } i \in \{1, \ldots, n\}, \quad x^{n+1} = R \sin w, \tag{10.6}$$

where $w \in [-\pi/2, \pi/2]$ is a "latitude"-like parameter and u^i are "spherical" coordinates on the standard unit sphere $\mathbb{S}^{n-1}$:

$$\mathbb{S}^{n-1} = \left\{ (u^1, u^2, \ldots, u^n) \mid {u^1}^2 + {u^2}^2 + \ldots + {u^n}^2 = 1 \right\}. \tag{10.7}$$

The curves on $\mathbb{S}_n$ with $(u^1, u^2, \ldots, u^n)$ fixed are called *meridians*; the points with coordinates $(0, 0, \ldots, 0, -R)$ and $(0, 0, \ldots, 0, R)$ are called *poles* of $\mathbb{S}_n$, and the intersection of $\mathbb{S}_n$ and the hyperplane $x^{n+1} = 0$ is called the *equator*. Naturally, geodesics in $\mathbb{S}_n$ are great circles in $\mathbb{S}_n$ (i.e. the principal sections, that is, intersections of $\mathbb{S}_n$ with two-planes passing through the origin – the center $(0, 0, \ldots, 0)$ of $\mathbb{S}_n$.

Consider a one-parameter transformation $\pi_t \colon \mathbb{S}_n \to \mathbb{S}_n$ defined as follows.

a) The points M and M_t corresponding to each other belong to the same meridian.

b) The latitude w_t of M_t is derived from the latitude w of M according to the formula

$$\begin{aligned} \operatorname{ctg} w_t &= \mathrm{e}^t \operatorname{ctg} w \quad && \text{for } w \neq 0, \\ w_t &= 0 && \text{for } w = 0, \end{aligned} \tag{10.8}$$

$t \in \mathbb{R}$ being the parameter of the transformation π_t.

It is easy to check that

$$w_t = \begin{cases} \operatorname{arcctg}(\mathrm{e}^t \operatorname{ctg} w) - \pi, & w \in [-\pi/2, 0); \\ 0, & w = 0; \\ \operatorname{arcctg}(\mathrm{e}^t \operatorname{ctg} w), & w \in (0, \pi/2]. \end{cases}$$

Note that $\pi_t \colon \mathbb{S}_n \to \mathbb{S}_n$ leaves invariant the points of the equator and the poles. One easily verifies that π_t is a continuous transformation of $\mathbb{S}_n$. The parameter t is a natural parameter of the one-parameter Lie transformation group π_t (i.e. it holds $\pi_0 = id$ and $\pi_t \circ \pi_\tau = \pi_{t+\tau}$). We have the following

Theorem 10.6 *The map π_t is a nontrivial projective transformation.*

Proof. It is sufficient to show that π_t maps principal sections of $\mathbb{S}_n$ into principal sections.

1. It is obvious that principal sections of $\mathbb{S}_n$ contained in the equator are left invariant by π_t.

2. The sections of $\mathbb{S}_n$ passing through the poles are the arcs of meridians, and those are also transformed into themselves by virtue of property a) of the transformation.

3. Without loss of generality we may consider the intersection of $\mathbb{S}_n$ with the 2-plane ϱ containing the axis Ox^1:

$$x^a = k^a \, x^{n+1}, \quad a \in \{2, \dots, n\},$$

where $k^a \not\equiv 0$ are some coefficients. It is easy to verify that the circle $s = \varrho \cap \mathbb{S}_n$ is mapped by π_t into a circle $s_t \subset \varrho_t$, where ϱ_t is the 2-plane given by

$$x^a = \mathrm{e}^t \, k^a \, x^{n+1}, \quad a \in \{2, \dots, n\}.$$

In fact, let $M(x^1, x^2, \dots, x^{n+1}) \in \mathbb{S}_n$ be the point corresponding to the point $M_t(x_t^1, x_t^2, \dots, x_t^{n+1}) \in \mathbb{S}_n$. Then the corresponding coordinates $(u^1, u^2, \dots, u^n)$ of these two points coincide. Denoting w, w_t the latitudes of M and M_t, respectively, and taking (10.6) into account we get

$$x^i = R \, u^i \cos w, \;\; x^{n+1} = R \, \sin w; \quad x_t^i = R \, u^i \cos w_t, \;\; x_t^{n+1} = R \, \sin w_t.$$

A point M not belonging to the equator lies on the circle $s = \varrho \cap \mathbb{S}$ if and only if $k^a = u^a \;\operatorname{ctg} w$ $(w \neq 0)$. Taking (10.8) into account we can write the last equality as $\mathrm{e}^t \, k^a = u^a \;\operatorname{ctg} w_t$. This implies $M_t \in \varrho_t$. Therefore $M_t \in \varrho_t \cap \mathbb{S}_n = s_t$.

It remains to consider the case when $M \in s = \varrho \cap \mathbb{S}_n$ and at the same time it belongs to the equator. The points of the equator being fixpoints of π_t we get $M = M_t$. We can easily check that these conditions are satisfied just by two points, namely points with coordinates $(\pm R, 0, 0, \dots, 0)$. Clearly, the points $M \equiv M_t$ with the above coordinates belong to the 2-plane ϱ_t. Since for $t \neq 0$, π_t does not preserve the natural parametrization of meridians, π_t is not affine. Hence it is a nontrivial projective transformation. □

In other words, π_t maps great circles of $\mathbb{S}_n$ (circles of principal sections of the n-sphere) into great circles.

Note that π_t constructed above is neither isometric, nor conformal, nor affine transformation. Obviously, for a fixed t, π_t is a global geodesic map of the sphere $\mathbb{S}_n$ onto itself (for $t \neq 0$ it is a nontrivial geodesic map). So we have verified that an n-sphere, which is an n-dimensional compact orientable Riemannian space, admits projective transformations, and also global nontrivial geodesic mappings.

It is not difficult to show that the fundamental vector field $\xi(\xi^h(x))$ of the global projective Lie transformation group π_t on $\mathbb{S}_n$ is generated by the gradient of a globally defined function (invariant) $\xi(x)$, which is, in the coordinate system (10.6) on $\mathbb{S}_n$, given by

$$\xi = R^2 \cos 2w \quad \text{for} \quad w \in [-\pi/2, \pi/2].$$

That is, $\xi^h(x) = g^{hi}(x)\nabla_i \xi$[91)]. At the poles as well as on the equator, the vector $\xi^h(x)$ takes zero value while at any other point of $\mathbb{S}_n$, it is colinear to the tangent vector of the meridian.

This vector field generates an infinitesimal one-parameter projective group $\overline{\pi}_t \colon \mathbb{S}_n \to \mathbb{S}_n$ defined by the following two conditions.

i) The points M and $\overline{M}_t$ which correspond to each other belong to the same meridian.

ii) The latitude $\overline{w}_t$ of $\overline{M}_t$ is connected with the latitude w of M by the rule

$$\overline{w}_t = w + t^{1/2} \sin 2w,$$

t being the infinitesimal parameter of $\overline{\pi}_t$.

[91)] Here $g^{hi}(x)$ are elements of the inverse matrix of the matrix $g_{ij}(x)$ of the metric tenzor on $\mathbb{S}_n$.

10.2.2 Surface of revolution

Now let us show that compact n-dimensional surface of revolution $\mathbb{S}_n \subset \mathbb{E}_{n+1}$ admits global nontrivial geodesic maps. Moreover, for particular surfaces of revolution, which are homeomorphic to the n-sphere, we construct a smooth geodesic deformation (mapping geodesics onto geodesics).

Let $\mathbb{S}_n$ be a surface of revolution of the class C^s, $s \geq 1$, in the Euclidean $(n+1)$-space $\mathbb{E}_{n+1}$, which is homeomorphic to the n-sphere. In the Cartesian coordinate system $(x^1, x^2, \dots, x^{n+1})$, $\mathbb{S}_n$ can be given by the conditions

$$x^i = r(w)u^i, \quad i = 1, 2, \ \dots \ , n, \quad x^{n+1} = z(w) \tag{10.9}$$

where u^i are coordinates on the standard sphere $\mathbb{S}^{n-1}$, (10.7), $w \in [w_1, w_2]$ is a parameter of horizontal type, w_1, w_2 are fixed values, and $r(w) \in C^s[w_1, w_2]$ is a function with the following properties:

$$r(w) > 0 \quad \text{and} \quad |r'(w)| \leq 1 \quad \text{for} \quad w \in (w_1, w_2),$$

$$r(w_1) = r(w_2) = 0 \quad \text{and} \quad ||r'(w_1)|| = ||r'(w_2)|| = 1,$$

$$z(w) \stackrel{\text{def}}{=} \int_{w_1}^{w} e(\tau)\sqrt{1 - r'^2(\tau)}\ d\tau.$$

The function $e(\tau) = \pm 1$ is a piecewise continuous function on $[w_1, w_2]$, continuous in those points τ for which $||r'(\tau)|| \neq 1$, and there exists an interval on which $z(w)$ is increasing. Moreover, the function r satisfies some natural requirements around the endpoints w_1 and w_2.

The points of $\mathbb{S}_n$ corresponding to parameters $w = w_1$ or $w = w_2$, respectively, are called *poles*. The curves of $\mathbb{S}_n$ for which $(u^1, u^2, \dots, u^n)$ are fixed, are called *meridians*. Note that the meridians are (unparametrized) geodesics.

A surface of revolution $\mathbb{S}_n$ will be called *simple* if it is given by (10.9) and the function $z(w)$ is increasing (which is satisfied if and only if $e(t)$=1). $\mathbb{S}_n$ will be said *an absolutely simple surface of revolution* if the condition (10.9) holds, $||r'(w)|| \neq 1$ for $w \in (w_1, w_2)$, and its poles are not flat points.

For the surface $\mathbb{S}_n$ given by the function $r(w)$, a one-parameter family of surfaces $\overline{\mathbb{S}}_n^a \subset \mathbb{E}_{n+1}$ can be defined by

$$x^i = \overline{r}(w)u^i, \quad x^{n+1} = \overline{z}(w) \tag{10.10}$$

where

$$\overline{r}(w) = r(w)/\sqrt{1 + ar^2(w)},$$

$$\overline{z}(w) = \int_{w_1}^{w} \overline{e}(\tau)\sqrt{\frac{1 - r'^2(\tau) + ar^2(\tau)}{(1 + ar^2(\tau))^3}}\ d\tau,$$

and a is a constant with

$$1 + ar^2(w) > 0, \quad 1 - r'^2(w) + ar^2(w) \geq 0 \quad \text{for} \quad w \in [w_1, w_2].$$

The piecewise smooth function $\overline{e}(\tau) = \pm 1$ is continuous for every τ satisfying $1 - r'^2(\tau) + ar^2(\tau) \neq 0$, and there must exist an interval where $\overline{z}(w)$ is increasing. Let us denote

$$a_1 = \inf\left\{\frac{1}{r^2(w)}; w \in (w_1, w_2)\right\}, \quad a_2 = \inf\left\{\frac{1 - r'^2(w)}{r^2(w)}; w \in (w_1, w_2)\right\}$$

and introduce $a_{\min} := -\min\{a_1, a_2\}$. The surfaces $\overline{\mathbb{S}}_n^a$ can be constructed for any $a > a_{\min}$ provided $a_1 \leq a_2$, and for $a \geq a_{\min}$ provided $a_1 > a_2$. If $\mathbb{S}_n$ is not absolutely simple then $a_{\min} = 0$ while if $\mathbb{S}_n$ is absolutely simple then $a_{\min} < 0$.

Particularly, $\overline{\mathbb{S}}_n^0 \equiv S_n$ if $\overline{e}(t) \equiv e(t)$. Obviously, the surfaces $\overline{\mathbb{S}}_n^a$ constructed above are surfaces of revolution homeomorphic to the n-sphere.

Theorem 10.7 *The surface S_n of revolution admits a non-trivial global geodesic map onto the surface $\overline{\mathbb{S}}_n^a$, $a \neq 0$.*

Proof. Let us verify that the map $f\colon \mathbb{S}_n \to \overline{\mathbb{S}}_n^a$ $(a \neq 0)$, that maps a point $M \in \mathbb{S}_n$ with coordinates $(w, u^1, u^2, \ldots, u^n)$ into the point $\overline{M} \in \overline{\mathbb{S}}_n^a$ with the same coordinates on the second surface, is a global nontrivial geodesic mapping. It is obvious that f is of class C^s if $\mathbb{S}_n$ is of class C^s, and f is globally defined.

Meridians of the surface $\mathbb{S}_n$ as well as of $\overline{\mathbb{S}}_n^a$ are geodesics, and f maps meridians into meridians. Since all geodesics passing through the poles of $\mathbb{S}_n$ and $\overline{\mathbb{S}}_n^a$, respectively, are just meridians, we have verified that geodesics through the poles are carried into geodesics under f. It remains to prove an analogous statement for the remaining points.

In all points with exception of the poles, the metric forms ds^2 on $\mathbb{S}_n$ and $d\overline{s}^2$ on $\overline{\mathbb{S}}_n^a$ can be calculated in the coordinate system $(w, u^1, u^2, \ldots, u^n)$ with $w \in (w_1, w_2)$. We get

$$ds^2 = dw^2 + r^2(w)d\sigma^2, \tag{10.11}$$

$$d\overline{s}^2 = (1 + ar^2)^{-2}dw^2 + r^2(1 + ar^2)^{-1}d\sigma^2, \tag{10.12}$$

where $d\sigma^2 = du^{1\,2} + du^{2\,2} + \cdots + du^{n\,2}$ is the metric form of the unit sphere $\mathbb{S}_{n-1}$.

Spaces having metrics of the form (10.11) and (10.12), respectively, are in geodesic correspondence which can be verified directly using the equations of Levi-Civita. This correspondence is non-trivial since $r(w) \not\equiv \text{const}$ and $a \neq 0$. □

Analysing the structure of surfaces $\overline{\mathbb{S}}_n^a$ $(a > a_{\min})$ it is not difficult to find out that all these surfaces are absolutely simple. For a non-simple surface $\overline{\mathbb{S}}_n^{a_{\min}}$, we can build up a unique simple surface ${}^*\overline{\mathbb{S}}_n^{a_{\min}}$ which is globally isometric to the given $\overline{\mathbb{S}}_n^{a_{\min}}$.

A deformation of a surface is called *geodesic* if it preserves geodesics.

Theorem 10.8 *Simple surfaces of revolution $\mathbb{S}_n \subset \mathbb{E}_{n+1}$ homeomorphic to the unit n-sphere admit global smooth nontrivial geodesic deformations.*

Proof. The idea of the proof makes use of the fact that the family of surfaces $\overline{\mathbb{S}}_n^a$ $(a > a_{\min})$ for $\overline{e}(\tau) = 1$ is given by a smooth deformation of the surface $\mathbb{S}_n \subset \mathbb{E}_{n+1}$. On the other hand the surfaces are in nontrivial geodesic correspondence according to Theorem 10.7; a is the deformation parameter; particularly, $\overline{\mathbb{S}}_n^a \equiv \mathbb{S}_n$ if $a = 1$. Obviously, the deformation of $\mathbb{S}_n$ can be extended by a translation of $\overline{\mathbb{S}}_n^a$ in $\mathbb{E}_{n+1}$ and by a homothety. □

Doing small changes in (10.9) we can define any surface of revolution $\mathbb{S}_n$ which geodesically corresponds to the family of surfaces $\overline{\mathbb{S}}_n^a$ from (10.10), $a > 0$. Here $\mathbb{S}_n$ is homeomorphic to the n-torus, is not simple, and the corresponding surfaces $\overline{\mathbb{S}}_n^a$ are not homeomorphic to the n-torus. Therefore there is no global geodesic map of $\mathbb{S}_n$ onto $\overline{\mathbb{S}}_n^a$. Nevertheless, we can prove the following.

Theorem 10.9 *A surface of revolution S_n homeomorphic to an n-dimensional torus admits global nontrivial geodesic mappings.*

We need

Lemma 10.1 *The function λ on $\mathbb{S}_n$ given by the formula*

$$\lambda \equiv \int_0^w f(\tau)\, d\tau \tag{10.13}$$

defines a global 1-form (covector) $\lambda^*(\lambda_i)$ on $\mathbb{S}_n$ given locally by $\lambda_i \equiv \partial_i \lambda$, and a function

$$\varrho = f'(w)$$

also globally defined on S_n , such that the following holds on $\mathbb{S}_n$ globally :

$$\nabla_X \nabla_Y \lambda^* = \varrho g(X, Y)$$

(in components, $\nabla_i \lambda_j = \varrho g_{ij}$ where g_{ij} are components of the metric tensor of $\mathbb{S}_n$).

Note that for the surfaces S_n homeomorphic to the n-torus, the function λ is not an invariant, and is never a scalar on $\mathbb{S}_n$. Furthermore, the one-form λ^* is not exact in general (is not a gradient vector globally)[92)].

Proof of Theorem 10.9. Let us start with a more precise description of surfaces of revolution S_n which are homeomorphic to the n-torus. S_n can be defined by (10.9) with $r(w) > 0$ for any $w \in (-\infty, \infty)$,

$$z(w) \stackrel{\text{def}}{=} \int_0^w e(\tau)\sqrt{1 - r'^2(\tau)}\; d\tau,$$

provided $r(w)$ and $z(w)$ are periodic functions with period w_0. Now the proof follows from the fact that making use of construction of the one-form (covector) λ^*, we can check that a constant c exists such that

$$a_{ij} = c g_{ij} + \lambda_i \lambda_j$$

[92)] Remark that in comparison to the case of surfaces homeomorphic to the n-torus, λ given by the formula (10.13) on a surface S_n homeomorphic to the standard n-sphere is a globally defined function (invariant).

are components of a globally defined non-degenerate (with $|a_{ij}| \neq 0$) tensor field on $\mathbb{S}_n$. Moreover, the tensor a satisfies the fundamental equations of geodesic mappings: under the notation $A_i = \varrho\lambda_i$,

$$a_{ij,k} = A_i g_{jk} + A_j g_{ik}.$$

As a consequence of the above equations, we can state that $\mathbb{S}_n$ admits a global geodesic mapping. In the case $\varrho \not\equiv 0$ this mapping is nontrivial. □

10.2.3 On global geodesic mappings of ellipsoids

In this section we study geodesic deformations of ellipsoids of revolution, see I. Hinterleitner [469]. We present a one-parameter family of geodesic mappings that deform ellipsoids to surfaces of revolution, which are generally of a different type. Here we construct explicitely rotational surfaces, which arise from geodesic deformation of a rotational ellipsoid and show that these surfaces cannot be ellipsoids.

Assume a rotational surface $\mathcal{S}_2$ in the Euclidian 3-space E_3 given by the equations $x = r(w)\cos t$,
$y = r(w)\sin t$, $z = z(w)$, $w \in [w_1, w_2]$, $t \in [0, 2\pi)$. Its metric has the form

$$ds^2 = a(w)\,dw^2 + b(w)\,dt^2, \tag{10.14}$$

where $a(w)$ and $b(w)$ are the differentiable functions $a(w) = r'^2(w) + z'^2(w)$ and $b(w) = r^2(w)$.

As it is known [466,467], the surface $\mathcal{S}_2$ with the metric (10.14) maps geodesically onto surfaces $\overline{\mathcal{S}}_2$ with the metric

$$d\overline{s}^2 = \frac{p\,a(w)}{(1+qb(w))^2}\,dw^2 + \frac{p\,b(w)}{1+qb(w)}\,dt^2, \tag{10.15}$$

where p and q are real parameters, t and w are common coordinates.

Now we suppose that a certain one-parameter family of rotational surfaces $\overline{\mathcal{S}}_2$ with $x = \overline{r}(w)\cos t$, $y = \overline{r}(w)\sin t$, $z = \overline{z}(w)$ is obtained from the original surface $\mathcal{S}_2$ by the following particular transformatinos

$$\overline{r}(w) = \frac{r(w)}{\sqrt{1+ar^2(w)}}, \qquad \overline{z}(w) = \int_{w_1}^{w} \sqrt{\frac{1+ar^2(\tau)-r'^2(\tau)}{(1+ar^2(\tau))^3}}\,d\tau \tag{10.16}$$

with parameter a. It was proved that (10.16) describes a one-parameter family of geodesic deformations.

The coordinate w is the same as before, therefore it is *not* the length parameter of the curve $(\overline{r}, \overline{z})$. The functions $\overline{r}$ and $\overline{z}$ introduced above must satisfy the conditions of smoothness of the surfaces at the poles $w = w_1$ and $w = w_2$, where $\overline{r} = 0$, namely $\frac{d\overline{r}}{dw} = \pm 1$ and $\frac{d\overline{z}}{dw} = 0$. They hold, provided they are satisfied for r and z.

Application to rotational ellipsoids

In the foregoing section we have seen a class of nontrivial geodesic mappings between smooth surfaces of revolution, which are homeomorphic to a sphere. Now we take as a concrete example a rotational ellipsoid, embedded into the 3-dimensional Euclidian space, and investigate its deformation by the considered geodesic mappings. This is done in a local coordinate patch, covering one half of the surface. Rather than in terms of the arc length w we formulate it in terms of the angular variable φ,

$$r(\varphi) = k \sin\varphi, \qquad z(\varphi) = 1 - \cos\varphi. \tag{10.17}$$

The squared element of the arc length is $dw^2 = dr^2 + dz^2 = (k^2 \cos^2\varphi + \sin^2\varphi)\, d\varphi^2$. We choose $w_1 = w(\varphi = 0) = 0$, so that the origin of φ and the arc length coincide, then $w_2 = w(\varphi = \pi)$ is half of the circumference of the ellipse. The condition $r(w_1) = r(w_2) = 0$ is fulfilled and

$$\frac{dr}{dw} = \frac{dr}{d\varphi}\frac{d\varphi}{dw} = \frac{k\,\cos\varphi}{\sqrt{k^2\cos^2\varphi + \sin^2\varphi}},$$

so also $\frac{dr}{dw}(w_1) = 1$ and $\frac{dr}{dw}(w_2) = -1$ are satisfied.

The transformation (10.16) in terms of φ is

$$\overline{r}(\varphi) = \frac{k\,\sin\varphi}{\sqrt{1 + ak^2\sin^2\varphi}} \tag{10.18}$$

and

$$\overline{z}(\varphi) = \int_0^{\varphi} \sqrt{\frac{1 + ar^2(\varphi') - r'^2(\varphi')}{(1 + ar^2(\varphi'))^3}}\, \frac{dw}{d\varphi'}\, d\varphi'. \tag{10.19}$$

Note that here and in the following r' means always the derivative with respect to w, even when written as function of φ', so $r'(\varphi')$ is $\frac{dr}{d\varphi'}\,\frac{d\varphi'}{dw}$.

Explicitly we find

$$\overline{r}'(\varphi) = \frac{k\,\cos\varphi}{(k^2\cos^2\varphi + \sin^2\varphi)^{\frac{1}{2}}\,(1 + ak^2\sin^2\varphi)^{\frac{3}{2}}}, \tag{10.20}$$

the maximal value of $\overline{r}$, $\overline{r}_{max} = \frac{k}{\sqrt{1+ak^2}}$, occurs at $\varphi = \pi/2$, like for the original ellipsoid.

Instead of solving the integral of (10.20) explicitly, we consider the derivative $\frac{d\overline{z}}{d\overline{r}}$, which gives a differential equation for the curve, and eliminate the parameter φ. This is done in several steps: First we express $\frac{d\overline{z}}{d\overline{r}}$ in the form $\frac{d\overline{z}}{d\varphi}/\frac{d\overline{r}}{d\varphi}$. From (10.18) and (10.19) we get

$$\frac{d\overline{r}}{d\varphi} = \frac{k\cos\varphi}{(1 + ak^2\sin^2\varphi)^{\frac{3}{2}}} \quad \text{and} \quad \frac{d\overline{z}}{d\varphi} = \sqrt{\frac{1 + ar^2(\varphi) - r'^2(\varphi)}{(1 + ar^2(\varphi))^3}}\, \frac{dw}{d\varphi}.$$

Then from the definitions (10.17) and the explicit equation of the ellipse $(1-z)^2 + \frac{r^2}{k^2} = 1$ we express $\sin\varphi$ and $\cos\varphi$ in terms of r and find

$$\frac{d\overline{z}}{d\overline{r}} = \frac{r\sqrt{1+ak^4+ar^2-a^2k^2r^2}}{k\sqrt{k^2-r^2}}$$

in terms of r.

Now we insert the inverse of (10.18), $r = \dfrac{\overline{r}}{\sqrt{1-a\overline{r}^2}}$ to express this derivative in terms of $\overline{r}$,

$$\frac{d\overline{z}}{d\overline{r}} = \frac{\overline{r}\,\sqrt{\frac{1}{k^2}+ak^2-a(1+ak^2)\overline{r}^2}}{\sqrt{1-a\overline{r}^2}\sqrt{k^2-(1+ak^2)\overline{r}^2}}.$$

At last, for a direct comparison with the corresponding differential equation for an ellipse, $\dfrac{dz}{dr} = \dfrac{r}{k\sqrt{k^2-r^2}}$, we carry out a scale transformation $\hat{r} = \overline{r}\sqrt{1+ak^2}$, $\hat{z} = \overline{z}\sqrt{1+ak^2}$, so that the maximal value of $\hat{r}$ is equal to k, like the maximal value of r in the case of the ellipse and the radial extensions of both surfaces are the same. In terms of these variables, finally,

$$\frac{d\hat{z}}{d\hat{r}} = \frac{\hat{r}}{k\sqrt{k^2-\hat{r}^2}} \cdot \sqrt{\frac{1+ak^2(k^2-\hat{r}^2)}{1+a(k^2-\hat{r}^2)}}. \tag{10.21}$$

From this we can see that the transformed curve is of a different type than an ellipse. At the maximal values of the radial variables, i. e. at the "equator", both the derivatives $\frac{dz}{dr}$ for the ellipse and $\frac{d\hat{z}}{d\hat{r}}$ for the deformed curve go to infinity, corresponding to the fact that r and t provide only a local chart for one half of the surface.

An interesting feature of these transformations is that they leave circles ($k = 1$) invariant (up to a scale factor $\sqrt{1+a}$). In the limit of a large transformation parameter a the modification factor in (10.21) goes to k and the transformed curve approaches a circle.

The metric of the resulting surface of revolution is,

$$ds^2 = \left(1+\frac{d\hat{z}^2}{d\hat{r}^2}\right)d\hat{r}^2 + \hat{r}^2dt^2 = \frac{k^2+ak^4+\left(\frac{1}{k^2}-ak^2-1\right)\hat{r}^2}{(k^2-\hat{r}^2)(1+ak^2-a\hat{r}^2)}\,d\hat{r}^2 + \hat{r}^2\,dt^2.$$

This form of the metric in terms of $\hat{r}$ is local and applies only to the lower or the upper half of the surface. It can be generalized without problems to higher dimensions, when the circles with constant $\hat{z}$ are replaced by higher-dimensional spheres. Then dt has only to be replaced by the solid angle element $d\Omega$ of the corresponding dimension.

This metric can be pulled back to the original ellipsoid by simply expressing $\hat{r}$ in terms of r,

$$ds^2 = (1+ak^2)\left[\frac{k^2+\left(\frac{1}{k^2}-1\right)r^2}{(k^2-r^2)(1+ar^2)^2}\,dr^2 + \frac{r^2}{1+ar^2}\,dt^2\right], \tag{10.22}$$

whereas the metric on the original ellipsoid is

$$ds^2 = \frac{k^2 + (\frac{1}{k^2} - 1)r^2}{k^2 - r^2}\, dr^2 + r^2\, dt^2. \tag{10.23}$$

For an explicit expression of the deformed surfaces, we calculate the equations of the "meridians" in the form $\hat{z}(r)$, where $\hat{z}$ and $\hat{r}$ are cartesian coordinates of a cross-section through the rotation axis. For this purpose we integrate (10.21), from now on we drop the hats on r and z. We begin with the substitution $\sin^2\varphi = \frac{k^2-r^2}{k^2-\frac{1}{a}-r^2}$. Then

$$z(r) = -\frac{1}{k\sqrt{a}} \int_{\varphi(0)}^{\varphi(r)} \frac{\sqrt{1-(1-k^2)\sin^2\varphi'}}{\cos^2\varphi'}\, d\varphi', \tag{10.24}$$

where $\varphi(0) = \arcsin\sqrt{\frac{ak^2}{1+ak^2}}$ and $\varphi(r) = \arcsin\sqrt{\frac{a(k^2-r^2)}{1+ak^2-ar^2}}$. Integrating (10.24) by parts gives

$$\int_{\varphi(0)}^{\varphi(r)} \sqrt{1-(1-k^2)\sin^2\varphi'}\, \frac{d\varphi'}{\cos^2\varphi'} = \sqrt{1-(1-k^2)\sin^2\varphi}\, \tan\varphi \Big|_{\varphi(0)}^{\varphi(r)}$$
$$-\frac{1}{(1-k^2)} \int_{\varphi(0)}^{\varphi(r)} \sqrt{1-(1-k^2)\sin^2\varphi'}\, d\varphi' + \frac{1}{(1-k^2)} \int_{\varphi(0)}^{\varphi(r)} \frac{d\varphi'}{\sqrt{1-(1-k^2)\sin^2\varphi'}}$$

where the last two integrals are the standard elliptic integrals of the second and first kind [43] with arguments Φ and κ

$$E(\Phi,\kappa) = \int_0^\Phi \sqrt{1-\kappa^2\sin^2\varphi}\, d\varphi \quad \text{and} \quad F(\Phi,\kappa) = \int_0^\Phi \frac{d\varphi}{\sqrt{1-\kappa^2\sin^2\varphi}}.$$

Inserting back r gives finally

$$z(r) = -\frac{\sqrt{k^2-r^2}}{k}\sqrt{\frac{1+ak^4-ak^2r^2}{1+ak^2-ar^2}} + \sqrt{\frac{1+ak^4}{1+ak^2}} + \frac{1}{\sqrt{a}\,k(1-k^2)} \times$$
$$\times \left[E\left(\arcsin\sqrt{\frac{k^2-r^2}{k^2+\frac{1}{a}-r^2}}, \sqrt{1-k^2}\right) - E\left(\arcsin\frac{k}{\sqrt{k^2+\frac{1}{a}}}, \sqrt{1-k^2}\right) \right.$$
$$\left. -F\left(\arcsin\sqrt{\frac{k^2-r^2}{k^2+\frac{1}{a}-r^2}}, \sqrt{1-k^2}\right) + F\left(\arcsin\frac{k}{\sqrt{k^2+\frac{1}{a}}}, \sqrt{1-k^2}\right) \right].$$

We have considered two aspects of geodesic mappings of ellipsoids. The last equation and (10.21) describe the geodesic deformations in E_3. An interesting property is that on a sphere as a special case of an ellipsoid these transformations act as identity, whereas they act highly non trivially on general ellipsoids. In the limit of large transformation parameters the transformed surfaces approach a sphere as limiting surface.

The second aspect, represented by (10.22) and (10.23), concerns geodesic transformations of the metric on a manifold homeomorphic to the sphere, in accordance with [932], where it is shown by application of a classical theorem by Dini [379] that there is (up to homothety) a one-parameter family of geodesically equivalent metrics on $\mathcal{S}_2$. Our result can be summarized in form of a theorem.

Theorem 10.10 *Rotational ellipsoids admit global nontrivial geodesic deformations under which they remain rotational surfaces. The resulting surfaces are not ellipsoids.*

10.2.4 Compact orientable spaces L_n

Note that if we apply globally the transformation Γ which was introduced locally by Sinyukov in [170] we can construct new compact spaces admitting global geodesic maps, see [636].

Theorem 10.11 *There exist compact orientable spaces L_n homeomorphic to the n-sphere $\mathbb{S}^n$, characterized by the following condition on components R_{ij} of the Ricci tensor: there exist one-forms (covectors) $a(a_i)$, $b(b_i)$ such that*

$$\nabla_k R_{ij} = a_i g_{jk} + b_j g_{ik} + b_i g_{jk}.$$

10.2.5 Global geodesic mappings $\mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$ with boundary

Using the methods of A. Švec (see [1]), J. Mikeš [118, 119, 642] have obtained the following results:

Theorem 10.12 *Let a compact orientable proper Riemannian manifold $\mathbb{V}_n$ with a boundary $\partial\mathbb{V}$ admits a geodesic mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$. If for all $m \in \mathbb{V}_n$ the sectional curvature is nonpositive and for all $m \in \partial\mathbb{V}_n$ the condition $\overline{g}(X,Y) = f\, g(X,Y)$ for metrics of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ is satisfied, where X, Y are arbitrary tangent vectors, then this mapping is homothetic.*

The proof of the above mentioned theorem is based on the following assertion.

Theorem 10.13 *Let $\mathbb{V}_n$ be an orientable compact Riemannian manifold and a be a symmetric bilinear form on $\mathbb{V}_n$, let a function J be determined as follows*

$$J = g_{ij} g^{kl} g^{mp} \left(a_{ik,l} a_{jm,p} - a_{ik,m} a_{jp,l}\right) \tag{10.25}$$

where a_{ij} are local coordinates of a, g^{ij} are local components of the inverse matrix of metric in $\mathbb{V}_n$, and comma denotes the covariant derivative with respect to the connection on $\mathbb{V}_n$. Then, if

1. *all sectional curvatures at each point $m \in \mathbb{V}_n$ are nonpositive,*
2. *the function $J \geq 0$ at each point of $\mathbb{V}_n$,*
3. *there exists a function $f\colon \partial\mathbb{V}_n \to \mathbb{R}$ such that $a = f(M)\, g$ at each point $M \in \partial\mathbb{V}_n$*

then $J = 0$ at each point $M \in \mathbb{V}_n$.

In case that of all the sectional curvatures of $\mathbb{V}_n$ are negative, then under the conditions 2) and 3) there exists a function f such that $a = f(M)\, g$ at each point $M \in \partial\mathbb{V}_n$.

Theorem 10.13 is analogous to that of Švec, and it can be verified the following [1, p. 12].

Let us restrict our attention to the proof of Theorem 10.12. Suppose that $\mathbb{V}_n$ admits geodesic mappings onto $\overline{\mathbb{V}}_n$. Then Sinyukov equations (8.6) hold:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}.$$

We can verify that the invariant J may be written in the following way:

$$J = (n^2 + n - 2)\, \lambda^\alpha \lambda_\alpha.$$

It is obvious that $J \geq 0$ holds.

In virtue of (7.13) and (8.6a) the conditions at the boundary of $\partial\mathbb{V}_n$: $\overline{g}(X, Y) = f\, g(X, Y)$ generate the condition

$$a(X, Y) = f(M)\, \exp(2\Psi)\, g(X, Y).$$

Hence, all the conditions of Theorem 10.13 are satisfied and $J = 0$. By a direct analysis we can check that $J = 0$ if and only if $\lambda_i = 0$. The latter is equivalent to $\psi = 0$. Hence the geodesic mapping is affine. has been proved. Moreover since in some points $\overline{g} = k\, g$ holds, it follows that the affine mapping is homothetic, and Theorem 10.12 has been proved. □

10.2.6 GM and principal orthonormal basis

L.P. Eisenhart [50, pp. 113-114] introduced a *principal direction* in a Riemannian manifold (M, g), as an eigenvector of the Ricci tensor. He showed that at any point $x \in M$ there exists the orthonormal basis $\{e_1, \ldots, e_n\}$ in which

$$g_{ij} = \delta_{ij} \quad \text{and} \quad R_{ij} = \varrho_i \delta_{ij},$$

i.e. $e_1, \ldots, e_n$ are the vectors of the principal directions and $\varrho_1, \ldots, \varrho_n$ are their eigenvalues. This basis is called the *principal orthonormal basis*.

This means that the existence of this basis is a property only of the Riemannian manifold (M, g), independent of the solution a_{ij} of equation (8.5). Generally the set of principal orthonormal bases is a proper subset of the set of orthonormal bases.

Because the vector field λ_i is gradient-like, formula (8.5) implies (8.9): $a_{i\alpha} R^\alpha_j = a_{j\alpha} R^\alpha_i$. So the tensors a_{ij} and R_{ij} commute and have common eigenvectors. From this fact it follows that it exists the principal orthonormal basis in which $g_{ij} = \delta_{ij}$ and $a_{ij} = \alpha_i \delta_{ij}$ hold. This basic is called a *joint principal orthonormal basis*. Note that we do not restrict the signature of the Ricci tensor and the tensor a_{ij}.

I. Hinterleitner [471] proved the following theorem:

Theorem 10.14 *Assume a compact Riemannian manifold (M, g) without boundary of dimension $n \geq 2$. If at any point $x \in M$ the sectional curvature $K(e_i, e_j)$ is non-positive and if there is a certain point $x_0 \in M$, where the sectional curvature $K(e_i, e_j)$ is negative in any two-direction $e_i \wedge e_j$ of all the principal orthonormal basis $\{e_1, \ldots, e_n\}$ of vectors of the main directions of the Ricci tensor, then any geodesic mapping of (M, g) is homothetic.*

10.3 On geodesic mappings with certain initial conditions

In this part we investigate geodesic mappings of n-dimensional (pseudo-) Riemannian manifolds $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ satisfying at a finite number of points

$$\overline{g}(X, Y) = f\, g(X, Y).$$

It follows that even under this weak condition it is true that the geodesic mapping is homothetic, see Chudá, Mikeš [212, 346, 347, 349, 657, 658].

10.3.1 On geodesic mappings with certain initial conditions

Here we studied geodesic mapping f between Riemannian spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ with the following initial condition $\overline{g}(f(x_o)) = k \cdot g(x_o)$, where g and $\overline{g}$ are the metrics of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$. We proved, if at the point $x_o \in \mathbb{V}_n$ the Weyl tensor of the projective curvature does not vanish, then f is homothetic.

The system (7.5) has no more than one solution for initial conditions at the point $x_o \in U$: $\overline{g}_{ij}(x_o) = \overline{g}^o_{ij}$, $\psi_i(x_o) = \psi^o_i$, $\mu(x_o) = \mu^o$. Evidently, if

$$\overline{g}_{ij}(x_o) = k \cdot g_{ij}(x_o),\ k = \text{const} \neq 0,\ \psi_i(x_o) = 0,\ \mu(x_o) = 0, \tag{10.26}$$

then the initial condition correspond to a trivial solution $\overline{g} = k \cdot g$.

Elementary, this solution is uniquely on whole manifold. It means, that $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ are homothetic.

It was proved pp. 278-279, that in a neighbourhood U, where the preimage space $\mathbb{V}_n$ $(n > 2)$ is not projective flat (this is equivalent with a statement: the Weyl tensor of projective curvature $W(x) \not\equiv 0$, $\forall x \in U$), the covector $\psi_i(x)$ will be expressed as a function of the components $\overline{g}_{ij}(x)$ (formula (7.11)). Thus $\psi_i = \mathcal{Z}_i(\overline{g})$ on $U \subset \mathbb{V}_n$ are the functions which depend on geometric objects of $\mathbb{V}_n$ and also components of the unknown metric tensor $\overline{g}$ of $\overline{\mathbb{V}}_n$.

It is known that for $n = 2$ the tensor of projective curvature $W(x) \equiv 0$. Therefore, the next results for $n \geq 3$ hold. It means, that the equations (7.5) are reduced in this neighborhood in the following way:

$$\overline{g}_{ij,k} = 2\mathcal{Z}_k(\overline{g})\, \overline{g}_{ij} + \mathcal{Z}_i(\overline{g})\, \overline{g}_{jk} + \mathcal{Z}_j(\overline{g})\, \overline{g}_{ik}. \tag{10.27}$$

It means, that (10.27) is the set of partial differential equations with unknown functions $\overline{g}_{ij}(x)$.

We proved the following:

Lemma 10.2 *Let $\mathbb{V}_n = (M, g)$ and $\overline{\mathbb{V}}_n = (M, \overline{g})$ be Riemannian spaces of the class C^3, $n \geq 3$, and let the Weyl tensor of projective curvature $W(x)$ be nonzero for all $x \in U$ in the coordinate neighborhood $U \subset M$.*

If the condition $\overline{g} = k \cdot g$ is satisfied in the point $x_o \in U$ and the spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ have the same geodesics, then the metrics g and $\overline{g}$ in U are homothetics, i.e. $\overline{g}(x) = k \cdot g(x)$, for all $x \in U$.

Proof. Let us suppose that the assumptions of Lemma 10.2 hold in neighbourhood U. Then the equations (7.5) hold and we get from them the system of Cauchy type (10.27). This set is a closed system of partial differential equations of Cauchy type with respect to the unknown functions $\overline{g}_{ij}(x)$.

For initial conditions $\overline{g}_{ij}(x_o) = k \cdot g_{ij}(x_o)$, where $x_o \in U$ there is no more than one unique solution.

On the other hand, $\overline{g} = k \cdot g$, $k = \text{const}$ is a trivial solution of the equations (10.27) and it satisfies the initial conditions (10.26). The given mapping is homothetic. □

The Lemma 10.2 implies:

Theorem 10.15 (J. Mikeš and H. Chudá [657]) *Let f be a geodesic mapping between Riemannian spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ with the condition $\overline{g}(f(\overline{x}_o)) = k \cdot g(x_o)$, where g and $\overline{g}$ are metrics of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$. If the Weyl tensor of projective curvature does not vanish in the point $x_o \in \mathbb{V}_n$ then f is a homothetic mapping, i.e. $\overline{g} = k \cdot g$, $k = \text{const}$.*

Remark 10.1 The condition $\overline{g}(f(\overline{x}_o)) = k \cdot g(x_o)$ can be evidently replaced by the following condition $a(x_o) = k \cdot g(x_o)$ where a is the tensor from the Sinyukov equation (8.5).

Proof. Let f be a geodesic mapping between Riemannian spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$. We suppose, that $\mathbb{V}_n$, $\overline{\mathbb{V}}_n \in C^3$ and $\overline{g}(f(x_o)) = k \cdot g(x_o)$.

Because $W(x_o) \neq 0$, then there exists a neighborhood U of the point x_o, so that $W(x) \not\equiv 0$, for $\forall x \in U$. It follows from Lemma 10.2 that there is only one solution of the Levi-Civita equation in the form: $\overline{g}(x) = k \cdot g(x)$, $k = \text{const}$, $\forall x \in U$.

It means that in the neighbourhood U the system of equations (7.5) has solution:

$$\overline{g}_{ij}(x) = k \cdot g_{ij}(x), \quad \psi_i(x) = 0, \quad \mu(x) = 0, \quad \text{for } x \in U. \tag{10.28}$$

These conditions guarantee that the system of equations in the point x_o has the initial condition (10.26). If equations (7.5) fulfil the initial condition (10.26) in the point $x_o \in U$ then the Riemannian spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ are homothetic. It follows from this that the initial conditions globally generate only trivial solutions $\overline{g} = k\, g$, $k = \text{const}$. These equations characterize homothetic mappings.

10.3.2 The first quadratic integral of a geodesic

There still exists an open question in the case when the Weyl tensor of projective curvature is vanishing at the point $x_o \in \mathbb{V}_n$. In the following, we introduce the answer of such questions. It will be shown, that this procedure is suitable not only for geodesic mappings but also for holomorphically projective mappings. This solution is connected with the existence of the first quadratic integral of geodesics.

Let $\mathbb{A}_n$ be a manifold with affine connection ∇. In $\mathbb{A}_n$ a *geodesic path* γ is defined; a geodesic path γ: $x = x(s)$ is an itegral curve on $\mathbb{A}_n$ satisfying equations (2.26): $\nabla_{\dot{x}}\dot{x} = 0$, where $\dot{x}$ is a tangent vector of γ, " $\dot{}$ " is a differentiation with respect to an affine parametr s of γ.

We consider the case when the equations of geodesics admit a (homogeneous) *first quadratic integral*, namely

$$\omega(\dot{x}, \dot{x}) = \text{const},$$

where ω is a symmetric bilinear form on $\mathbb{A}_n$. This bilinear form admits the *first quadratic integral of the geodesic* if only if it holds [9, 51]:

$$\nabla_X \omega(Z, Y) + \nabla_Y \omega(X, Z) + \nabla_Z \omega(Y, X) = 0 \tag{10.29}$$

for any vectors fields X, Y, Z on A_n.

If $\mathbb{A}_n$ is a (pseudo-) Riemannian manifold $\mathbb{V}_n$ with a metric tensor g, then evidently $\text{const} \cdot g$ generates the first quadratic integral of geodesics. This integral is called *trivial*.

If an equaffine manifold $\mathbb{A}_n$ admits non trivial geodesic mappings onto the Riemannian manifold $\overline{\mathbb{V}}_n$, then in $\mathbb{A}_n$ there exists the first quadratic integral of geodesic:

$$\omega = \exp(-4\,\Psi)\ \overline{g}, \tag{10.30}$$

where $\overline{g}$ is the metric tensor of $\overline{\mathbb{V}}_n$ and Ψ is a function generating form ψ $(= \nabla\Psi)$ from the Levi-Civita equation (7.1).

Moreover, if a Riemannian manifold $\mathbb{V}_n$ admits non trivial geodesic mappings onto $\overline{\mathbb{V}}_n$, then in $\mathbb{V}_n$ there exist two independent nontrivial first quadratic integrals of geodesic:

$$\omega = \exp(-4\,\Psi)\ \overline{g}, \quad \text{and} \quad \omega = a - 2\Lambda\, g. \tag{10.31}$$

Here g is a metric of $\mathbb{V}_n$, a is a symmetric form from the solution of the Sinyukov equation (8.6), and Λ is a function generating the form λ $(= \nabla\,\Lambda)$ from this equation.

10.3.3 On first quadratic integral of geodesics with initial conditions

In this section we consider first quadratic integral of geodesics on a domain V of (pseudo-) Riemannian manifold $\mathbb{V}_n$. We suppose that V is connected and ∂V is the Lipschitz boundary, i.e. the domain V lies on one side of ∂V.

We can restrict ourselves to the study of special domains V of the Riemannian manifold $\mathbb{V}_n$ with a metric g and V admitting a first quadratic integral of geodesics.

Let x be a point of int V. The metric g of $\mathbb{V}_n$ at this point is a bilinear mapping $g_x\colon\ T_x \times T_x \to \mathbb{R}$, where T_x is the tangent manifold at x, $\dim T_x = n$. On the other hand $g_x\colon\ S^2T_x \to \mathbb{R}$, where S^2T_x is the second symmetric power of T_x. Evidently, $\dim S^2T_x = N = \frac{1}{2}\,n(n+1)$.

We choose vectors $v_1,\ v_2,\ \ldots,\ v_N \in T_x$ in such a way that

1. $v_1 \circ v_1,\ v_2 \circ v_2,\ \ldots,\ v_N \circ v_N$ is a basis of S^2T_x,
2. $g_x(v_i, v_i) \neq 0$, for $\forall i = 1, 2, \ldots, N$.

Evidently it follows $\omega_x(v_i, v_i) = k\,g_x(v_i, v_i),\ \text{for } \forall i = 1, \ldots, N \iff \omega_x = k\,g_x$.

We shall construct N geodesics path $\gamma_i(s)$, $i = 1, \ldots, N$, for which $x \in \gamma_i$ and the vectors v_i are the tangent vectors of γ_i at the point x. We shall choose a point x_i on each of these geodesics γ_i and we suppose that all geodesic arcs (x, x_i) are subsets of V.

We denote by γ_{ij} the geodesic arc joing the points x_i and x_j. We suppose that γ_{ij} is a non isotropic geodesic, the arc (x_i, x_j) is a subset of V. The system of these geodesics γ_{ij} can be non complete. Howeover, every two points x_i and x_j can be joint by geodesic arcs of this systems.

Further we naturally suppose that there exists some neighbourhood $U \subset V$ of the point x such that for all $y \in U$ there exist non isotropic geodesic arcs $(\subset V)$ for which tangent vectors form a basis of S^2T_y. We get

Lemma 10.3 *If at the points x_i, $i = 1, 2, \ldots, N$, we have $\omega_{x_i} = k_i\,g_{x_i}$, then at the point x and in a neighbourhood U its holds $\omega = k(x)\,g$, where k is function, i.e. U admits only a trivial first quadratic integral of geodesics.*

From Lemma 10.3 evidently follows the more general Theorem.

Theorem 10.16 *Let V described above be a domain of the (pseudo-) Riemannian manifold $\mathbb{V}_n$ and there exist points $x, x_1, x_2, \ldots, x_N \in V$ as above. If V admits a first quadratic integral of geodesics ω, and in the points x_i, $i = 1, 2, \ldots, N$, satisfy conditions $\omega_{x_i} = k_i\,g_{x_i}$, then this first quadratic integral of geodesics is trivial, i.e. $\omega = \text{const} \cdot g$.*

From Theorem 10.16 and (10.30) it follows immediatelly

Theorem 10.17 *Let V described above be a domain of the (pseudo-) Riemannian manifold $\mathbb{V}_n$ and there exist the points $x, x_1, x_2, \ldots, x_N \in V$ as above. If $\mathbb{V}_n$ admits a geodesic mapping onto $\overline{\mathbb{V}}_n$, and in points x_i, $i = 1, 2, \ldots, N$, the metrics of $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ satisfy the conditions $\overline{g}_{x_i} = k_i\,g_{x_i}$, then this mapping is homothetic, $\overline{g} = \text{const}\ g$.*

10.4 Geodesic deformations of hypersurfaces in Riemannian spaces

As mentioned above, the theory of geodesic mappings of Riemannian spaces deals with diffeomorphisms which send every geodesic of one space just onto (an unparametrized) geodesic of the other (recall that we do not impose any restrictions on the signature of the Riemannian space under consideration).

In this chapter, we will be interested in such infinitesimal deformations of Riemannian (sub)spaces under which any geodesic is mapped to a curve approximating a geodesic (of the deformed subspace) with a given precision, i.e. up to some "precision under control" (the exact definition follows below). Keeping in mind possible situations which might occur, and their eventual mathematical models, this approach seems to be more appropriate for applications than the standard one. For example, for simulating real physical situations when evolution of gravity fields (electromagnetic fields, mechanical systems etc.) is considered.

10.4.1 Infinitesimal deformations of Riemannian spaces

Let $\mathbb{W}_m = (N_m, a)$ be a Riemannian manifold with local coordinates denoted by $(y^1, \dots, y^m)$, and metric tensor a. Let $f^1, \dots, f^m$ be (differentiable) functions of variables $x^1, \dots, x^n$, $n < m$, such that the Jacobi matrix is of rank n. Let

$$y^\alpha = f^\alpha(x^1, \dots, x^n), \qquad \operatorname{rank} \left\| \frac{\partial f^\alpha}{\partial x^i} \right\| = n < m \tag{10.32}$$

be local equations of an immersed submanifold $\mathbb{V}_n \subset \mathbb{W}_m$ with an induced Riemannian metric g. That is, $\mathbb{V}_n = (M_n, g)$ is a submanifold of $\mathbb{W}_m = (N_m, a)$. If $a_{\alpha\beta}$ and g_{ij} denote components of metric tensor a and g, respectively, then ([50]) the components are related by

$$g_{ij} = a_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}. \tag{10.33}$$

Hereafter the Greek indices $\alpha, \beta, \dots$ run through $\{1, \dots, m\}$, and the Latin indices $i, j, \dots$ run through $\{1, 2, \dots, n\}$.

Let ξ be a restriction onto $\mathbb{V}_n$ of some vector field defined on a Riemannian space W_m, the components being $\xi^\alpha(x^1, \dots, x^n)$. The equations

$$\widetilde{y}^\alpha = y^\alpha(x) + \varepsilon \xi^\alpha(x), \tag{10.34}$$

where ε is a small real parameter, define some Riemannian suspace $\widetilde{\mathbb{V}}_n$ of $\mathbb{W}_m$ which will be called an *infinitesimal deformation* of $\mathbb{V}_n$. The field $\xi(x)$ can be naturally called the *displacement field*, or the *displacement vector*, [427].

When considering $\widetilde{\mathbb{V}}_n$, we drop terms of order ε^2 and of higher orders assuming that either ε is sufficiently small, or the precision is sufficient.

Thus we consider a first order infinitesimal deformation. As usual, we will drop the words "of first order" to be short.

If $A = A(x^1, \dots, x^n)$ is a geometric object belonging to $\mathbb{V}_n$ (mostly, a tensor field, or a differential form) then the corresponding object belonging to $\widetilde{\mathbb{V}}_n$ will be denoted by $\widetilde{A} = \widetilde{A}(x^1, \dots, x^n)$. It can be presented in the form

$$\widetilde{A}(x, \varepsilon) = A(x) + \delta A \cdot \varepsilon + \delta^2 A \cdot \varepsilon^2 + \dots . \tag{10.35}$$

The value of the parameter of deformation ε in the formula (10.34) must be chosen so that the series (10.35) be convergent, whatever is characterized by $A(x)$.

Coefficients $\delta A, \delta^2 A, \dots$ of the decomposition (10.35) will be called the *first, second* etc. *variation of* A under an infinitesimal deformation of $\mathbb{V}_n$, [189]. As announced above, we will be interested in the theory in which such series are supposed to be "cutted" after the second member.

Thus consider a first order infinitesimal deformation (in short, deformation). For the metric tensor of the space $\widetilde{\mathbb{V}}_n$ we have

$$\tilde{g} = g + \varepsilon \delta g, \qquad \tilde{g}^{ij} = g^{ij} + \varepsilon \delta g^{ij}. \tag{10.36}$$

Let us find the tensor δg. Note that from (10.34) we get $\delta y^\alpha = \xi^\alpha$, and

$$a_{\alpha\beta}(\widetilde{y}) = a_{\alpha\beta}(y + \varepsilon\, \xi) = a_{\alpha\beta}(y) + \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} \xi^\gamma \cdot \varepsilon + \cdots ,$$

therefore $\delta a_{\alpha\beta} = \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} \xi^\gamma$. With respect to the coordinate transformations in $\mathbb{V}_n$, the functions y^α and ξ^α are invariant, hence their partial derivatives coincide with the covariant derivatives. By (10.33), we get

$$\begin{aligned} \delta g_{ij} &= \delta(a_{\alpha\beta} y^\alpha_{,i} y^\beta_{,j}) = \delta a_{\alpha\beta} y^\alpha_{,i} y^\beta_{,j} + a_{\alpha\beta} \delta y^\alpha_{,i} y^\beta_{,j} + a_{\alpha\beta} y^\alpha_{,i} \delta y^\beta_{,j} \\ &= \tfrac{\partial a_{\alpha\beta}}{\partial y^\gamma} \xi^\gamma y^\alpha_{,i} y^\beta_{,j} + a_{\alpha\beta} (\xi^\alpha_{,i} y^\beta_{,j} + y^\alpha_{,i} \xi^\beta_{,j}). \end{aligned} \tag{10.37}$$

The expression (10.37) holds for any infinitesimal deformation with respect to any coordinate system.

Definition 10.1 An infinitesimal deformation of $\mathbb{V}_n$ is said to be *geodesic* if this deformation preserves the geodesic curves of $\mathbb{V}_n$.

In other words, $\mathbb{V}_n$ and $\widetilde{\mathbb{V}}_n$, referred to the common coordinate system (x^i), admit geodesic maps onto each other. Then $\mathbb{V}_n$ and $\widetilde{\mathbb{V}}_n$ satisfy the Levi-Civita equations (8.1) and the equality (8.2),

$$\widetilde{g}_{ij,k} = 2\psi_k \widetilde{g}_{ij} + \psi_i \widetilde{g}_{jk} + \psi_j \widetilde{g}_{ik}, \tag{10.38}$$

where ψ_i is a gradient vector determined by the invariant (function)

$$\Psi = \frac{1}{2(n+1)} \ln \left| \frac{\widetilde{\mathcal{G}}}{\mathcal{G}} \right| , \quad \text{where} \quad \mathcal{G} \overset{\text{def}}{=} \det \|g_{ij}\| \quad \text{and} \quad \widetilde{\mathcal{G}} \overset{\text{def}}{=} \det \|\widetilde{g}_{ij}\|.$$

In our case

$$2(n+1)\Psi = \ln \left| \frac{\widetilde{\mathcal{G}}}{\mathcal{G}} \right| = \ln \left| 1 + \frac{\delta \mathcal{G}}{\mathcal{G}} \varepsilon \right| = \ln \left(1 + \frac{\delta \mathcal{G}}{\mathcal{G}}\, \varepsilon \right) = \varepsilon\, \frac{\delta \mathcal{G}}{\mathcal{G}} + \cdots ,$$

so we should change ψ_i in (10.38) into $\varepsilon\, \psi_i$.

By (10.36), the Levi-Civita equations give

$$\delta g_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}. \tag{10.39}$$

Conversely, if $\widetilde{g}_{ij} = g_{ij} + \varepsilon \delta g_{ij}$ satisfies (10.39), then this tensor also satisfies (10.38). For appropriate values of ε, the tensor $\widetilde{g}$ is nondegenerate. Thus, (10.39) holds true if and only if the deformation of $\mathbb{V}_n$ is geodesic. For convenience, the symmetric tensor δg will be denoted by h. Thus we have proven:

Theorem 10.18 [426, M.L. Gavrilchenko] *A Riemannian space $\mathbb{V}_n$ admits infinitesimally small deformations if and only if on $\mathbb{V}_n$, there exists a type $(0,2)$ symmetric tensor h such that*

$$\nabla h_Z(X,Y) = 2\psi(Z)g(X,Y) + \psi(X)g(Y,Z) + \psi(Y)g(X,Z) \tag{10.40}$$

for an exact 1-form ψ; i.e. $h_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}$ for a gradient vector ψ_i.

10.4.2 Geodesic deformations and geodesic maps

If $\psi_i = 0$ in (10.40), then $h_{ij,k} = 0$, which is possible in the following two cases:

1. $h_{ij} = c\, g_{ij}$, $c = \text{const}$, then, see (10.36), $\widetilde{g}_{ij} = g_{ij}(1 + \varepsilon\, c)$, i.e. the deformation is an infinitesimal homothety.
2. $h_{ij} \neq c\, g_{ij}$. Then there exists a covariantly constant symmetric tensor on $\mathbb{V}_n$.

From (10.38), it follows that in both cases $\widetilde{g} = c\, g$, i.e. the deformation is an area homothety. Let us call these cases *trivial*. The nontrivial case is characterized by $\psi \neq 0$. Then we can rewrite (10.40) as

$$(h_{ij} - 2\psi g_{ij})_{,k} = \psi_i g_{jk} + \psi_j g_{ik},$$

i.e., in $\mathbb{V}_n$ the tensor $a_{ij} = h_{ij} - 2\psi g_{ij}$ satisfies the basic equations of the theory of geodesic maps (8.6) for $\lambda_i = \psi_i$.

On the other hand, the tensor $h_{ij} = a_{ij} + 2\lambda g_{ij}$, where a_{ij} is a solution to (8.6), satisfies (10.40), hence the metric tensor of $\widetilde{\mathbb{V}}_n$ can be written as $\widetilde{g}_{ij} = (1 + 2\lambda\varepsilon)g_{ij} + \varepsilon a_{ij}$. Combining this with Theorem 10.18, we get

Theorem 10.19 *A Riemannian space $\mathbb{V}_n$ admits nontrivial infinitesimal geodesic deformations if and only if $\mathbb{V}_n$ admits nontrivial geodesic maps.*

This theorem was first proven in [823] but only for Riemannian space of first class (1971). From Theorem 10.19, using theorems 7.12, 7.14 due to Sinyukov and Mikeš, we get

Theorem 10.20 *Symmetric spaces, recurrent spaces, double symmetric spaces, double recurrent spaces, m-recurrent spaces, and semisymmetric spaces D_n^m of non-constant curvature do not admit nontrivial geodesic deformations.*

Note that the spaces D_n^m were introduced in [80].

To the contrary, for spaces of constant curvature and for equidistant spaces, geodesic deformations exist. In $\mathbb{E}_3$, only the Liouville surfaces admit geodesic deformations.

10.4.3 Geodesic deformations of subspaces of Riemannian spaces

The main problem on geodesic deformations is to study relations between the Riemannian space $\mathbb{V}_n$, its ambient space $\mathbb{W}_m$, and the displacement field $\xi(\xi^\alpha(x))$. If one can find the field ξ, the situation is very good. Note that we can easily write down equations for ξ^α. Indeed, substituting (10.37) to (10.39), we get

$$\begin{aligned}\frac{\partial^2 a_{\alpha\beta}}{\partial y^\gamma \partial y^\nu} y^\alpha_{,i} y^\beta_{,j} \xi^\gamma y^\nu_{,k} &+ \frac{\partial a_{\alpha\beta}}{\partial y^\gamma}\left(y^\alpha_{,i} y^\beta_{,j} \xi^\gamma_{,k} + y^\alpha_{,i} y^\beta_{,jk} \xi^\gamma + y^\alpha_{,ik} y^\beta_{,j} \xi^\gamma\right) \\ &+ \frac{\partial a_{\alpha\beta}}{\partial y^\gamma}\left(\xi^\alpha_{,i} y^\beta_{,j} + y^\alpha_{,i} \xi^\beta_{,j}\right) y^\gamma_{,k} \\ &+ a_{\alpha\beta}\left(\xi^\alpha_{,ik} y^\beta_{,j} + \xi^\alpha_{,i} y^\beta_{,jk} + y^\alpha_{,i} \xi^\beta_{,jk} + y^\alpha_{,ik} \xi^\beta_{,j}\right) \\ &= 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}.\end{aligned} \tag{10.41}$$

We have used the fact that $a_{\alpha\beta}$ is invariant in $\mathbb{V}_n$.

The equations (10.41) are rather complicated by the fact that the second covariant derivative $\xi^\alpha_{,ik}$ of the displacement field ξ is involved twice. We can avoid this situation by the following operation on i, j, k: $(i, j, k) + (i, k, j) - (j, k, i)$. Then we get

$$\begin{aligned}a_{\alpha\beta} y^\alpha_{,i} \xi^\beta_{,jk} + a_{\alpha\beta} \xi^\alpha_{,i} y^\beta_{,jk} + \Gamma_{\gamma\beta\alpha}(y^\alpha_{,i} y^\beta_{,j} \xi^\gamma_{,k} + y^\alpha_{,i} \xi^\beta_{,j} y^\gamma_{,k} + \xi^\alpha_{,i} y^\beta_{,j} y^\gamma_{,k}) + \\ + \xi^\gamma\left(\frac{\partial a_{\alpha\beta}}{\partial y^\gamma} y^\alpha_{,i} y^\beta_{,jk} + \frac{\partial \Gamma_{\nu\beta\alpha}}{\partial y^\gamma} y^\alpha_{,i} y^\beta_{,j} y^\nu_{,k}\right) = \psi_k g_{ij} + \psi_j g_{ik},\end{aligned} \tag{10.42}$$

where $\Gamma_{\alpha\beta\gamma}$ are the Christoffel symbols of $\mathbb{W}_m$, which are constructed via $a_{\alpha\beta}$. Equations (10.41) and (10.42) are equivalent. They provide necessary and sufficient conditions for infinitesimal geodesic deformations of $\mathbb{V}_n \subset \mathbb{W}_m$ with displacement field ξ.

10.4.4 Basic equations of geodesic deformations of hypersurfaces

Let $m = n + 1$, i.e. $\mathbb{V}_n$ is a hypersurface in $\mathbb{W}_m$. By (10.32), we can choose vectors $y^\alpha_{,i}$ for a basis of the tangent plane of $\mathbb{V}_n$. We assume that $\det \|g_{ij}\| \neq 0$, hence the normal to the hypersurface $\eta^\alpha(x)$ is non-isotropic [50], i.e.,

$$a_{\alpha\beta} y^\alpha_{,i} \eta^\beta = 0, \quad a_{\alpha\beta} \eta^\alpha \eta^\beta = e, \quad e = \pm 1. \tag{10.43}$$

The vectors $\{y^\alpha_{,i}, \eta^\alpha\}$ form a basis of the tangent plane of $\mathbb{W}_m$. Hence the displacement vector ξ^α can be uniquely written as

$$\xi^\alpha(x) = \lambda^i y^\alpha_{,i} + \mu\eta^\alpha, \tag{10.44}$$

where $\lambda(\lambda^i(x))$ is a vector, $\lambda^*(\lambda_j(x))$ is its dual (1-form), and $\mu(x)$ is a function ("invariant") in $\mathbb{V}_n$, with components related by

$$\lambda_j = g_{ij}\lambda^i = a_{\alpha\beta}\xi^\alpha y^\beta_{,j}, \quad \mu = e a_{\alpha\beta}\xi^\alpha \eta^\beta \tag{10.45}$$

(the conditions (10.45) give their geometric meaning).

Since $\mathbb{V}_n$ is a hypersurface in V_{n+1}, we have ([50]) the *Gauss' equations*:

$$y^{\alpha}_{,ij} = -\Gamma^{\alpha}_{\mu\nu} y^{\mu}_{,i} y^{\nu}_{,j} + e\Omega_{ij}\eta^{\alpha}$$

and the *Weingarten equations*:

$$\eta^{\alpha}_{,i} = -\Gamma^{\alpha}_{\mu\nu} y^{\mu}_{,i}\eta^{\nu} - \Omega^{k}_{i} y^{\alpha}_{,k},$$

where $\Omega(\Omega_{ij})$ is the second fundamental tensor of $\mathbb{V}_n$, $\Omega^{k}_{i} \overset{\text{def}}{=} \Omega_{ij}g^{jk}$, and $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols of $\mathbb{W}_m$.

Using (10.44), these formulas anable us to express $\xi^{\alpha}_{,i}$ and $\xi^{\alpha}_{,ij}$ with respect to the basis $\{y^{\alpha}_{,i}, \eta^{\alpha}\}$. For example,

$$\xi^{\alpha}_{,i} = \left(\lambda^{s}_{,i} - \mu\Omega^{s}_{i}\right) y^{\alpha}_{,s} - \lambda^{s}\Gamma^{a}_{\mu\nu} y^{\mu}_{,s} y^{\nu}_{,i} + \left(e\lambda^{s}\Omega_{si} + \mu_{,i}\right)\eta^{\alpha} - \mu\Gamma^{\alpha}_{\mu\nu} y^{\mu}_{,i}\eta^{\nu}. \quad (10.46)$$

After simple but cumbersome calculations, substituting all these expressions to (10.42) and taking into account that $\dfrac{\partial a_{\alpha\beta}}{\partial y^{\gamma}} = \Gamma_{\alpha\gamma\beta} + \Gamma_{\beta\gamma\alpha}$, we get

$$\lambda_{i,jk} = -\lambda_s R^{s}_{kij} + \psi_j g_{ik} + \psi_k g_{ij} + (\mu\Omega_{ij})_{,k} + (\mu\Omega_{ik})_{,j} - (\mu\Omega_{jk})_{,i}. \quad (10.47)$$

Theorem 10.21 *A Riemannian space $\mathbb{V}_n \subset \mathbb{W}_{n+1}$ admits infinitesimal deformations if and only if on $\mathbb{V}_n$ there exist a 1-form $\lambda^{*}(\lambda_i(x))$ ("vector" λ_i) and a function (invariant) μ satisfying (10.47).*

The system (10.47) coincides with (10.42), written with respect to the basis $\{y^{\alpha}_{,i}, \eta^{\alpha}\}$, hence (10.47) gives necessary and sufficient conditions for geodesic deformations of a hypersurface $\mathbb{V}_n$. We call (10.47) the *basic system*, [427].

By symmetrization with respect to indices i and j, (10.47) gives

$$(\lambda_{i,j} + \lambda_{j,i} - 2\mu\Omega_{ij})_{,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}. \quad (10.48)$$

By calculations similar to those for (10.41), we can prove that (10.47) is equivalent to (10.48). And (10.48) shows that for

$$a_{ij} = \lambda_{i,j} + \lambda_{j,i} - 2\mu\Omega_{ij} - 2\psi g_{ij},$$

(10.48) turns into (8.6). This again proves Theorem 10.19.

For the tensor h, given by $h_{ij} = \lambda_{i,j} + \lambda_{j,i} - 2\mu\Omega_{ij}$, (10.48) take the form (10.40). Finally, let us note the following. If $\mathbb{V}_n$ coincides with $\mathbb{W}_m$, then the equations (10.34) determine infinitesimal transformations of $\mathbb{V}_n$, which have been studied in [50, 80]. In this case $y^{\alpha} = x^{\alpha}$, in (10.44) $\mu = 0$, $\xi^{\alpha} = \lambda^{\alpha}$, and from (10.47) we get the well-known equation system for infinitesimal geodesic transformations:

$$\xi_{i,jk} = -\xi_s R^{s}_{kij} + \psi_i g_{jk} + \psi_j g_{ik}. \quad (10.49)$$

10.4.5 A system of equations of Cauchy type for geodesic deformations of a hypersurface

In order to find infinitesimal geodesic deformations of a hypersurface, we need to solve non-linear system (10.47) whose right-hand side contains, besides λ_i, the partial derivatives of unknown functions μ and ψ_i. This system can be reduced to the (equivalent) mixed tensor system of Cauchy type in n^2+3n+2 unknown functions. In [427], such a system was obtained for the case of non-degenerate second fundamental tensor $\Omega(\Omega_{ij})$ of a hypersurface.

In this section we deduce a system of Cauchy type and moreover show here, that this system exists under weaker conditions.

In what follows we assume that

$$\operatorname{rank}\|\Omega_{ij}\| > 2.$$

First, let us set

$$\lambda_{ij} = \lambda_{i,j}\,, \qquad \mu_i = \mu_{,i}\,. \tag{10.50}$$

By (10.50), the formula (10.47) can be rewritten as

$$\begin{aligned}\lambda_{ij,k} = {} & -\lambda_s R^s_{kij} + g_{ij}\psi_k + g_{ik}\psi_j + \mu(\Omega_{ij,k} + \Omega_{ik,j} - \Omega_{jk,i}) \\ & +\mu_k\Omega_{ij} + \mu_j\Omega_{ik} - \mu_i\Omega_{jk}.\end{aligned} \tag{10.51}$$

The integrability conditions for (10.51) are

$$\begin{aligned}\lambda_{sj}R^s_{ikl} + \lambda_{si}R^s_{jkl} = {} & \lambda_{s[k}R^s_{l]ij} + \lambda_s(R^s_{lij,k} - R^s_{kij,l}) + g_{ik}\psi_{j,l} - g_{il}\psi_{j,k} \\ & +\mu\Omega_{s(i}R^s_{j)kl} + (\mu\Omega_{ik})_{,jl} + (\mu\Omega_{jl})_{,ik} - (\mu\Omega_{il})_{,jk} - (\mu\Omega_{jk})_{,il},\end{aligned} \tag{10.52}$$

where the parentheses and brackets stand for symmetrization and alternation, respectively.

If we symmetrize (10.52) with respect to i and j, and then contract the result with g^{il} ($\|g^{ij}\| = \|g_{ij}\|^{-1}$), we get

$$n\psi_{j,k} = h_{js}R^s_k - h_{ps}R^s{}_{jk}{}^p + \omega g_{jk}, \tag{10.53}$$

where $R^s_k = R^s_{ijk}g^{ij}$, ω is a function, the indices are raised by the tensor g^{ij}, and $h_{ij} = \lambda_{(ij)} - 2\mu\,\Omega_{ij}$.

Let us find conditions for the unknown invariant ω. The integrability conditions for (10.53) reduce to

$$(n+3)\psi_s R^s_{jkl} = \psi_s R^s_{[k}g_{l]j} + h_{js}R^s_{[k,l]} - h_{sp}R^s{}_{j[k\cdot,l]}{}^p + g_{j[k}\omega_{,l]}.$$

Contracting the last expression with g^{jk}, we get

$$(n-1)\omega_{,l} = 2(n+1)\psi_s R^s_l + h_{js}R^s{}_{\cdot\,l,\cdot}{}^j - h_{sp}R^{sk}{}_{\cdot\,l\,\cdot,k}{}^p. \tag{10.54}$$

Now, using (10.53), we eliminate $\psi_{j,k}$ from (10.52), and get

$$\begin{aligned}&\lambda_{sp}A^{sp}_{ijkl} + \lambda_s B^s_{ijkl} + \mu_s C^s_{ijkl} + \omega D_{ijkl} + \mu E_{ijkl} \\ &= \mu_{,l}\Omega_{ik} - \mu_{,k}\Omega_{il} - \mu_{,l}\Omega_{jk} + \mu_{,k}\Omega_{jl},\end{aligned} \tag{10.55}$$

where

$$A^{sp}_{ijkl} \overset{\text{def}}{=} \delta^p_j R^s_{ikl} + \delta^s_i R^p_{jkl} + \delta^p_l R^s_{kij} + \delta^s_k R^p_{lji} - \frac{1}{n} g_{ik}(\delta^{(s}_j R^{p)}_l - R^{(s\ \ p)}_{\cdot\, jl\, \cdot}) + \tfrac{1}{n} g_{il}(\delta^{(s}_j R^{p)}_k - R^{(s\ \ p)}_{\cdot\, jk\, \cdot}),$$

$$B^s_{ijkl} \overset{\text{def}}{=} R^s_{kij,l} - R^s_{lij,k}, \qquad C^s_{ijkl} \overset{\text{def}}{=} \delta^s_{[i}\Omega_{j]\,[k,l]} + \delta^s_{[k}\Omega_{l]\,[i,j]},$$

$$D_{ijkl} \overset{\text{def}}{=} -\frac{1}{n}(g_{ik}g_{jl} - g_{il}g_{jk}),$$

$$E_{ijkl} \overset{\text{def}}{=} -\Omega_{s(i}R^s_{j)kl} - \Omega_{k\,[i,j]l} + \Omega_{l\,[i,j]k} + \frac{2}{n}\Omega_{js}g_{i[k}R^s_{l]} + \frac{2}{n}\Omega_{sp}R^{s\ \ p}_{\cdot\, j]k\, \cdot}\, g_{l]i}.$$

It is important to note that the last five tensors are determined only by the metric of $\mathbb{V}_n$, or by the metric and the tensor Ω.

Since $\operatorname{rank}\|\Omega_{ij}\| > 2$, it is evident that in $\mathbb{V}_n$ we have at least three non-coplanar vector fields a^i, b^i, c^i such that

$$\Omega_{ij}a^ia^j = e_a = \pm 1; \quad \Omega_{ij}b^ib^j = e_b = \pm 1;$$
$$\Omega_{ij}c^ic^j = e_c = \pm 1; \quad \Omega_{ij}a^ib^j = \Omega_{ij}a^ic^j = \Omega_{ij}b^ic^j = 0.$$

Note that these vectors are determined by Ω_{ij}. They are not determined uniquely, however we can consider them as objects of the space $\mathbb{V}_n$, i.e., defined by the metric and the second fundamental form of $\mathbb{V}_n$.

Let us multiply (10.55) in turn by a^ia^k, b^ib^k, and c^ic^k, and then sum it up by i and k. We get

$$e_a\mu_{j,l} = \overset{a}{\mu}_j\overset{a}{\Omega}_l + \overset{a}{\mu}_l\overset{a}{\Omega}_j - \overset{a}{\mu}\Omega_{jl} + \overset{1}{T}_{jl}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega), \qquad (4.25a)$$

$$e_b\mu_{j,l} = \overset{b}{\mu}_j\overset{b}{\Omega}_l + \overset{b}{\mu}_l\overset{b}{\Omega}_j - \overset{b}{\mu}\Omega_{jl} + \overset{2}{T}_{jl}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega), \qquad (4.25b)$$

$$e_c\mu_{j,l} = \overset{c}{\mu}_j\overset{c}{\Omega}_l + \overset{c}{\mu}_l\overset{c}{\Omega}_j - \overset{c}{\mu}\Omega_{jl} + \overset{3}{T}_{jl}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega), \qquad (4.25c)$$

where

$$\overset{a}{\mu}_j \overset{\text{def}}{=} \mu_{j,k}a^k; \quad \overset{b}{\mu}_j \overset{\text{def}}{=} \mu_{j,k}b^k; \quad \overset{c}{\mu}_j \overset{\text{def}}{=} \mu_{j,k}c^k;$$
$$\overset{a}{\mu} \overset{\text{def}}{=} \mu_{j,k}a^ja^k; \quad \overset{b}{\mu} \overset{\text{def}}{=} \mu_{j,k}b^jb^k; \quad \overset{c}{\mu} \overset{\text{def}}{=} \mu_{j,k}c^jc^k;$$
$$\overset{a}{\Omega}_l \overset{\text{def}}{=} \Omega_{lk}a^k; \quad \overset{b}{\Omega}_l \overset{\text{def}}{=} \Omega_{lk}b^k; \quad \overset{c}{\Omega}_l \overset{\text{def}}{=} \Omega_{lk}c^k;$$

and $\overset{\sigma}{T}$ $(\sigma = 1, 2, \ldots,)$ are tensors which linearly depend on their arguments. The coefficients of these tensors are determined by the metric g_{ij} and the second fundamental form Ω_{ij} of $\mathbb{V}_n$, as well as by the vectors a^i, b^i, c^i, which are also determined by Ω_{ij}. Using (4.25b), we eliminate the tensor $\mu_{j,l}$ from (4.25a), and get

$$e_a\left(\overset{a}{\mu}_j\overset{a}{\Omega}_l + \overset{a}{\mu}_l\overset{a}{\Omega}_j\right) - e_b\left(\overset{b}{\mu}_j\overset{b}{\Omega}_l + \overset{b}{\mu}_l\overset{b}{\Omega}_j\right) - \left(e_a\overset{a}{\mu} - e_b\overset{b}{\mu}\right)\Omega_{jl} + e_a\overset{1}{T}_{jl} - e_b\overset{2}{T}_{jl} = 0. \qquad (10.56)$$

Contracting (10.56) with c^jc^l, we get $(e_a\overset{a}{\mu} - e_b\overset{b}{\mu}) = e_c(e_a\overset{1}{T}_{jl} - e_b\overset{2}{T}_{jl})c^jc^l$. Then, contracting (10.56) with a^ja^l we have $\overset{a}{\mu} = \overset{4}{T}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega)$, and from the

above we get $\overset{b}{\mu} = \overset{5}{T}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega)$. Finally, contracting (10.56) with a^l, we arrive at

$$\overset{a}{\mu}_j = \alpha \overset{b}{\Omega}_j + \overset{6}{T}_j(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega), \tag{10.57}$$

where α is an invariant. In a similar way, from (4.25a) and (4.25c) we obtain

$$\overset{a}{\mu}_j = \beta \overset{c}{\Omega}_j + \overset{7}{T}_j(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega). \tag{10.58}$$

By subtracting (10.58) from (10.57), we get $\alpha\overset{b}{\Omega}_j - \beta\overset{c}{\Omega}_j + \overset{6}{T}_j - \overset{7}{T}_j = 0$. Contracting this equality with b^j, we find that $\alpha = \overset{8}{T}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega)$. Finally, (21a) takes the form:

$$\mu_{j,l} = \overset{9}{T}_{jl}(\lambda_{sp}, \lambda_s, \mu_s, \mu, \omega). \tag{10.59}$$

The system of equations (10.50), (10.51), (10.53), (10.54), and (10.59), call it (A) for short, gives linear expression for the first covariant derivatives with respect to $\lambda_{ij}, \lambda_i, \psi_i, \mu_i, \mu$, and ω in terms of the same tensors with coefficients uniquely determined by $\mathbb{V}_n$ and $\mathbb{W}_{n+1}$, i.e., we obtain a system of Cauchy type. Note that (A) is invariant under the choice of coordinates in $\mathbb{V}_n$. Thus, we have proved

Theorem 10.22 *A Riemannian subspace $\mathbb{V}_n \subset \mathbb{W}_{n+1}$, $\operatorname{rank}\|\Omega_{ij}\| > 2$, admits nontrivial geodesic deformations if and only if the equation system (A) has nontrivial solution.*

The integrability conditions for the equations in (A) are linear homogeneous algebraic equations in λ_{ij}, λ_i, ψ_i, μ_i, μ, ω. Let us denote the system of these equations by (B). The differential prolongations of any order for these equations also have the same form since (A) is linear. Let us denote the prolongations by (B_1), (B_2), etc.

By analogy with the corresponding result in the theory of geodesic maps [170], we obtain

Theorem 10.23 *A Riemannian space $V_n \subset V_{n+1}$ ($\operatorname{rank}\|\Omega_{ij}\| > 2$) admits nontrivial infinitesimal geodesic deformations if and only if the system of linear homogeneous algebraic equations (B), (B_1), (B_2), …, (B_s) ($s < n^2 + 3n + 2$) in λ_{ij}, λ_i, $\psi_i (\not\equiv 0)$, μ_i, μ, and ω has a solution.*

Note that the bound $n^2 + 3n + 2$ can be essentially decreased in case $\mathbb{V}_n$ has not constant curvature (see [118, 170, 220, 221, 542]).

11 APLICATIONS OF GEODESIC MAPPINGS

11.1 Applications of geodesic mappings to general relativity

11.1.1 Agreement on terminology

In this section we will consider an n-dimensional Riemannian manifold (M, g). We will use this terminology also for those manifolds on which the metric tensor g is not positive-definite. In general, those manifolds for which the metric g is not positive-definite are called pseudo-Riemannian, see Definition 3.1. We prefer here the term ***Riemannian*** because the properties we will assume apply to the proper Riemannian as well as to pseudo-Riemannian geometries. For physical reasons, a four-dimensional Riemannian manifold (M, g) with a metric tensor g of signature $(-+++)$ is called a ***space-time***. We will use this concept.

11.1.2 Killing-Yano tensors

The *Killing vector field* is defined on an n-dimensional Riemannian manifold $\mathbb{V}_n$ in Section 4.4.4. A reader can find a variety of applications of Killing vector fields in the classical monographs on the general relativity, see [67, 126, 133, 158] and etc.

The *Killing p-form* or, in another words, the *Killing-Yano tensor* of rank p $(1 \leq p \leq n-1)$ on $\mathbb{V}_n$ is a covariant skew-symmetric tensor field ω whose local components $\omega_{i_1 \cdots i_p}$ satisfy the *Killing-Yano equations*

$$\nabla_j \omega_{k\, i_2 \cdots i_p} + \nabla_k \omega_{j\, i_2 \cdots i_p} = 0.$$

This condition is in turn the generalization of the Killing equations to the case $p \geq 2$.

The equations of Killing-Yano tensor fields have been studied intensively in the physics literature in connection with its role generating quadratic first integrals of the geodesic equation (for example [321], [182, pp. 560-563]) and with its role in dynamics of spin 1/2 fermions in spinning particle models, supersymmetries and supergravity theories (see [434, 435]). In addition, a review of Killing-Yano tensors in relativity is given in [452].

In the next part of this section, we will solve the problem of finding integrals of equations determining the Killing-Yano tensor of rank p on an n-dimensional $(1 \leq p \leq n-1)$ Riemannian manifold of constant curvature using geodesic mappings.

11.1.3 Geodesic mappings and integrals of the Killing-Yano equations

As before, let $f\colon \overline{\mathbb{V}}_n \to \mathbb{V}_n$ be a smooth mapping of a Riemannian manifold $\overline{\mathbb{V}}_n = (\overline{M}, \overline{g})$ onto another Riemannian manifold $\mathbb{V}_n = (M, g)$ and let f_* be the differential of this mapping. For any covariant p-tensor field ω on $\mathbb{V}_n$, we can define the covariant p-tensor field $f^*\omega$ on $\overline{\mathbb{V}}_n$ by the equation $(f^*\omega)(X_1, \ldots, X_p) = \omega(f_*X_1, \ldots, f_*X_p)$ for arbitrary vector fields $X_1, \ldots, X_p$ on $\mathbb{V}_n$. If $\dim \overline{M} = \dim M$ and $f\colon \overline{\mathbb{V}}_n \to \mathbb{V}_n$ is a geodesic mapping then we have the following lemma (see [860]).

Lemma 11.1 *Let $f\colon \overline{\mathbb{V}}_n \to \mathbb{V}_n$ be a geodesic mapping of n-dimensional Riemannian manifolds and ω be a Killing-Yano tensor of* $\operatorname{rank} p$ $(1 \le p \le n-1)$ *on $\mathbb{V}_n$. Then the tensor field $\overline{\omega} = e^{-(p+1)\,\varphi}(f^*\omega)$ for $\varphi = \frac{1}{2(n+1)} \ln|\det g/\det \overline{g}|$ is a Killing-Yano tensor of rank p on $\overline{\mathbb{V}}_n$.*

It was proved in [859] that in Cartesian coordinates $\overline{x}^1, \ldots, \overline{x}^n$ of some neighbourhood $\overline{U} \subset \overline{M}$ of a locally flat manifold $\overline{\mathbb{V}}_n$ the components of a Killing-Yano tensor ω of rank p has the structure $\omega_{i_1\cdots i_p} = A_{k\,i_1\cdots i_p}\overline{x}^k + B_{i_1\cdots i_p}$ for arbitrary constant tensor $A_{k\,i_1\cdots i_p}$ and $B_{i_1\cdots i_p}$ which are skew-symmetric in all indices.

It is well known that spaces of constant curvature and only these spaces are geodesically diffeomorphic to locally flat manifolds (see Theorem 9.4, p. 319, and in addition, [50, pp. 134-135]). Then due to the Lemma 11.1, we have proved that the components of an arbitrary Killing-Yano tensor of rank p on Riemannian manifold $\mathbb{V}_n$ of constant curvature have the structure (see [860])

$$\omega_{i_1\cdots i_p} = e^{(p+1)\varphi}\,(A_{k\,i_1\cdots i_p}\overline{x}^k + B_{i_1\cdots i_p}), \tag{11.1}$$

where $\varphi = \frac{1}{2(n+1)} \ln|\det g|$. To prove this fact we choose $\overline{\mathbb{V}}_n$ to be a locally flat manifold with the metric tensor $\overline{g}_{ij} = \varepsilon\,\delta_{ij}$ where $\varepsilon = \pm 1$ and δ_{ij} is the Kronecker symbol. As a result $\mathbb{V}_n$ of Lemma 11.1 we obtain a manifold with constant curvature. We can formulate now the following theorem.

Theorem 11.1 *An arbitrary Killing-Yano tensor of* $\operatorname{rank} p$ $(1 \le p \le n-1)$ *has the form (11.1) in a special local coordinate system of an n-dimensional Riemannian manifold $\mathbb{V}_n$ of constant curvature.*

In particular, from (11.1) we conclude that the following corollary is true, see [695, 861].

Corollary 11.1 *On an n-dimensional Riemannian manifold $\mathbb{V}_n$ of constant curvature the dimension of the space of Killing-Yano tensors of* $\operatorname{rank} p$ $(1 \le p \le n-1)$ *is equal to*

$$\frac{(n+1)!}{(p+1)!\,(n-p)!}.$$

11.1.4 Closed conformal Killing-Yano tensors

We denote by η the volume element of Riemannian manifold $\mathbb{V}_n$ such that $\eta = \sqrt{|\det g|}\; dx^1 \wedge \ldots \wedge dx^n$ in the local coordinates $x^1, \ldots, x^n$. Then we

can define the Hodge operator $*$ setting $\omega \wedge (*\omega') = g(\omega \wedge \omega')\eta$ for any covariant skew-symmetric tensor fields ω and ω' of rank p, see [126, pp. 87-88], [23, p. 33]. Then, as was proved in [859], for an arbitrary Killing-Yano tensor field ω of rank p $(1 \leq p \leq n-1)$, the tensor field $\theta = *\omega$ is a closed conformal Killing-Yano tensor field of rank $(n-p)$. The converse statement is also true.

We recall that a closed conformal Killing-Yano tensor of rank p is determined by the equations

$$\nabla_k \omega_{i_1 i_2 \cdots i_p} = \frac{p}{n-p+1}\, g_{k[i_1} \nabla^j \omega_{|j|i_2 \cdots i_p]}.$$

In particular, for $p = 1$ the form ω is dual to a special concircular vector (see Definition 3.11, p. 140).

Closed conformal Killing-Yano tensors proved to be of great relevance in higher-dimensional General Relativity. Indeed, the equations of closed conformal Killing-Yano tensors has an important role for the study of hidden symmetries of higher-dimensional black holes (see [58, pp. 414, 426], [558]), the complete integrability of geodesic motion in Kerr-NUT-(anti)de Sitter spacetime (see [496]) and electrodynamics in the General Relativity theory (see [696, 859, 860]).

It was found in [859] that on a Riemannian manifold $\mathbb{V}_n$ of constant nonzero curvature C, an arbitrary closed conformal Killing tensor ω of rank p $(1 \leq p \leq n-1)$ has the form $\omega = -\frac{1}{pC}\nabla\theta$ for some Killing-Yano tensor θ of rank $p-1$. Then from the above identity (11.1) we obtain the following equations

$$\omega_{i_1 i_2 \cdots i_p} = -\frac{1}{C}\, e^{p\varphi}\, (\varphi_{[i_1} A_{|k|i_2 \cdots i_p]} \overline{x}^k + \varphi_{[i_1} B_{i_2 \cdots i_p]} + \frac{1}{p}\, A_{i_1 i_2 \cdots i_p}), \qquad (11.2)$$

where $\varphi = \frac{1}{2(n+1)} \ln |\det g|$, $\varphi_i = \partial_i \varphi$ and $A_{i_1 i_2 \cdots i_p}$ and $B_{i_2 \cdots i_p}$ are arbitrary constant tensors which skew-symmetric in all indices. In the view of this we conclude that we have the following theorem.

Theorem 11.2 *An arbitrary closed conformal Killing-Yano tensor of* rank p $(1 \leq p \leq n-1)$ *has the form (11.2) in a special local coordinate system of an n-dimensional Riemannian manifold $\mathbb{V}_n$ of constant curvature.*

In addition, from (11.2) we conclude that the following corollary is true (see [695, 861]).

Corollary 11.2 *On an n-dimensional Riemannian manifold $\mathbb{V}_n$ of nonzero constant curvature the dimension of the space of closed conformal Killing-Yano tensors of* rank p $(1 \leq p \leq n-1)$ *is equal to*

$$\frac{(n+1)!}{p!\,(n-p+1)!}.$$

11.1.5 Conformal Killing-Yano tensors

A skew-symmetric tensor field of rank p is called a conformal Killing-Yano tensor if it obeys the following equations:

$$\nabla_k \omega_{i_1 i_2 \cdots i_p} = \nabla_{[k} \omega_{i_1 i_2 \cdots i_p]} + 2 g_{k[i_1} \theta_{i_2 \cdots i_p]},$$

where $\theta_{i_2 \cdots i_p}$ is some skew-symmetric tensor of rank $p-1$. Contracting the latter equation with $g^{k i_1}$ one can see that θ is essentially the divergence of ω,

$$\theta_{i_2 \cdots i_p} = \frac{p}{2(n-p+1)} \nabla^k \omega_{k i_2 \cdots i_p}.$$

The definition of conformal Killing forms was first proposed by Tachibana and Kashiwada (see [529, 883]) as a generalization of the definition of conformal Killing vector fields (see p. 250 and [50, pp. 230-233], [567]). Since then these forms have had wide applications in physics related to hidden symmetries, conserved quantities, symmetry operators, or separation of variables, see [295, 326, 510, 862] [58, pp. 414, 426] and etc. In particular, Kamran and McLenaghan in [527] have shown that a symmetry operator for the massless Dirac equation can be made from a conformal Killing-Yano tensor.

At the same time, the conformal Killing tensors have been studied by many geometers, see [520, 552, 591, 782, 864] and etc.

For manifold $\mathbb{V}_n$ with constant sectional curvature $C \neq 0$, in [529, 883] it was obtained, by direct calculation, a decomposition of anarbitrary conformal Killing-Yano tensor field ω of rank p into direct sum $\omega = \omega_1 + \omega_2$ of a Killing-Yano tensor field ω_1 of rank p and of closed conformal Killing-Yano tensor field ω_2 of rank p such that $\omega_2 = -\frac{1}{pC} \nabla \theta$ for some Killing-Yano tensor field θ of rank $p-1$.

Based on a pointwise decomposition $\omega = \omega_1 + \omega_2$ the expression of an arbitrary conformal Killing-Yano tensor field ω in some special local coordinates $x^1, \ldots, x^n$ on a manifold $\mathbb{V}_n$ with nonzero constant curvature $C \neq 0$ read

$$\begin{aligned} \omega_{i_1 i_2 \cdots i_p} = e^{(p+1)\varphi} \Big(& A_{k\, i_1 \cdots i_p} \overline{x}^k + B_{i_1 \cdots i_p} - \\ & \frac{1}{C} \left(\varphi_{[i_1} D_{|k| i_2 \cdots i_p]} \overline{x}^k + \varphi_{[i_1} E_{i_2 \cdots i_p]} + \tfrac{1}{p}\, D_{i_1 i_2 \cdots i_p} \right) \Big), \end{aligned} \tag{11.3}$$

where $A_{k i_1 i_2 \cdots i_p}$, $B_{i_1 i_2 \cdots i_p}$, $D_{i_1 i_2 \cdots i_p}$ and $E_{i_2 \cdots i_p}$ are local components of constant skew-symmetric tensors. Now, we can compute the dimension of the vector space of conformal Killing-Yano tensor fields of rank p. Namely, the following theorem holds

Theorem 11.3 *An arbitrary conformal Killing-Yano tensor of* $\operatorname{rank} p$ $(1 \leq p \leq n-1)$ *has the form (11.3) in a special locally coordinate system of an n-dimensional Riemannian manifold $\mathbb{V}_n$ of constant curvature.*

In particular, from (11.3) we conclude that the following corollary is true, see [695, 861].

Corollary 11.3 *On an n-dimensional Riemannian manifold* $\mathbb{V}_n$ *of nonzero constant curvature the dimension of the space of conformal Killing-Yano tensors of* $\operatorname{rank} p$ $(1 \le p \le n-1)$ *is equal to*

$$\frac{(n+2)!}{(p+1)!\,(n-p+1)!}\,.$$

If ω is a conformal Killing-Yano tensor on Riemannian manifold with metric tensor g, then $\overline{\omega}$ is a conformal Killing-Yano tensor with conformally scaled metric $\overline{g}$, where the conformal weights are related by $\overline{g} = e^{2\varphi} g$ and $\overline{\omega} = e^{(p+1)\varphi}\omega$. It is well known, that a conformally flat Riemannian manifold is an Einstein manifold if and only if it is a manifold of constant sectional curvature. Using these propositions we can prove the following corollary.

Corollary 11.4 *On an n-dimensional conformally flat Einstein manifold* $\mathbb{V}_n$ *the dimension of the space of conformal Killing-Yano tensors of* $\operatorname{rank} p$ $(1 \le p \le n-1)$ *is equal to*

$$\frac{(n+2)!}{(p+1)!\,(n-p+1)!}.$$

11.1.6 The pre-Maxwell equations

Now let $\mathbb{V}_4$ be a space-time. The pre-Maxwell equations of relativistic electrodynamics are the equations of the following form (see [696, 862])

$$\nabla_i F_{jk} = \frac{4\pi}{3}\,(J_j g_{ik} - J_k g_{ij}), \tag{11.4}$$

where F_{ij} are the components of the electromagnetic field tensor and J_j are the components of the four-vector of the current J for $i, j, k, l, \ldots = 0, 1, 2, 3$. In particular, the equations (11.4) automatically imply the Maxwell equations in the presence of charge, $\nabla^k F_{kj} = 4\pi J_j$.

In the flat space-time with the local Lorentz coordinate system x^0, x^1, x^2, x^3, the integrals of Maxwell equations (11.4) are

$$F_{ij} = \frac{4\pi}{3}\,(C_j x_k - C_k x_j + C_{jk}),$$

where $J = \{C_0, C_1, C_2, C_3\}$, C_k are arbitrary constants, and $C_{jk} = -C_{kj}$. The vectors of the electric and magnetic field strengths E and H are subject to conditions

$$\frac{dE}{dt} = \{-C_1, -C_2, -C_3\}, \quad \frac{dH}{dt} = \{0, 0, 0\},$$

where $x^0 = t$. In particular. It was proved in [696] that the equations (11.4) are completely integrable and the tensor F of the electromagnetic field is

$$F_{jk} = \frac{4\pi\varepsilon}{3r^2}\nabla_j J_k \tag{11.5}$$

in the case of the de Sitter space-time, which is a space-time of constant curvature $C = \varepsilon/r^2$ with $\varepsilon = \pm 1$ depending on the kind of the space.

The equations (11.4) imply that, first, the tensor F is a closed conformal Killing-Yano tensor of rank 2 and, second, the four-vector J of current is a Killing vector field. Using Killing vector fields and closed conformal Killing-Yano tensors explicitly found above, we can then write integrals of the pre-Maxwell equations (11.4) in a de Sitter space-time in the form

$$F_{jk} = \frac{4\pi\varepsilon}{3r^2}\ e^{2\varphi}(\varphi_j(A_{kl}x^l + B_k) - \varphi_k(A_{jl}x^l + B_l) + A_{kj}),$$

where $J_j = \ e^{2\varphi}(A_{jk}x^k + B_j)$, $\varphi = \ln|\det g|/10$ and $\varphi_j = \partial_j\varphi$. Here, we must note that the dual electromagnetic tensor $*F$ then becomes the Killing-Yano tensor.

11.1.7 Operators of symmetries of Dirac equations

We call the equations of the form (see [789, 790])

$$D\Psi = \gamma^k P_k \Psi = m\Psi, \tag{11.6}$$

where D is the *Dirac operator*, m is a constant, $P_k = i(\nabla_k + \Gamma_k)$, and Γ_k is the spinor density on which we impose the conditions $[P_k, \gamma_i] = 0$ and $Sp(\Gamma_k) = 0$, the *Dirac equation* in the four-dimensional Riemannian manifold $\mathbb{V}_n$. The Dirac matrices are defined as an arbitrary, but fixed, solution of the system $\gamma^i\gamma^j + \gamma^j\gamma^i = 2\,g^{ij}\,\mathrm{Id}_{TM}$.

To study the algebra of the Dirac equation, we must find symmetry operators L that, by equation, commute with the Dirac operator, i.e. $[D, L] = 0$. The general form of such operators for Dirac equation (11.6) is a linear combination of the independent operators

$$\begin{aligned}
L_1 &= \mathrm{Id}_{TM}\xi^k P_k - \tfrac{i}{4}\ \gamma^{kl}\nabla_l\xi_k,\\
L_2 &= (*\gamma_j)\omega^{kj}P_k + \tfrac{i}{3}\ \gamma_j\nabla_k(*\omega^{kj}),\\
L_3 &= 2(*\gamma)\gamma^{lk}\theta_k P_l + \tfrac{3i}{4}\ \gamma\nabla_k\theta^k,
\end{aligned}$$

where $\gamma^{kl} = \frac{1}{2}\ [\gamma^k, \gamma^l]$, $*\gamma = -\frac{1}{4!}\ \eta_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l$, $*\gamma_i = -\frac{1}{3!}\ \eta_{ijkl}\gamma^j\gamma^k\gamma^l$ and, eventually, $*\omega_{ij} = \frac{1}{2}\ \eta_{ijkl}\omega^{kl}$.

The vector field ξ is Killing in the operator L_1, the tensor field ω is Killing-Yano in the operator L_2, and the vector field θ is the specially concircular vector field in the operator L_3, see Definition 3.9.

Using Killing vectors and Killing-Yano tensors explicitly found above, we can construct symmetry operators L and study the structure of algebra of symmetries of Dirac equation in a four-dimensional Riemannian manifold of constant curvature (see [546]).

Remark. Conformal Killing-Yano tensors also give rise to symmetry operators for Maxwell's equations. In four dimensions Kalnins, McLenaghan and Williams have founded that such operators are naturally formulated in terms of conformal Killing vectors, conformal Killing-Yano tensors and spinors, see [526].

11.2 Three invariant classes of the Einstein equations and geodesic mappings

11.2.1 The Einstein equations

Let (M, g) be a space-time with its Einstein equations

$$R_{ij} - \frac{R}{2} g_{ij} = T_{ij} \tag{11.7}$$

where R_{ij} are components of Ricci tensor Ric, $R = R_{ij}g^{ij}$ is scalar curvature and T_{ij} are components of the *energy-momentum tensor* T, see [182]. The equations (11.7) are complemented by *conservation laws* which are derived from the Einstein equations on the basis of the Bianchi identities, see p. 84.

In addition, we remark that ∇T belongs to the vector space $T_x^* \otimes S^2 T_x^* M$ at each point x of (M, g), where $S^2 T_x^* M$ is a vector space of covariant symmetric tensors of rank 2.

11.2.2 Invariantly defined seven classes of the Einstein equations

Let $\mathbb{V}$ be a finite dimensional pseudo-Euclidian vector space over $\mathbb{R}$ and

$$\Omega(\mathbb{V}) = \{T \in \mathbb{V}^* \otimes S^2\mathbb{V} | T(a,b,c) = T(a,c,b) \colon \sum_{i=1}^{n} \Omega(e_i, e_i, c) = 0\}$$

for $a, b, c \in \mathbb{V}$ and orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{V}$. Next put

$$\Omega_1(\mathbb{V}) = \{T \in \Omega(\mathbb{V}) \colon T(a,b,c) + T(b,c,a) + T(c,a,b) = 0\},$$
$$\Omega_2(\mathbb{V}) = \{T \in \Omega(\mathbb{V}) \colon T(a,b,c) = T(b,a,c)\},$$
$$\Omega_3(\mathbb{V}) = \{T \in \Omega(\mathbb{V}) \colon T(a,b,c) = \mu(a)\langle b,c\rangle + \mu(b)\langle a,c\rangle + \mu(c)\langle a,b\rangle\}.$$

Theorem 11.4 (see [857]) *The space $\Omega(\mathbb{V})$ is the orthogonal direct sum of the subspaces $\Omega_i(\mathbb{V})$, $i = 1, 2, 3$. Moreover these spaces are invariant and irreducible under the action of pseudo-orthogonal group of $\mathbb{V}$.*

This theorem implies that there are in general six invariant subspaces and so this leads to

Definition 11.1 (see [857]) Let $J(\mathbb{V})$ be invariant subspace of $\Omega(\mathbb{V})$. We say that the *energy-momentum tensors* J of (M, g) is of type J when $T(x) \in J(T_x M)$ for all $x \in M$.

Based on the above, we can define the six classes of the Einstein equations Ω_1, Ω_2, Ω_3, $\Omega_1 \oplus \Omega_2$, $\Omega_2 \oplus \Omega_3$ and $\Omega_1 \oplus \Omega_3$. In addition, the seven class is defined by the following condition $\nabla T = 0$, see [327].

In the next sections we will consider three interesting classes of the Einstein equations (11.7), which is denote by Ω_1, Ω_2 and Ω_3. An arbitrary space-time with the Einstein equation (11.7) from these classes is an *Einstein like manifold*, see [23, Chapter 16].

11.2.3 Einstein like manifolds of Killing type

The class Ω_1 of the Einstein equations is selected via condition

$$\nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ki} = 0, \tag{11.8}$$

it means that T is a *symmetric Killing tensor*, see [182]. In this case, a space-time (M, g) is called *Einstein-like manifold of Killing type*, see [863].

From (11.8) we obtain that the scalar curvature $R = \text{const}$. In this case (11.8) can be rewritten in the form.

$$\nabla_k R_{ij} + \nabla_i R_{jk} + \nabla_j R_{ki} = 0. \tag{11.9}$$

The converse is obvious, that is, from the equations (11.9) we obtain that the scalar curvature R is constant, and the equations (11.9) are satisfied automatically. From (11.9) we conclude that an arbitrary geodesic line $x^k = x^k(s)$ admits the following first quadratic integral $R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{const}$, see p. 364. Then we have the following theorem, see [857] and [863].

Theorem 11.5 *A space-time (M, g) is an Einstein like manifold of Killing type if and only if equations of an arbitrary geodesic line $x^k = x^k(s)$ admit the following first quadratic integral*

$$R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \text{const}\,.$$

11.2.4 Einstein like manifolds of Codazzi type

The class Ω_2 of the Einstein equations is selected via condition

$$\nabla_k T_{ij} = \nabla_i T_{kj}, \tag{11.10}$$

it means that T is a *Codazzi tensor*, [23, pp. 436-440]. In this case (M, g) is called *Einstein-like manifold of Codazzi type*, see [863]. From (11.9) we obtain that the scalar curvature $R = \text{const}$. In this case (11.10) can be rewritten in the form

$$\nabla_k R_{ij} = \nabla_i R_{kj}. \tag{11.11}$$

Let W be the *Weyl projective curvature tensor* of (M, g), see Definition 6.2. As well known, the Weyl projective curvature tensor W is an *invariant of geodesic mappings*, see Theorem 6.5, p. 266.

We will say that (M, g) has *harmonic Weyl projective curvature tensor* if W is a coclosed and closed tensor field, i.e. $\delta W = 0$ and $dW = 0$, see [23, p. 447]. If W^l_{ijk} are local components of W, then the condition $\delta W = 0$ is equivalent to $\nabla^i W^l_{ijk} = 0$. In this case, the components R_{ij} of the Ricci tensor Ric satisfy the equations (11.9). The converse is also true. In turn, the condition $dW = 0$ is equivalent to the following equations $\nabla_l W^h_{ijk} + \nabla_j W^m_{ikl} + \nabla_k W^h_{ilj} = 0$, which is a consequence of (11.9).

We can formulate the following theorem as a consequence of the above arguments.

Theorem 11.6 *A space-time (M, g) is an Einstein like manifold of Codazzi type if and only if its Weyl projective curvature tensor is harmonic.*

In conclusion, we remark that if (M, g) is an Einstein like manifold of Codazzi type which admits projective transformations then these transformations are affine, see [118].

11.2.5 The class of Einstein like manifolds of Sinyukov type

The Class of the Einstein equations is selected via condition

$$\nabla_k T_{ij} = \mu_k g_{ij} + \nu_i g_{jk} + \nu_j g_{ik},$$

it means that (M, g) is called an *Einstein-like manifold of Sinyukov type,* see [863]. An arbitrary Einstein-like manifold of Sinyukov type is a non trivial example of a Riemannian manifold with non-constant sectional curvature that always admits a non-trivial geodesic mapping onto another Riemannian manifold, see (8.6), p. 297, and [170, Chapter III]. Moreover, the metric form ds^2 of (M, g) has the form of a *warped-product* space-time

$$ds^2 = g_{00}(x^0)^2 + f(x^0) \cdot \sum_{a,b=1^3} g_{ab}(x^1, x^2, x^3)\ dx^a dx^b,$$

where the function $f(x^o)$, components $g_{oo}(x^o)$ and $g_{ab}(x^1, x^2, x^3)$ of the metric form ds^2 satisfy some particular properties which can be found in the papers [413] and [170].

12 F-PLANAR MAPPINGS AND TRANSFORMATIONS

12.1 On F-planar mappings of spaces with affine connections

In many papers geodesic mappings and their generalizations, like quasigeodesic, holomorphically-projective, F-planar, 4-planar mappings, were considered. One of the basic tasks appears to be deriving the fundamental equations of these mappings. They were found in various ways, see [19, 118, 119, 122, 170, 198, 404, 743], etc.

Unless otherwise specified, all spaces, connections and mappings under consideration are differentiable of a sufficiently high class. The dimension n of the considered spaces being is higher than **two**, as a rule. This fact is not explicitely stipulated. All spaces are assumed to be connected.

Here we show a method that simplifies and generalizes many of the results. Our results are valid also for infinite dimensional spaces with Banach bases.

12.1.1 Definitions of F-planar curves and F-planar mappings

We consider an n-dimensional manifold $\mathbb{A}_n$ with a torsion-free affine connection ∇, and an affinor structure F, i.e. a tensor field of type (1,1).

Definition 12.1 (J. Mikeš, N.S. Sinyukov [690]**)** A curve ℓ, which is given by the equations

$$\ell = \ell(t), \quad \lambda(t) = d\ell(t)/dt \ (\neq 0), \quad t \in I \tag{12.1}$$

where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains, under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

In accordance with this definition, ℓ is F-planar if and only if the following condition holds [690]:

$$\nabla_{\lambda(t)}\,\lambda(t) = \varrho_1(t)\,\lambda(t) + \varrho_2(t)\,F\lambda(t), \tag{12.2}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

In particular, if $F = \varrho\,\mathrm{Id}$ or a function $\varrho_2 \equiv 0$ we obtain the definition of a geodesic parametrized by an arbitrary parameter. Here ϱ is a function and Id is the identity operator.

A.Z. Petrov's quasi-geodesic curves [743], the analytic curves of Kähler, hyperbolic Kähler, and parabolic Kähler spaces provide examples of F-planar curves [119, 690], see p. 418.

We suppose two spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ with torsion-free affine connections ∇ and $\overline{\nabla}$, respectively. Affine structures F and $\overline{F}$ are defined on $\mathbb{A}_n$, resp. $\bar{\mathbb{A}}_n$.

Definition 12.2 (J. Mikeš, N.S. Sinyukov [690]**)** A diffeomorphism f between two manifolds $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ with affine connections is called *F-planar* if any F-planar curve in $\mathbb{A}_n$ is mapped onto an $\overline{F}$-planar curve in $\bar{\mathbb{A}}_n$.

Due to the diffeomorphism f we always suppose that ∇, $\overline{\nabla}$, and the affinors F, $\overline{F}$ are defined on M where $\mathbb{A}_n = (M, \nabla, F)$ and $\bar{\mathbb{A}}_n = (M, \overline{\nabla}, \overline{F})$. Moreover, we always identify a given curve $\ell\colon I \to \mathbb{A}_n$ and its tangent vector function $\lambda(t)$ with their images $\overline{\ell} = f \circ \ell$ and $\overline{\lambda} = f_*(\lambda(t))$ in $\bar{\mathbb{A}}_n$.

Two principially different cases are possible for the investigation:

$$\text{a)} \quad \overline{F} = a\,F + b\,Id; \tag{12.3}$$

$$\text{b)} \quad \overline{F} \neq a\,F + b\,Id, \tag{12.4}$$

a, b are some functions.

Naturally, case a) characterizes F-planar mappings *which preserve F-structures*. In case b) the structures of F and $\overline{F}$ are essentially distinct. The following holds.

Theorem 12.1 *An F-planar mapping f from $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ preserves F-structures and is characterized by the following condition*

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,FY + \varphi(Y)\,FX \tag{12.5}$$

for any vector fields X, Y, where $P = \overline{\nabla} - \nabla$ is the deformation tensor field of f, ψ, φ are some linear forms.

In local notation formula (12.5) has the following form:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i \tag{12.6}$$

where Γ^h_{ij}, $\overline{\Gamma}^h_{ij}$, F^h_i, ψ_i, φ_i are components of ∇, $\overline{\nabla}$, F, ψ, and φ.

Let us recall that on each tangent space $T_x\mathbb{A}_n$, $P(X,Y)$ is a symmetric bilinear mapping $T_x\mathbb{A}_n \times T_x\mathbb{A}_n \to T_x\mathbb{A}_n$, and a tensor field of type $(1,2)$.

Theorem 12.1 was proved by J. Mikeš and N.S. Sinyukov [690] for finite dimension $n > 3$. I. Hinterleitner and J. Mikeš [478] found a more concise proof for $n > 3$ and a proof for $n = 3$. Here we can show a more rational proof of this Theorem for $n \geq 3$. A counterexample was shown for $n = 2$, see [478]. For this reason in the following we assume that dimension $n \geq 3$.

This means that we are using an alternative definition of F-planar mappings:

Definition 12.3 A diffeomorphism f from $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ is called an *F-planar mapping* if it satisfies the condition (12.5).

12.1.2 Preliminary lemmas of linear and bilinear forms and operators

Inspired by the Proof 3, p. 262, we will formulate the following lemmas, by which we can effectively prove the fundamental equations of F-planar mappings.

Let V be a real n-dimensional vector space, $n \geq 2$.

Lemma 12.1 *Let a and b be linear forms on V, and ϱ a function on V. If $a(\lambda) = \varrho(\lambda)\, b(\lambda)$ for any vector λ, then $a = \alpha \cdot b$, where α is a constant, and $\varrho(\lambda) = \alpha$ for all λ for which $b(\lambda) \neq 0$.*

Proof. We can suitably transform coordinates, so that $\tilde{b}_i = \delta_i^1$ and the equation $a(\lambda) = \varrho(\lambda)\, b(\lambda)$ has the form: $\tilde{a}_i \tilde{\lambda}^i = \varrho(\lambda)\, \delta_i^1 \tilde{\lambda}^i$. We sequentially consider $\tilde{\lambda}^i = \delta_j^i$ for $j = 1, 2, \ldots, n$ and obtain $\tilde{a}_i = \alpha\, \delta_i^1$ $(= \alpha\, \tilde{b}_i)$. It follows that $a = \alpha\, b$. Evidently, $\varrho(\lambda) = \alpha$ for all λ for which $b(\lambda) \neq 0$. The case $b = 0$ is trivial. □

Lemma 12.2 *Suppose $Q\colon V \to V$ and $F\colon V \to V$ are linear mappings, $\varrho(\lambda)$ is a function on V. If $Q(\lambda) = \varrho(\lambda)\, F(\lambda)$ for any vector $\lambda \in V$ holds, then there is a constant α such that the condition $Q = \alpha\, F$ holds and $\varrho(\lambda) = \alpha$ for all λ for which $F(\lambda) \neq 0$.*

Proof. Our formula has the following coordinate expression $Q_\alpha^h \lambda^\alpha = \varrho(\lambda)\, F_\alpha^h \lambda^\alpha$, where λ^i, F_i^h and Q_i^h are the components of λ, F and Q.

This formula is contracted with the vector u_h for which $u_h F_i^h \neq 0$. We obtain $(u_h Q_\alpha^h)\lambda^\alpha = \varrho(t)\, (u_h F_\alpha^h)\lambda^\alpha$. From Lemma 12.1 we get $\varrho(\lambda) = \alpha = \text{const}$, for all λ with $(u_h F_\alpha^h)\lambda^\alpha \neq 0$. Hence $Q = \alpha \cdot F$. Moreover holds $\varrho = \alpha$ for all λ for which $F_\alpha^h \lambda^\alpha \neq 0$. □

Lemma 12.3 *Suppose $Q\colon V \to V$ and $F\colon V \to V$ are linear mappings, $\varrho_1(\lambda)$, $\varrho_2(\lambda)$ are functions on V, $n \geq 3$. If, for each vector $\lambda \in V$*

$$Q(\lambda) = \varrho_1(\lambda)\, \lambda + \varrho_2(\lambda)\, F(\lambda) \tag{12.7}$$

holds, then there are constants α and β such that the condition $Q = \alpha\, \mathrm{Id} + \beta\, F$ holds, moreover, $\varrho_1(\lambda) = \alpha$ and $\varrho_2(\lambda) = \beta$ for all λ for which $F(\lambda) \neq 0$.

Proof. Formula (12.7) has the following coordinate expression

$$Q_\alpha^h \lambda^\alpha = \varrho_1(\lambda)\, \lambda^h + \varrho_2(\lambda)\, F_\alpha^h \lambda^\alpha, \tag{12.8}$$

where λ^i, F_i^h and Q_i^h are the components of λ, F and Q.

1. Suppose F_i^h has an eigenvector u_h with eigenvalue $\varrho \in \mathbb{R}$, i.e. $u_h F_i^h = \varrho\, u_i$. Contracting formula (12.8) with u_h we obtain

$$u_h Q_\alpha^h \lambda^\alpha = (\varrho_1(\lambda) + \varrho\varrho_2(\lambda))\, u_\alpha \lambda^\alpha.$$

From Lemma 12.1 we get $\varrho_1(\lambda) + \varrho\varrho_2(\lambda) = \alpha$, where α is a constant, for all λ with $u_\alpha \lambda^\alpha \neq 0$. Then (12.8) has the form $(Q_\alpha^h - \alpha\, \delta_\alpha^h)\lambda^\alpha = \varrho_2(\lambda)(F_\alpha^h - \varrho\, \delta_\alpha^h)\lambda^\alpha$, and again with the help of Lemma 12.2 follows that $Q_\alpha^h - \alpha\, \delta_\alpha^h = \beta\, (F_\alpha^h - \varrho\, \delta_\alpha^h)$ and $\varrho_2(\lambda)$ is a constant β. From this follows the validity of the Lemma.

It is crucial that the dimension n is odd, then there always exists an eigenvector of F with eigenvalue $\alpha \in \mathbb{R}$. Among others, therefore the Lemma holds for $n = 3$.

2. We can not use 1, we proceed in the following way. It is natural to assume that there exists some vector ξ^h such that $F^h \overset{\text{def}}{=} \xi^\alpha F^h_\alpha$ and ξ^h are non collinear, otherwise ξ is an eigenvector with a real eigenvalue, and we can assume that $n > 3$.

By multiplying (12.8) with $\lambda^i F^j_\alpha \lambda^\alpha$ and antisymmetrizing the indices h, i and j we obtain

$$\left\{ Q^{[h}_\alpha \delta^i_\beta F^{j]}_\gamma \right\} \lambda^\alpha \lambda^\beta \lambda^\gamma = 0, \tag{12.9}$$

where square brackets denote the alternation of indices. The term in curly brackets does not depend on λ and (12.9) holds for any vector $\lambda \in V$, therefore

$$Q^{[h}_{(\alpha} \delta^i_\beta F^{j]}_{\gamma)} = 0 \tag{12.10}$$

holds, where the round brackets denote symmetrization of indices.

Introducing $Q^h \overset{\text{def}}{=} Q^h_\alpha \xi^\alpha$, we contract (12.10) with $\xi^\alpha \xi^\beta \xi^\gamma$. Since $F^h \nparallel \xi^h$, we obtain $Q^h = 2a\,\xi^h + 2b\,F^h$, where a, b are certain constants. Contracting (12.10) with $\xi^\beta \xi^\gamma$, and taking into account the precending, we have

$$Q^h_i = a\,\delta^h_i + b\,F^h_i + a_i\,\xi^h + b_i\,F^h, \tag{12.11}$$

where a_i, b_i are some components of some linear forms.

In case that $a_i = b_i = 0$, evidently from (12.19) we obtain the formula $Q = \alpha \mathrm{Id} + \beta\,F$.

Now we will suppose that either $a_i \neq 0$, or $b_i \neq 0$. Since ξ^h and F^h are non collinear, it is evident that

$$\xi^h a_i + F^h b_i \neq 0. \tag{12.12}$$

By virtue of (12.11) formula (12.10) has the form

$$\Omega^{[hi}_{(\alpha\beta} F^{j]}_{\gamma)} = 0, \tag{12.13}$$

where $\Omega^{hi}_{\alpha\beta} = (\xi^h a_\alpha + F^h b_\alpha)\delta^i_\beta - (\xi^i a_\alpha + F^i b_\alpha)\delta^h_\beta$. It is possible to show that there exists some vector ε^h for which $\Omega^{hi}_{\alpha\beta}\varepsilon^\alpha\varepsilon^\beta \neq 0$, otherwise (12.12) would be violated.

Contracting (12.13) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma$, we have $F^h_\alpha \varepsilon^\alpha = a\,\xi^h + b\,F^h + c\,\varepsilon^h$, with a, b, c being constants. Analogously, contracting (12.13) with $\varepsilon^\beta\varepsilon^\gamma$, we obtain that F^h_i is represented in the following manner: $F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h + c_i\,\varepsilon^h$, where a_i, b_i, c_i are components of 1-forms.

Because for $n > 3$ exists a non-zero vector u_h with $u_h\xi^h = u_h F^h = u_h\varepsilon^h = 0$, then from the last formula follows $u_\alpha F^\alpha_i = a\,u_i$ and the vector u_h is an eigenvector of F, which in this case we do not expect. Evidently Lemma 12.3 is proved. □

Lemma 12.4 *Let a and b be a bilinear and a linear form on V, and ϱ is a function on V. If $a(\lambda,\lambda) = \varrho(\lambda)\, b(\lambda)$ for any vector λ, then $a(\lambda,\lambda) = \tilde{\varrho}(\lambda)\, b(\lambda)$ where $\tilde{\varrho}$ is a linear form, and $\varrho(\lambda) = \tilde{\varrho}(\lambda)$ for λ where $b(\lambda) \neq 0$.*

Proof. We write the condition $a(\lambda,\lambda) = \varrho(\lambda)\, b(\lambda)$ in coordinates: $a_{ij}\lambda^i\lambda^j = \varrho\, b_i\lambda^i$. Let $b_1 \neq 0$ (otherwise we suitably rearrange indices). We introduce on V new coordinates $\Lambda^1 = b_i\lambda^i$ and $\Lambda^i = \lambda^i$ for $i > 1$. Then this condition is written in the following form $\tilde{a}_{ij}\Lambda^i\Lambda^j = \varrho\,\Lambda^1$, where $\tilde{a}_{ij}$ are new components of a. From $\Lambda^1 = 0$ and arbitrary Λ^i, $i > 1$, follows that $\tilde{a}_{ij} = 0$ for all $i, j > 1$. We have $\tilde{a}_{11}\Lambda^1\Lambda^1 + 2\sum\limits_{j=2}^{n} \tilde{a}_{1j}\Lambda^1\Lambda^j = \varrho\,\Lambda^1$. Finally, $\varrho = \tilde{a}_{11}\Lambda^1 + 2\sum\limits_{j=2}^{n} \tilde{a}_{1j}\Lambda^j = \tilde{\varrho}_i\Lambda^i$, and $\tilde{\varrho}$ is a linear form on V, and $a(\lambda,\lambda) = \tilde{\varrho}(\lambda)\, b(\lambda)$. □

In the sequel we shall need the following lemma:

Lemma 12.5 *Let V be an n-dimensional vector space, $n \geq 3$, $Q\colon V \times V \to V$ be a symmetric bilinear mapping and $F\colon V \to V$ a linear mapping.*

If for each vector $\lambda \in V$

$$Q(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\, F(\lambda) \tag{12.14}$$

holds, where $\varrho_1(\lambda)$, $\varrho_2(\lambda)$ are functions on V, then there are linear forms ψ and φ such that the condition

$$Q(X,Y) = \psi(X)\, Y + \psi(Y)\, X + \varphi(X)\, F(Y) + \varphi(Y)\, F(X) \tag{12.15}$$

holds for any $X, Y \in V$.

Proof. For $F = \alpha\mathrm{Id}$ this lemma is true for $\varphi = 0$. Therefore we suppose that $F \neq \alpha\mathrm{Id}$.

Formula (12.14) has the following coordinate expression

$$Q^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_1(\lambda)\,\lambda^h + \varrho_2(\lambda)\, F^h_\alpha\lambda^\alpha, \tag{12.16}$$

where $\lambda^i, F^h_i, Q^h_{ij}$ are the components of λ, F, Q.

1. Assume F^h_i has an eigenvector u_h with eigenvalue $\alpha \in \mathbb{R}$, i.e. $u_hF^h_i = \alpha\, u_i$. Contracting formula (12.16) with u_h we obtain

$$u_hQ^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = (\varrho_1(\lambda) + \alpha\varrho_2(\lambda))\, u_hF^h_\alpha\lambda^\alpha.$$

From Lemma 12.4 we get that $\varrho_1(\lambda)+\alpha\varrho_2(\lambda)$ is a linear form, e.g. $2\tilde{\psi}_\alpha\lambda^\alpha$. Then (12.16) has the form $(Q^h_{\alpha\beta} - \tilde{\psi}_{(\alpha}\delta^h_{\beta)})\lambda^\alpha\lambda^\beta = \varrho_2(\lambda)\,(F^h_\alpha - \alpha\,\delta^h_\alpha)\lambda^\alpha$, and again with the help of Lemma 12.4 follows that $\varrho_2(\lambda)$ is a linear form.

Finally, $\varrho_1(\lambda)$ and $\varrho_2(\lambda)$ are linear forms. When $\varrho_1(\lambda) = 2\psi_\alpha\lambda^\alpha$ and $\varrho_2(\lambda) = 2\varphi_\alpha\lambda^\alpha$ we obtain (12.15) and lemma is true.

It is important that the dimension n is odd, then always exists an eigenvector u_h with eigenvalue $\alpha \in \mathbb{R}$. Among others, therefore Lemma holds also for $n = 3$.

2. If we can not use 1, we do the following. It is natural to assume that there exists some vector ξ^h which is not an eigenvector of F^h_α with a real eigenvalue, and we can assume that $n > 3$.

By multiplying (12.16) with $\lambda^i F^j_\alpha \lambda^\alpha$ and antisymmetrizing the indices h, i and j we obtain

$$\left\{ Q^{[h}_{\alpha\beta}\delta^i_\gamma F^{j]}_\delta \right\} \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta = 0, \tag{12.17}$$

where square brackets denote the alternation of indices. The term in curly brackets does not depend on λ and (12.17) holds for any vector $\lambda \in V$, therefore

$$Q^{[h}_{(\alpha\beta}\delta^i_\gamma F^{j]}_{\delta)} = 0 \tag{12.18}$$

holds, where the round brackets denote symmetrization of indices.

Introducing $P^h_i \overset{\text{def}}{=} P^h_{i\alpha}\xi^\alpha$ and $P^h \overset{\text{def}}{=} P^h_\alpha \xi^\alpha$, we contract (12.18) with $\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta$. Since $F^h \nparallel \xi^h$, we obtain $P^h = 2a\,\xi^h + 2b\,F^h$, where a, b are certain constants. Contracting (12.18) with $\xi^\beta\xi^\gamma\xi^\delta$, and taking into account the preceding, we have $P^h_i = a\,\delta^h_i + b\,F^h_i + a_i\,\xi^h + b_i\,F^h$, where a_i, b_i are some components of some linear forms. Analogously, contracting (12.18) with $\xi^\gamma\xi^\delta$, we have

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i + \xi^h a_{ij} + F^h b_{ij}, \tag{12.19}$$

where ψ_i, φ_i are components of 1-forms ψ, φ, defined on V, and a_{ij}, b_{ij} are components of symmetric 2-forms defined on V.

In case that $a_{ij} = b_{ij} = 0$, evidently from (12.19) we obtain the formula (12.15).

Now we will suppose that either $a_{ij} \neq 0$, or $b_{ij} \neq 0$. Since ξ^h and F^h are noncollinear, it is evident that

$$\xi^h a_{ij} + F^h b_{ij} \neq 0. \tag{12.20}$$

By virtue of (12.19) formula (12.18) has the form

$$\Omega^{[hi}_{(\alpha\beta\gamma}F^{j]}_{\delta)} = 0, \tag{12.21}$$

where $\Omega^{hi}_{\alpha\beta\gamma} = (\xi^h a_{\alpha\beta} + F^h b_{\alpha\beta})\delta^i_\gamma - (\xi^i a_{\alpha\beta} + F^i b_{\alpha\beta})\delta^h_\gamma$. It is possible to show that there exists some vector ε^h for which $\Omega^{hi}_{\alpha\beta\gamma}\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma \neq 0$, otherwise (12.20) would be violated.

Contracting (12.21) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we have $F^h_\alpha\varepsilon^\alpha = a\,\xi^h + b\,F^h + c\,\varepsilon^h$, with a, b, c being constants. Analogously, contracting (12.21) with $\varepsilon^\beta\varepsilon^\gamma\varepsilon^\delta$, we obtain that F^h_i is represented in the following manner: $F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h + c_i\,\varepsilon^h$,where a_i, b_i, c_i are components of 1-forms.

Because $n > 3$, there exists a non-zero vector u_h which satisfies $u_h\xi^h = u_hF^h = u_h\varepsilon^h = 0$. If it satisfies the formula $u_hF^h = a\,u_i$ and the vector u_h is an eigenvector of F, which in this case we do not expect. Evidently Lemma 12.5 is proved. □

12.1.3 F-planar mappings which preserve F-structures

First we prove the following

Theorem 12.2 *An F-planar mapping f from $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ which preserves F-structures is characterized by the condition (12.5).*

Proof. It is obvious that geodesics are a special case of F-planar curves. Let a geodesic ℓ in $\mathbb{A}_n$, which satisfies the equations (12.1) and $\nabla_\lambda\lambda = 0$, be mapped onto an F-planar curve $\overline{\ell} = f(\ell)$ in $\overline{\mathbb{A}}_n$, which satisfies equations (12.1) and

$$\overline{\nabla}_\lambda\lambda = \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)F\lambda,$$

where $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

Because the deformation tensor satisfies $P(\lambda,\lambda) = \overline{\nabla}_\lambda\lambda - \nabla_\lambda\lambda$, we have

$$P(\lambda(t),\lambda(t)) = \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)F\lambda.$$

It follows from the previous formula that at each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)F\lambda$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions depending on λ.

By Lemma 12.5, it follows that there exist linear forms ψ and φ, for which formula (12.5) holds.

12.1.4 F-planar mappings which do not preserve F-structures

We now assume that the structures F and $\overline{F}$ are essentially distinct, i.e.

$$\overline{F}_i^h \neq a\delta_i^h + b\,F_i^h,$$

where a and b are constants.

a) It is obvious that geodesics are a special case of F-planar curves. Let a geodesic in $\mathbb{A}_n$, which satisfies the equations (12.1) and $\nabla_\lambda\lambda = 0$, be mapped onto an $\overline{F}$-planar curve in $\overline{\mathbb{A}}_n$, which satisfies the equations (12.1) and

$$\overline{\nabla}_\lambda\lambda = \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)\overline{F}\lambda.$$

Here $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t),\lambda(t)) = \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)\overline{F}\lambda$. It follows from the previous formula that at each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\overline{F}\lambda$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions depending on λ.

By Lemma 12.5 we obtain that there are linear forms ψ and φ for which the following formula is valid:

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,\overline{F}Y + \varphi(Y)\,\overline{F}X. \tag{12.22}$$

b) Let a special F-planar curve in A_n, which satisfies the equations (12.1) and $\nabla_\lambda\lambda = F\lambda$, be mapped onto an $\overline{F}$-planar curve in $\overline{A}_n$, which satisfies the equations (12.1) and

$$\overline{\nabla}_\lambda\lambda = \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)\overline{F}\lambda.$$

Here $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have $P(\lambda(t),\lambda(t)) = F\lambda + \overline{\varrho}_1(t)\,\lambda + \overline{\varrho}_2(t)\overline{F}\lambda$ It follows from the previous formula that at each point $x \in A_n$

$$P(\lambda,\lambda) = F\lambda + \varrho_1(\lambda)\,\lambda + \varrho_2(\lambda)\overline{F}\lambda$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions depending on λ.

Applying (12.22) we obtain

$$F\lambda = \tilde{\varrho}_1(\lambda)\,\lambda + \tilde{\varrho}_2(\lambda)\overline{F}\lambda.$$

By Lemma 12.3, there exist functions a and b for which formula (12.3) holds. In this way we proved

Theorem 12.3 *Any F-planar mapping from a space with affine connection $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ preserves F-structures.*

12.1.5 F-planar mappings for dimension $n = 2$

It is easy to see that for $n = 2$ Theorems 12.1 and 12.2 do not hold. If they held, the functions ϱ_1 and ϱ_2, appearing in (12.14), would be linear in λ.

In the case

$$F_i^h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for example, these functions have the forms

$$\varrho_1(\lambda) = \frac{\lambda^1 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^2 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2} \quad \text{and} \quad \varrho_2(\lambda) = \frac{\lambda^1 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta - \lambda^2 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2},$$

which are not linear in general.

On the other hand, any diffeomorphism from A_2 onto $\overline{A}_2$ is an F-planar mapping with (12.14) being valid for the above functions ϱ_1 and ϱ_2.

12.1.6 F-planar mappings with covariantly constant structure

As we said before, F-planar mappings generalize the whole series of previously studied mappings. We list below some conditions under which the F-planar mapping will be one of the mappings studied earlier by the authors.

Let us recall that an affinor F is said to be an *e-structure* if the relation [170, 171]

$$F^2 = e\,Id, \quad \text{where} \quad e = \pm 1, 0, \tag{12.23}$$

is satisfied.

The affinor $\overset{*}{F}$ is *equivalent to an e-structure* F if there exist numbers α, β such that the following relationship is satisfied

$$\overset{*}{F} = \alpha\,F + \beta\,Id. \tag{12.24}$$

We have the following theorem.

Theorem 12.4 *Let a diffeomorphism $\mathbb{A}_n \to \bar{\mathbb{A}}_n$ be a non-affine F-planar mapping. If the structures F and $\overline{F}$ are covariantly constant and* $\operatorname{rank} \|F - \varrho\, Id\| \geq 4$, *then this mapping is almost geodesic of type $\pi_2(e)$ and the structures are covariantly constant equivalent e-structures.*

Proof. Let $\mathbb{A}_n$ admit a non-affine F-planar mapping onto $\bar{\mathbb{A}}_n$, and let the structures F and $\overline{F}$ be covariantly constant in $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively, and $\operatorname{rank} \|F - \varrho\, Id\| \geq 4$. Then the formulas (12.5) and (12.3) hold.

We express the covariant derivative of F in $\bar{\mathbb{A}}_n$: $F^h_{i|j} \equiv \delta_j F^h_i + \overline{\Gamma}^h_{\alpha j} F^\alpha_i - \overline{\Gamma}^\alpha_{ij} F^h_\gamma$, where $\delta_i = \delta/\delta x^i$. Using formula (12.5) we obtain:

$$F^h_{i|j} = F^h_{i,j} + F^\alpha_i \psi_\alpha \delta^h_j + (F^\alpha_i \varphi_\alpha - \psi_i) F^h_j - \varphi_i F^h_\alpha F^\alpha_j \,, \tag{12.25}$$

where "$,$" and "$|$" are covariant derivatives in $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

We differentiate formulas (12.5) in $\bar{\mathbb{A}}_n$ covariantly. As we have assumed $F^h_{i,j} = 0$ and $\overline{F}^h_{i|j} = 0$, by substitution of (12.24) we obtain

$$\delta_j a\, F^h_i + \delta_j b\, \delta^h_i + a(F^\alpha_i \psi_\alpha \delta^h_j + (F^\alpha_i \varphi_\alpha - \psi_i) F^h_j - \varphi_i F^h_\alpha F^\alpha_j) = 0. \tag{12.26}$$

By $\delta_j a \neq 0$ we get a contradiction with $\operatorname{rank} \|F - \varrho\, Id\| \geq 4$. Thus $a \equiv \text{const}$. Analogously, for $n > 3$ formulae (12.26) imply that $b \equiv \text{const}$.

Since $a \neq 0$, formula (12.26) can be simplified:

$$F^\alpha_i \psi_\alpha \delta^h_j + (F^\alpha_i \varphi_\alpha - \psi_i) F^h_j - \varphi_i F^h_\alpha F^\alpha_j = 0. \tag{12.27}$$

The mapping $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ is not affine and hence $\psi_i \neq 0$ or $\varphi_i \neq 0$. If $\varphi_i = 0$, then for $\psi_i \neq 0$ it follows from (12.27) that $F = \varrho\, Id$, which is a contradiction.

So we have $\psi_i \neq 0$. Then the relation (12.27) we obtain $F^2 = c\, Id + d\, F$, where c, d are functions. By covariant derivations of this relation in $\mathbb{A}_n$, we can show that c, d are constants. Then we can easily see that we can choose numbers α a β such that for the affinor structure $\overset{*}{F} = \alpha\, F + \beta\, Id$, it holds $\overset{*}{F}{}^2 = e\, Id$, where $e = \pm 1, 0$. This means that the affinor F is equivalent to an e-structure.

Since in our case a and b in (12.3) are constant, we can prove analogously that the structure $\overline{F}$ is also equivalent to an e-structure. Moreover, both structures F and $\overline{F}$ are simultaneously covariantly constant in $\mathbb{A}_n$ and in $\bar{\mathbb{A}}_n$.

It follows from the facts mentioned above that in formulae (12.5) the original structures can be substituted by equivalent covariantly constant e-structures. That is why for an F-planar mappings $f\colon \mathbb{A}_n \to \bar{\mathbb{A}}_n$ the formulae (12.5), $\nabla F = 0$ and $F^2 = e\, Id$ are satisfied. These conditions show that the mapping f is an almost geodesic mapping of type $\pi_2(e)$ in the sense of N.S. Sinyukov [170, 171]. The proof of Theorem 12.4 is now complete. □

From formula (12.25) we obtain the following as a consequence

Theorem 12.5 *Let a diffeomorphism $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ be a non-affine F-planar mapping. If the covariant derivative of the structure F is preserved ($F^h_{i,j} = F^h_{i|j}$) then $F^2 = \alpha\, F + \beta\, Id$, $\quad \psi_\alpha F^\alpha_i = \beta \varphi_i \quad$ and $\quad \psi_i = F^\alpha_i \varphi_\alpha + \alpha \varphi_i$.*

12.2 F-planar mappings onto Riemannian manifolds

12.2.1 Fundamental equations of F-planar mappings onto Riemannian manifolds

In this section we consider F-planar mappings from manifolds with affine connection onto (pseudo-) Riemannian manifolds. We present equations controlling these mappings in the form of a system of Cauchy equations under some very general conditions. These results generalize the results obtained for geodesic, holomorphically projective and special F-planar mappings of Riemannian and Kählerian manifolds, obtained by N.S. Sinyukov, J. Mikeš, V.V. Domashev, I.N. Kurbatova, V.E. Berezovsky, M. Shiha, see [118, 121, 170, 208, 235, 382, 569, 645, 650, 651, 817].

Let us consider the manifold $\mathbb{A}_n$ with a torsion-free affine connection ∇, and an affinor structure F.

Theorem 12.6 *A manifold $\mathbb{A}_n$ with affine connection admits an F-planar mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$ with metric $\overline{g}$ if and only if the following equations are satisfied:*

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi_j \overline{g}_{ik} + \varphi_k \overline{F}_{(ij)} + \varphi_i \overline{F}_{jk} + \varphi_j \overline{F}_{ik}, \tag{12.28}$$

or equivalently, for any vector fields $X, Y, Z \in \mathcal{X}(\mathbb{A}_n)$:

$$\begin{aligned} &\nabla_Z \overline{g}(X,Y) = 2\psi(Z)\overline{g}(X,Y) + \psi(X)\overline{g}(Y,Z) + \psi(Y)\overline{g}(X,Z) \\ &+ \varphi(Z)(g(FX,Y) + g(X,FY)) + \varphi(X)g(FY,Z) + \varphi(Y)g(FX,Z). \end{aligned} \tag{12.29}$$

Here ", " is the covariant derivative relative the connection ∇ on $\mathbb{A}_n$, ψ and φ are one-forms, and ψ_i and φ_i are their components, $\overline{F}_{ij} = \overline{g}_{i\alpha} F^{\alpha}_{j}$, $\overline{g}_{ij}$ and F^h_i are components of the metric $\overline{g}$ and of the structure F. For the structure $\overline{F}$ the formula (12.3) holds.

Proof. The proof of this theorem follows from the more general Theorem 4.1, p. 183, and the shape of the deformation tensor P, which must be in the form (12.5). □

12.2.2 Equations of F-planar mappings in Cauchy form

In any chart of a manifold $\mathbb{A}_n$ equations (12.28) form a system of differential equations with covariant derivatives relative to components of the unknown tensors $\overline{g}_{ij}$, ψ_i and φ_i. Under the condition $|\overline{g}_{ij}| \neq 0$, the solution of (12.28) generates a Riemannian manifold $\overline{\mathbb{V}}_n$ with the metric tensor $\overline{g}_{ij}$on which the manifold $\mathbb{A}_n$ admits an F-planar mapping, where the structure $\overline{F}^h_i$ in $\overline{\mathbb{V}}_n$ is (non-uniquely) defined by formulae (12.3).

We shall prove that the general solution of the system (12.28) on the given manifold $\mathbb{A}_n$ depends on a finite number of parameters. From this it follows that from equations (12.28) we can find a fundamental system describing the F-planar mappings in Cauchy form. We have ([646]):

Theorem 12.7 *Let $\mathbb{A}_n$ be a manifold with affine connection ∇ and let an affinor F be defined such that* $\operatorname{rank}\|F-\varrho\,Id\|>5$. *Then $\mathbb{A}_n$ admits an F-planar mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$ if and only if the system of differential equations of Cauchy type:*

$$\begin{array}{lrcl}
\text{(a)} & \overline{g}_{ij,k} & = & 2\psi_k\overline{g}_{ij}+\psi_{(i}\overline{g}_{j)k}+\varphi_k\overline{F}_{(ij)}+\varphi_{(i}\overline{F}_{j)k};\\
\text{(b)} & \psi_{i,j} & = & \alpha\,\overline{g}_{ij}+\beta\,\overline{F}_{ij}+\overset{1}{Q}_{ij}(\overline{g},\psi,\varphi);\\
\text{(c)} & \varphi_{i,j} & = & \beta\,\overline{g}_{ij}+\gamma\,\overline{F}_{ij}+\overset{2}{Q}_{ij}(\overline{g},\psi,\varphi);\\
\text{(d)} & \alpha_{,i} & = & \overset{3}{Q}_i(\overline{g},\psi,\varphi,\alpha,\beta,\gamma);\\
\text{(e)} & \beta_{,i} & = & \overset{4}{Q}_i(\overline{g},\psi,\varphi,\alpha,\beta,\gamma);\\
\text{(f)} & \gamma_{,i} & = & \overset{5}{Q}_i(\overline{g},\psi,\varphi,\alpha,\beta,\gamma);
\end{array}\tag{12.30}$$

has a solution in $\mathbb{A}_n$ for the unknown tensor $\overline{g}_{ij}(x)$ ($\overline{g}_{ij}=\overline{g}_{ji}$, $|\overline{g}_{ij}|\neq 0$), the covectors $\psi_i(x)$, $\varphi_i(x)$ and the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$.

Here $\overset{\sigma}{Q}$ ($\sigma=\overline{1,5}$) are tensors which are expressed as functions of the listed arguments, and also of objects defined in $\mathbb{A}_n$, namely the affine connection and the affinor F_i^h.

Proof. Let $\mathbb{A}_n$ be a manifold with affine connection endowed with an affinor $F_i^h(x)$ satisfying

$$\operatorname{rank}\|F_i^h-\varrho\delta_i^h\|>5,\tag{12.31}$$

where ϱ is a function. Let the manifold $\mathbb{A}_n$ admit an F-planar mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$. Then in $\mathbb{A}_n$, the equation (12.28) holds.

We shall investigate the integrability conditions of these equations. Let them differentiate covariantly by x^l and then alternate the indices k and l. With respect to the Ricci identity and equations (12.28) we find the following:

$$\begin{array}{c}
2\psi_{[kl]}\overline{g}_{ij}+\psi_{il}\overline{g}_{jk}+\psi_{jl}\overline{g}_{ik}-\psi_{ik}\overline{g}_{jl}-\psi_{jk}\overline{g}_{il}+\\
+\varphi_{[kl]}\overline{F}_{(ij)}+\varphi_{il}\overline{F}_{jk}+\varphi_{jl}\overline{F}_{ik}-\varphi_{ik}\overline{F}_{jl}-\varphi_{jk}\overline{F}_{il}=\overset{6}{Q}_{ijkl}(\overline{g},\psi,\varphi),
\end{array}\tag{12.32}$$

where $[kl]$ is the alternation in k and l without division, $\psi_{ij}\overset{\text{def}}{=}\psi_{i,j}$; $\varphi_{ij}\overset{\text{def}}{=}\varphi_{i,j}$. The tensor $\overset{6}{Q}$ has a form analogous to the previous tensors $\overset{\sigma}{Q}$, where $\sigma=\overline{1,5}$. Its concrete form is as follows:

$$\overset{6}{Q}\overset{\text{def}}{=}\overline{g}_{i\alpha}Q^{\alpha}_{jkl}+\overline{g}_{j\alpha}Q^{\alpha}_{ikl},$$

where

$$\begin{array}{c}
Q^h_{ikl}\overset{\text{def}}{=}R^h_{ikl}+F^h_\alpha F^\alpha_l\varphi_i\varphi_k-F^h_\alpha F^\alpha_k\varphi_i\varphi_l+F^h_{i,[k}\varphi_{l]}-F^h_{[k,l]}\varphi_i+\\
+\delta^h_k(\psi_i\psi_l+\psi_\alpha F^\alpha_i\varphi_l+\psi_\alpha F^\alpha_l\varphi_i)+F^h_k(\varphi_\alpha F^\alpha_i\varphi_l+\varphi_\alpha F^\alpha_l\varphi_i)-\\
-\delta^h_l(\psi_i\psi_k+\psi_\alpha F^\alpha_i\varphi_k-\psi_\alpha F^\alpha_k\varphi_i)-F^h_l(\varphi_\alpha F^\alpha_i\varphi_k+\varphi_\alpha F^\alpha_k\varphi_i)
\end{array}$$

and R^h_{ijk} is the Riemannian tensor.

Let us investigate the system of homogeneous equations in the form

$$\begin{aligned} &2\overset{*}{\psi}_{[kl]}\overline{g}_{ij}+\overset{*}{\psi}_{il}\overline{g}_{jk}+\overset{*}{\psi}_{jl}\overline{g}_{ik}-\overset{*}{\psi}_{ik}\overline{g}_{jl}-\overset{*}{\psi}_{jk}\overline{g}_{il}+\\ &+\overset{*}{\varphi}_{[kl]}\overline{F}_{(ij)}+\overset{*}{\varphi}_{il}\overline{F}_{jk}+\overset{*}{\varphi}_{jl}\overline{F}_{ik}-\overset{*}{\varphi}_{ik}\overline{F}_{jl}-\overset{*}{\varphi}_{jk}\overline{F}_{il}=0, \end{aligned} \tag{12.33}$$

with unknowns $\overset{*}{\psi}_{ij}$ and $\overset{*}{\varphi}_{ij}$. We shall prove that this equation has, by the condition (12.31), solutions in the form

$$\text{(a)}\quad \overset{*}{\psi}_{ij}=\alpha\overline{g}_{ij}+\beta\overline{F}_{ij};\quad \text{(b)}\quad \overset{*}{\varphi}_{ij}=\beta\overline{g}_{ij}+\gamma\overline{F}_{ij}, \tag{12.34}$$

where α, β, γ are numbers.

a) Let us assume that there exists a vector ε^h such that the vectors $\varepsilon^\alpha\overset{*}{\varphi}_{\alpha i}$, $\varepsilon^\alpha\overline{g}_{\alpha i}$ and $\varepsilon^\alpha\overline{F}_{\alpha i}$ are linearly independent.

Then there exists a vector η^i such that $\varepsilon^\alpha\eta^\beta\overset{*}{\varphi}_{\alpha\beta}=1$, $\varepsilon^\alpha\eta^\beta\overline{g}_{\alpha\beta}=0$, $\varepsilon^\alpha\eta^\beta\overline{F}_{\alpha\beta}=0$. Contracting (12.33) with $\varepsilon^i\varepsilon^j\eta^l$ we see that the vector $\varepsilon^\alpha\overline{F}_{\alpha i}$ is a linear combination of the following vectors $\eta^\alpha\overset{*}{\psi}_{[k\alpha]}$, $\eta^\alpha\overset{*}{\varphi}_{[k\alpha]}$, $\varepsilon^\alpha\overline{g}_{k\alpha}$.

After contraction of (12.33) with $\varepsilon^j\eta^l$ and elimination of the vector $\varepsilon^\alpha\overline{F}_{\alpha i}$ with $\varepsilon^j\eta^l$, we see that rank $\|\overline{F}_{ij}-\alpha g_{ij}\|\leq 5$, which is a contradiction to (12.31).

Therefore the vectors $\varepsilon^\alpha\overset{*}{\varphi}_{\alpha i}$, $\varepsilon^\alpha\overline{g}_{\alpha i}$ and $\varepsilon^\alpha\overline{F}_{\alpha i}$ are linearly dependent for any vector ε^h. From this fact it follows that for any ε^h the equation $\overset{*}{\varphi}{}^{[i}_{\alpha}\delta^{j}_{\beta}F^{k]}_{\gamma}\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma=0$ holds where $\overset{*}{\varphi}{}^{h}_{i}\overset{\text{def}}{=}\overline{g}^{h\alpha}\overset{*}{\varphi}_{\alpha i}$. This condition is equivalent to

$$\overset{*}{\varphi}{}^{[i}_{(\alpha}\delta^{j}_{\beta}F^{k]}_{\gamma)}=0, \tag{12.35}$$

where $[ijk]$ and $(\alpha\beta\gamma)$ denote the alternation and the symmetrization of the mentioned indices, respectively.

Since $F^h_i\neq\alpha\delta^h_i$, there exists a vector ε^i such that ε^i and $\xi^i\overset{\text{def}}{=}\varepsilon^\alpha F^i_\alpha$ are linearly independent. Contracting (12.35) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma$, we see that the vector $\varepsilon^\alpha\overset{*}{\varphi}{}^{i}_{\alpha}$ is a linear combination of the vectors ε^i and ξ. Then, after the contraction of (12.35) with $\varepsilon^\beta\varepsilon^\gamma$, we obtain that $\overset{*}{\varphi}{}^{i}_{\alpha}=\beta\delta^i_\alpha+\gamma F^i_\alpha+a_\alpha\varepsilon^i+b_\alpha\xi^i$, where a_α, b_α are covectors and β, γ are functions. Under the assumption that a_α or b_α is non-zero, after the substitution of $\overset{*}{\varphi}{}^{i}_{\alpha}$ into (12.35) we get a contradiction with (12.31).

Hence $\overset{*}{\varphi}{}^{i}_{\alpha}=\beta\delta^i_\alpha+\gamma F^i_\alpha$. From these formulae, (12.34b) follow immediately.

b) Analogously, let us suppose the existence of a vector ε^h such that the vectors $\varepsilon^\alpha\overset{*}{\psi}_{\alpha i}$, $\varepsilon^\alpha\overline{g}_{\alpha i}$ and $\varepsilon^\alpha\overline{F}_{\alpha i}$ are linearly independent. However, this assumption is in contradiction with (12.31) and the regularity of the metric tensor $\overline{g}_{ij}$. That is why the vectors $\varepsilon^\alpha\overset{*}{\psi}_{\alpha i}$, $\varepsilon^\alpha\overline{g}_{\alpha i}$ and $\varepsilon^\alpha\overline{F}_{\alpha i}$ are linearly dependent for any vector ε^h.

From this it follows that $\overset{*}{\psi}_{ij} = \alpha\overline{g}_{ij} + \overline{\beta F}_{ij}$, where α, $\overline{\beta}$ are numbers. Substituting this relation and (12.34) into (12.33), we see that $\overline{\beta} = \beta$.

In this way we proved that the general solution of the homogeneous system of equations (12.33) is of the form (12.34). Therefore the conditions (12.32) imply the equations (12.31b) and (12.30b).

Further, we shall investigate the integrability conditions of equations (12.30b). Differentiating the equations (12.30b) covariantly by x^k and then alternating over j and k, using the Ricci identity and (12.30a, b, c), we obtain

$$\overline{g}_{ij}\alpha_{,k} - \overline{g}_{ik}\alpha_{,j} + \overline{F}_{ij}\beta_{,k} - \overline{F}_{ik}\beta_{,j} = \overset{7}{Q}_{ijk}(\overline{g}, \psi, \varphi, \alpha, \beta, \gamma). \tag{12.36}$$

The homogeneuos equations

$$\overline{g}_{ij}\,\overset{*}{\alpha}_k - \overline{g}_{ik}\,\overset{*}{\alpha}_j + \overline{F}_{ij}\,\overset{*}{\beta}_k - \overline{F}_{ik}\,\overset{*}{\beta}_j = 0$$

with unknowns $\overset{*}{\alpha}_i$ and $\overset{*}{\beta}_i$ have only the trivial solution $\overset{*}{\alpha}_i = 0$, $\overset{*}{\beta}_i = 0$ if the conditions (12.31) are satisfied. That is why the equations (12.30d) follow from the condition (12.36).

Similarly, the last equation (12.30f) of the system (12.30) can be obtained using the integrability conditions of equations (12.30c).

Evidently, the system (12.30) is closed with respect to the unknown tensors $\overline{g}_{ij}$, ψ_i, φ_i, α, β, γ. The Theorem 12.7 is proved. □

We know from the theory of differential equations that the initial value problem (12.30) with initial conditions

$$\overline{g}_{ij}(x_o) = \overset{o}{\overline{g}}_{ij};\ \psi_i(x_o) = \overset{o}{\psi}_i;\ \varphi_i(x_o) = \overset{o}{\varphi}_i;\ \alpha(x_o) = \overset{o}{\alpha};\ \beta(x_o) = \overset{o}{\beta};\ \gamma(x_o) = \overset{o}{\gamma},$$

has at most one solution. As the tensor $\overline{g}$ is symmetric, the general solution of this system depends on

$$r_{_F} \leq \tfrac{1}{2}n(n+5) + 3$$

real parameters.

From this the following theorem follows.

Theorem 12.8 *Let $\mathbb{A}_n$ be a manifold with affine connection, where an affinor F is defined such that* $\operatorname{rank}\|F - \varrho\, Id\| > 5$. *The set of all Riemannian manifolds $\overline{\mathbb{V}}_n$, for which $\mathbb{A}_n$ admits F-planar mappings, depends on at most $\frac{1}{2}n(n+5)+3$ real parameters.*

By a detailed analysis of the proof we can see in both theorems 12.7 and 12.8 that the condition $\operatorname{rank}\|F - \varrho\, Id\| > 5$ can be substituted by the assumptions $n > 8$ and $\operatorname{rank}\|F - \varrho\, Id\| > 4$.

Theorems 12.7 and 12.8 generalize similar results obtained by N.S. Sinyukov [170] for geodesic mappings of Riemannian manifolds, J. Mikeš and V.E. Berezovski [651] for geodesic mappings of manifolds with affine connection onto

Riemannian manifolds, V.V. Domashev and J. Mikeš [119, 381, 382], for holomorphically projective mappings of Kählerian manifolds, I.N. Kurbatova [569] for holomorphically projective mappings of hyperbolic Kählerian manifolds, K- and H-manifolds and M. Shiha [235, 799] for holomorphically projective mappings of m-parabolic Kählerian manifolds (see [118, 119, 170]).

12.2.3 Special F-planar mappings onto Riemannian manifolds

The section deals with so-called F_1-planar mappings from $\mathbb{A}_n$ with affine connection ∇ onto Riemannian manifolds $\overline{\mathbb{V}}_n$ with metric $\overline{g}$. F_1-planar mappings are F-planar mappings and for which the condition $\overline{g}(X, FX) = 0$ for all $X \in T\mathbb{A}_n$ is satisfied for the affinor F. These mappings generalize the quasigeodesic and holomorphically projective mappings introduced by A.Z. Petrov, T. Otsuki, I. Tashiro, and M. Prvanović. The main equations of F_1-planar mappings have been shown to be representable as a closed system of differential equations of Cauchy-type in covariant derivatives of $(n + 1)(n + 2)/2$ unknown functions. Particular cases are illustrated, in which the main equations are represented as a linear system.

Besides the affine connection ∇ and $\overline{\nabla}$, let the structural affinors F and $\overline{F}$ be also defined in $\mathbb{A}_n$ and $\overline{\mathbb{A}}_n$.

A mapping of $\mathbb{A}_n$ onto $\overline{\mathbb{A}}_n$ is F-planar if and only if the conditions (12.3) and (12.5) are satisfied in a common coordinate system with respect to the mapping. By virtue of conditions (12.3), the structure of the tensor F can be assumed to be preserved under F-planar mappings.

Let us introduce the following classes of F-planar mappings [644].

Definition 12.4 An F-planar mapping of a manifold $\mathbb{A}_n$ with affine connection onto a Riemannian manifold $\overline{\mathbb{V}}_n$ is called an *F_1-planar mapping* if the metric tensor satisfies the condition

$$\overline{g}(X, FX) = 0, \text{ for all } X. \tag{12.37}$$

Definition 12.5 An F_1-planar mapping $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ is called an *F_2-planar mapping* if the one form ψ is gradient-like, i.e. $\psi(X) = \nabla_X \Psi$, where Ψ is a function on $\mathbb{A}_n$.

Definition 12.6 An F_1-planar mapping $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ is called an *F_3-planar mapping* if the one-forms ψ and φ are related by

$$\psi(X) = \varphi(FX). \tag{12.38}$$

The signature of the metric of the manifold $\overline{\mathbb{V}}_n$ is assumed to be arbitrary. If $\varphi = 0$ or $F = \varrho\, Id$ then the F-planar mapping is a geodesic map. In the sequel, we assume that

$$\varphi \neq 0 \quad \text{and} \quad F \neq \varrho\, Id. \tag{12.39}$$

It is readily seen that F_3 and F_2 are subsets of F_1 which in turn is a subset of F. Contracting (12.5) with respect to the indices h and j, we find that if $\mathbb{A}_n$ is an equiaffine manifold and (12.38) holds, then the one-form ψ is gradient-like;

consequently, $F_3 \subset F_2$. Note that one type of the Petrov quasigeodesic mappings $\mathbb{V}_4 \to \overline{\mathbb{V}}_4$ of pseudo-Riemannian manifolds with the Minkowski metric [743] is a particular case of F_1-planar mappings.

Holomorphically projective mappings of Kähler manifolds, hyperbolical and parabolical Kähler manifolds, and K- and H-manifolds, in turn, are F_3-planar mappings (see [170, 569, 570, 645, 690, 737, 754].). We may note that the F_1-planar mappings F_1: $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ are defined by relations (12.28) and (12.37). It is a simple matter to verify that a manifold $\mathbb{A}_n$ admits an F_1-planar mapping onto a Riemannian manifold $\overline{\mathbb{V}}_n$ with a metric tensor if and only if the conditions

$$\text{(a) } \overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k} + \varphi_{(i}\overline{F}_{j)k}, \quad \text{(b) } \overline{g}_{\alpha(i}F^{\alpha}_{j)} = 0 \tag{12.40}$$

hold where $\overline{F}_{ij} = \overline{g}_{i\alpha}F^{\alpha}_{j}$.

Let us consider an F-planar mapping of $\mathbb{A}_n$ onto $\overline{\mathbb{V}}_n$ obeying the conditions

$$\overline{g}_{i\alpha}F^{\alpha}_{j} + \overline{g}_{j\alpha}F^{\alpha}_{i} = 2\,\mu\,\overline{g}_{ij} \tag{12.41}$$

where μ is a function.

F-planar mappings of Riemannian manifolds $\mathbb{V}_n$ onto $\overline{\mathbb{V}}_n$ obeying conditions (12.41) and other additional conditions are studied in [834].

Contracting (12.41) with $\overline{g}^{ij}$ (components of the inverse matrix of the matrix $\|\overline{g}_{ij}\|$), we obtain $\mu = 1/n\, F^{\alpha}_{\alpha}$. Put

$$\overset{*}{F}{}^{h}_{i} = F^{h}_{i} - \frac{1}{n}\, F^{\alpha}_{\alpha}\, \delta^{h}_{i}. \tag{12.42}$$

Then, from (12.41) we obtain $\overline{g}_{i\alpha}\overset{*}{F}{}^{\alpha}_{j} + \overline{g}_{j\alpha}\overset{*}{F}{}^{\alpha}_{i} = 0$. Substitution of (12.42) into (12.3) does not change the form of this equation. Consequently, the special F-planar mapping under consideration is an F_1-planar mapping relative to the affinor $\overset{*}{F}$, which is uniquely defined by formulas (12.41) in terms of the affinor F. On the other hand, any F_1-planar mapping defined by the affinor $\overset{*}{F}$ generates a family of F-planar mappings, for which conditions (12.41) are satisfied for the affinor $F = a\overset{*}{F} + b\,Id$, where $a(\neq 0)$ and b are functions.

12.2.4 Fundamental equations of F_1-planar mappings

Determination of all Riemannian manifolds $\overline{\mathbb{V}}_n$, on which a given manifold $\mathbb{A}_n$ with affine connection admits F_1-planar mappings, consists in solving a system of equations in covariant derivatives (12.40) for the unknown regular symmetric tensor $\overline{g}$ and the unknown one-forms ψ and φ. The tensor $\overline{g}$ is the metric tensor of the manifolds $\overline{\mathbb{V}}_n$ we are seeking.

Let us show that the main equations of the F_1-planar mappings can be reduced to a Cauchy-type closed system of equations with a finite number of unknown functions. We have

Theorem 12.9 *The manifold $\mathbb{A}_n$ with the affinor F satisfying the conditions*

$$F^2 \neq \varrho\, Id \tag{12.43}$$

where ϱ is a function, admits F_1-planar mappings onto the Riemannian manifolds $\overline{\mathbb{V}}_n$ if and only if the following Cauchy system has a solution for the components of the unknown regular symmetric tensor $\overline{g}_{ij}$ that obeys condition (12.37):

$$\overline{g}_{ij,k} = \theta_{ijk}(\overline{g}) \tag{12.44}$$

where $\theta_{ijk}(\overline{g})$ is a tensor specially constructed with the help of the components of $\overline{g}_{ij}$, the components of the affine connection of $\mathbb{A}_n$ and the structure F.

Proof. We show that the one-forms ψ and φ in Eqs. (12.40a) are expressed only through $\overline{g}$, ∇ and F. Covariantly differentiating (12.40b) with respect to x^k, by virtue of (12.40a), we find

$$\overline{\psi}_{(i}\overline{g}_{j)k} + \chi_{(i}\overline{F}_{j)k} + \varphi_{(i}\overset{2}{F}_{j)k} + \overline{g}_{\alpha(i}F^{\alpha}_{j),k} = 0 \tag{12.45}$$

where

$$\chi_i = \psi_i - \overline{\varphi}_i, \quad \overline{\psi}_i = \psi_\alpha F^\alpha_i, \quad \overline{\varphi}_i = \varphi_\alpha F^\alpha_i, \quad \overset{2}{F}_{ij} = \overline{F}_{i\alpha}F^\alpha_j.$$

Let us contract conditions (12.45) with F^k_l, and then symmetrize them with respect to i, j and l. Since F_{ij} and $\overset{2}{F}_{i\alpha}F^\alpha_j$ are skew tensors, we find

$$\chi_{(i}\overset{2}{F}_{jk)} = \theta_{ijk}(\overline{g}). \tag{12.46}$$

As before, let $\theta(\overline{g})$ denotes a tensor constructed with the help of $\overline{g}$ and the objects defined in $\mathbb{A}_n$, i.e. the components Γ^h_{ij} and F^h_i. Note that the tensors θ in (12.44) and (12.46) are distinct.

By virtue of (12.43), the conditions $\overset{2}{F}_{ij} \neq 0$ are satisfied. Then, from (12.46) we find $\chi_i = \theta_i(\overline{g})$. Consequently,

$$\psi_i = \overline{\varphi}_i = \theta_i(\overline{g}). \tag{12.47}$$

Now conditions (12.45) can be simplified. Since $\overset{2}{F}_{ij} \neq a\, g_{ij}$, the relations already obtained yield $\varphi_i = \theta_i(\overline{g})$. By virtue of (12.47), we conclude that $\psi_i = \theta_i(\overline{g})$, and the system of equations (12.40a) takes the form (12.44). Clearly, (12.37) and (12.44) form a Cauchy system relative to the tensor $\overline{g}$. This completes the proof of the theorem. □

The following theorem holds:

Theorem 12.10 *The manifold $\mathbb{A}_n$ with the affinor F satisfying the conditions*

$$F^2 = \varrho\, Id \tag{12.48}$$

where ϱ is a nonzero function, admits F_1-planar mappings onto the Riemannian manifolds $\overline{\mathbb{V}}_n$ if and only if the following Cauchy system has a solution for the unknown nondegenerate symmetric tensor $\overline{g}_{ij}(x)$ (obeying condition (12.37)), the vector $\varphi_i(x)$ and the function $\mu(x)$:

$$\begin{aligned}
&(a)\ \ \overline{g}_{ij,k} = 2\psi_k\overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k} + \varphi_{(i}\overline{F}_{j)k} + \theta_{ijk}(\overline{g});\\
&(b)\ \ \varphi_{i,j} = \mu\overline{F}_{ij} + \theta_{ij}(\overline{g},\varphi);\\
&(c)\ \ \mu_{,i} = \theta_i(\overline{g},\varphi,\mu).
\end{aligned} \tag{12.49}$$

The tensors θ depend on $\overline{g}_{ij}$, φ_i and μ according to our reasoning, and also on the objects defined in $\mathbb{A}_n$.

Remark 12.1 Without loss of generality, the affinor F can be normalized, so $\varrho = e = \pm 1$. Hence, this affinor generates an e-structure ($e = \pm 1$) in $\mathbb{A}_n$.

Proof. As in proving Theorem 12.9, here too we obtain the formulas (12.45)-(12.47). Then Eqs. (12.40a) take the form (12.49a), and their integrability conditions take the form

$$2\,\overline{\varphi}_{[kl]}\overline{g}_{ij} + \overline{g}_{k(i}\overline{\varphi}_{j)l} - \overline{g}_{l(i}\overline{\varphi}_{j)k} - F_{k(i}\varphi_{j)l} + F_{l(i}\varphi_{j)k} = \theta_{ijkl}(\overline{g},\varphi) \tag{12.50}$$

where $\varphi_{ij} = \varphi_{i,j}$ and $\overline{\varphi}_{ij} = \varphi_{\alpha j}F_i^{\alpha}$.

Contracting (12.50) with respect to $\overline{g}^{ij}$, we find

$$\overline{\varphi}_{[kl]} = \theta_{kl}(\overline{g},\varphi). \tag{12.51}$$

After contracting (12.50) with $\overline{g}^{jl}$, we obtain

$$(n+1)\,\overline{\varphi}_{ik} + \varphi_{i\alpha}F_k^{\alpha} + \mu_1\overline{g}_{ik} + \mu_2\overline{F}_{ik} = \theta_{ik}(\overline{g},\varphi) \tag{12.52}$$

where μ_1 and μ_2 are functions.

Contracting (12.51) with $F_{i'}^{i}$ and $F_{k'}^{k}$ eliminating primes, and substituting $\varphi_{i\alpha}F_k^{\alpha}$ into (12.52), we obtain

$$\overline{\varphi}_{ik} = \mu\,\overline{g}_{ik} + \mu_3\overline{F}_{ik} + \theta_{ik}(\overline{g},\varphi).$$

Alternating this relation, we find $\mu_3 = \theta(\overline{g},\varphi)$. Thus, the last expression takes the form (12.49b). From the conditions of integrability of these equations, we obtain Eq. (12.49c). Clearly, the system (12.49) is dosed with respect to the unknowns $\overline{g}_{ij}$, φ_i and μ. This completes the proof of Theorem 12.10. □

The following theorem holds:

Theorem 12.11 *A manifold $\mathbb{A}_n$ where a structure F satisfying the nilpotency condition*

$$F^2 = 0 \tag{12.53}$$

is defined, admits F_1-planar mappings onto Riemannian manifolds if and only if there exists a solution to the Cauchy system of the type (12.49) for the unknown symmetric regular tensor $\overline{g}_{ij}(x)$ (obeying condition (12.37)), the vector $\varphi_i(x)$ and the function $\mu(x)$.

Proof. Relations (12.45), which also hold for the present case, are simplified as follows:

$$\overline{\psi}_{(i}\overline{g}_{j)k} + \chi_{(i}\overline{F}_{j)k} + \overline{g}_{\alpha(i}F^{\alpha}_{j),k} = 0.$$

Hence it follows that $\psi_i = \overline{\varphi}_i + \theta_i(\overline{g})$ and $\overline{\psi}_i = \theta_i(\overline{g})$. Then Eqs. (12.40a) take the form (12.49a), and their integrability conditions are of the form (12.50).

Conditions (12.51) are derived from them along analogous lines. Contracting (12.50) with $F^k_{k'}$ we readily find that

$$-F_{k(i}\overline{\varphi}_{j)l} + F_{l(i}\overline{\varphi}_{j)k} = \theta_{ijkl}(\overline{g}, \varphi).$$

Symmetrizing the last expression with respect to the indices i, j and l, and since F_{ij} is antisymmetric, we obtain $F_{k(i}\overline{\varphi}_{jl)} = \theta_{ijkl}(\overline{g}, \varphi)$.

By virtue of (12.51), the last expression yields $\overline{\varphi}_{ij)} = \theta_{ij}(\overline{g}, \varphi)$. Then (12.50) takes the form

$$F_{l(i}\varphi_{j)k} + F_{k(i}\varphi_{j)l} = \theta_{ijkl}(\overline{g}, \varphi).$$

Hence, it is a simple matter to derive (12.49b), and conditions (12.49c) are derived from (12.49b). This completes the proof of Theorem 12.11. □

Since our investigations are local, conditions (12.43), (12.48) and (12.53) cover all possible cases. Thus, Theorems 12.9, 12.10 and 12.11 guarantee that the main equations of F_1-planar mappings $\mathbb{A}_n \to \overline{\mathbb{V}}_n$ can be represented as a Cauchy-type system of equations in covariant derivatives.

Such systems do not have more than one solution for the initial values. Consequently, the general solution of the system of equations (12.44) depends on at mostly parameters, while the solution of system (12.49) depends on not more than $n(n+1)/2 + n + 1$ parameters.

But the algebraic conditions (12.37) add at least n more relations, whereas relations (12.48) and (12.53) still more reduce the number of parameters on which the general solution of system (12.49) depends.

It is a simple matter to verify validity of the following theorem.

Theorem 12.12 *The general solution of F_1-planar mappings of a manifold $\mathbb{A}_n$ with affine connection onto Riemannian manifold $\overline{\mathbb{V}}_n$ depends on not more than*

$\frac{1}{2}\, n(n-1)$ *parameters under the condition (12.43),*

$\frac{1}{4}\, n^2 + n + 1$ *parameters under condition (12.48), and*

$\frac{1}{2}\, (n+2)(n+1) - m(n-m+1)$ *parameters under condition (12.53), where* $m = \operatorname{rank} \|F\|$.

This theorem implies the following

Corollary 12.1 *The family of Riemannian manifolds $\overline{\mathbb{V}}_n$ on which a given manifold $\mathbb{A}_n$ with affine connection admits F_1-planar mappings depends on the number of parameters stated in Theorem 12.12.*

12.2.5 Fundamental linear equations of F_2-planar mappings

The Cauchy-type systems of differential equations stated in Theorems 12.9-12.11 are not linear. Now let us examine whether the main equations of the F_1-planar mappings can be reduced to a linear system.

The following theorem holds:

Theorem 12.13 *If a manifold $\mathbb{A}_n$ with affine connection admits F_2-planar mapping onto the Riemannian manifold $\overline{\mathbb{V}}_n$, then the following conditions are satisfied:*

$$\text{(a) } a^{ij}{}_{,k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k + \xi^i F^j_k + \xi^j F^i_k; \quad \text{(b) } a^{\alpha(i} F^{j)}_\alpha = 0, \tag{12.54}$$

where a^{ij} is a regular symmetric tensor, and λ^i and ξ^i are vectors.

Proof. Differentiating the relation $\overline{g}_{i\alpha}\overline{g}^{\alpha j} = \delta^j_i$, we readily find

$$\overline{g}^{ij}{}_{,k} = -\overline{g}_{\alpha\beta,k}\,\overline{g}^{\alpha i}\overline{g}^{\beta j}. \tag{12.55}$$

Put

$$\text{(a) } a^{ij} = \mathrm{e}^{2\Psi}\overline{g}^{ij}; \quad \text{(b) } \lambda^i = -a^{i\alpha}\psi_\alpha; \quad \text{(c) } \xi^i = -a^{i\alpha}\varphi_\alpha \tag{12.56}$$

where Ψ is a function which generates the gradient $\psi_i = \delta_i \Psi$.

Now differentiating a^{ij} covariantly with respect to x^k, and by virtue of (12.55) and (12.56), we find that (12.40) yields conditions (12.54). This completes the proof of Theorem 12.13. □

As was already mentioned, F_2-planar mappings form a particular case of F_1-planar mappings. For this case, from Eqs. (12.40) we get conditions (12.54). In general the converse of this assertion is not true, i.e. the solution of Eqs. (12.40) may not correspond to the solution of Eqs. (12.54). The following theorem holds:

Theorem 12.14 *An equiaffine manifold $\mathbb{A}_n$ admits F_3-planar mappings onto the Riemannian manifold $\overline{\mathbb{V}}_n$ if and only if the following conditions are satisfied:*

$$\text{(a) } a^{ij}{}_{,k} = \overline{\xi}^i \delta^j_k + \overline{\xi}^j \delta^i_k + \xi^i F^j_k + \xi^j F^i_k; \quad \text{(b) } a^{\alpha(i} F^{j)}_\alpha = 0, \tag{12.57}$$

where a^{ij} is a regular symmetric tensor, and ξ^i and $\overline{\xi}^i = \xi^\alpha F^i_\alpha$ are vectors.

Proof. Putting $a^{ij} = \mathrm{e}^{2\Psi}\overline{g}^{ij}$ and $\xi^i = -a^{i\alpha}\varphi_\alpha$, we find as before that Eqs. (12.40), by virtue of (12.38), yield relations (12.57).

We prove the converse. Assume that Eqs. (12.57) have a solution. Covariantly differentiating $a^{i\alpha}\tilde{g}_{\alpha j} = \delta^i_j$, where $\|\tilde{g}_{ij}\| \equiv \|a^{ij}\|^{-1}$, we find $\tilde{g}_{ij,k} = -a^{\alpha\beta}{}_{,k}\,\tilde{g}_{\alpha i}\tilde{g}_{\beta j}$; and, by virtue of (12.57),

$$\tilde{g}_{ij,k} = \tilde{\varphi}_{(i}\tilde{g}_{j)k} - \varphi_{(i}F^\alpha_{j)}\tilde{g}_{\alpha k}; \quad F^\alpha_{(i}\tilde{g}_{j)\alpha} = 0, \tag{12.58}$$

where $\varphi_i = -\tilde{g}_{i\alpha}\xi^\alpha$ and $\tilde{\varphi} = \varphi_\alpha F^\alpha_i$.

The tensor $\tilde{g}_{ij}$ can be considered as the metric tensor of some Riemannian manifold $\tilde{\mathbb{V}}_n$. The Christoffel symbols of the second kind $\tilde{\Gamma}^h_{ij}$ obey the conditions $\tilde{\Gamma}^\alpha_{\alpha k} = \frac{1}{2}\, a^{\alpha\beta}(\tilde{g}_{\alpha\beta,k} + \Gamma^\gamma_{k\alpha}\tilde{g}_{\gamma\beta} + \Gamma^\gamma_{k\beta}\tilde{g}_{\gamma\alpha})$. By virtue of (12.58), we obtain $\tilde{\varphi}_k = \frac{1}{2}\,(\tilde{\Gamma}^\alpha_{\alpha k} - \Gamma^\alpha_{\alpha k})$. Since $\mathbb{A}_n$ and $\tilde{\mathbb{V}}_n$ are equiaffine manifolds, we have $\tilde{\varphi}_k = \delta_k \Psi$, where Ψ is a function. Then, putting $\overline{g}_{ij} = \mathrm{e}^{2\Psi}\tilde{g}_{ij}$, we find that Eqs. (12.40) follow from (12.57). This completes the proof of Theorem 12.14. □

We can demonstrate that Eqs. (12.54) as well as Eqs. (12.57) can be augmented to form a Cauchy-type closed system of equations in covariant derivatives. This system would be linear, and its solution, in principle, lies in studying linear algebraic equations.

Thus, the determination of all the Riemannian manifolds $\mathbb{V}_n$ onto which a given manifold $\mathbb{A}_n$ with affine connection admits F_2- and F_3-planar mappings consists in studying linear algebraic systems.

Note that holomorphically projective mappings of elliptic, hyperbolic, and parabolic Kähler spaces, K-spaces and H-spaces are particular cases of F_3-planar mappings. For these cases the main equations in the form stated above are derived in [226, 235, 382, 569, 628, 645, 799, 802, 803].

12.2.6 Generating F-planar mappings

Suppose a Riemannian manifold $\mathbb{V}_n$ admits an F_2-planar mappings onto $\overline{\mathbb{V}}_n$. Then conditions (12.6), (12.37), and $\psi_i = \Psi_{,i}$ are satisfied.

We construct the metric tensors of the manifolds $\tilde{\mathbb{V}}_n$ and $\tilde{\overline{\mathbb{V}}}_n$ as follows:

$$\tilde{g}_{ij} = \mathrm{e}^{2\Psi}\, \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j} \quad \text{and} \quad \tilde{\bar{g}}_{ij} = \mathrm{e}^{2\Psi}\, g_{ij}.$$

Computing the covariant derivative of the tensor $\tilde{g}_{ij}$ in the manifold $\tilde{\overline{\mathbb{V}}}_n$, we readily find that $\tilde{\overline{\mathbb{V}}}_n$ admits an F_1-planar mapping onto $\tilde{\overline{\mathbb{V}}}_n$ such that the conditions

$$\tilde{\bar{\Gamma}}^h_{ij} = \tilde{\Gamma}^h_{ij} + \psi_{(i}\delta^h_{j)} + \overset{*}{\varphi}_{(i}\overset{*}{F}{}^h_{j)}$$

are satisfied, where $\overset{*}{\varphi}_i = \tilde{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$; $\quad \overset{*}{F}{}^h_i = g^{h\alpha}\tilde{g}_{\alpha\beta} F^\beta_i$.

12.3 Infinitesimal F-planar transformations

12.3.1 Definition of infinitesimal F-planar transformations

Let us consider an n-dimensional manifold $\mathbb{A}_n$ with symmetric affine connection ∇, where, along with the object of affine connection ∇, an affinor structure F is given. Denote by $x = (x^1, x^2, \dots, x^n)$, a coordinate system on $\mathbb{A}_n$. In what follows we suppose that $n > 2$.

A curve ℓ: $x^h = x^h(t)$ is F-planar if (12.2) hold, i.e.

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}(x(t))\lambda^\alpha\lambda^\beta = \varrho_1(t)\lambda^h + \varrho_2(t)F^h_\alpha\lambda^\alpha \tag{12.59}$$

where $\Gamma^h_{ij}(x)$ are the components of ∇, $\lambda^h \equiv dx^h(t)/dt$ is the tangent vector of ℓ, and $\varrho_1(t)$, $\varrho_2(t)$ are functions of the parameter t.

Definition 12.7 *An infinitesimal transformation* of a manifold $\mathbb{A}_n$ with affine connection is given with respect to the coordinates as follows:

$$\overline{x}^h = x^h + \varepsilon\,\xi^h(x), \tag{12.60}$$

where x^h are the coordinates of a point in $\mathbb{A}_n$ and $\overline{x}^h$ are the coordinates of its image, ε is an infinitesimal parameter which does not depend on x^h, and ξ^h is the displacement vector.

An infinitesimal transformation (12.60) of the manifold $\mathbb{A}_n$ will be said to be *F-planar*, if it maps F-planar curves of $\mathbb{A}_n$ onto curves which are F-planar in their *principal parts*.

If an object $\mathcal{A}$ depends not only on $x \in \mathbb{A}_n$ but also on the infinitesimal parameter ε, i.e., $\mathcal{A} = \mathcal{A}(x, \varepsilon)$, then the *principal part* of $\mathcal{A}$ is $\underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}(x)\,\varepsilon$ in the expansion in a series with respect to ε:

$$\mathcal{A}(x,\varepsilon) = \underset{0}{\mathcal{A}}(x) + \underset{1}{\mathcal{A}}(x)\,\varepsilon + \underset{2}{\mathcal{A}}(x)\,\varepsilon^2 + \dots\ .$$

In our case curves obtained by the transformation from F-planar curves satisfy the equation of F-planar curves under the condition that the terms containing higher powers of ε (i.e. ε^2, $\dots$) are dropped.

12.3.2 Basic equations of infinitesimal F-planar transformations

Theorem 12.15 *A differential operator $X = \xi^\alpha(x)\partial_a$ ($\partial_a = \partial/\partial x^a$) determines an infinitesimal F-planar transformation of a manifold $\mathbb{A}_n$ with affine connection if and only if*

$$(a)\quad \mathcal{L}_\xi\Gamma^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i; \quad (b)\quad \mathcal{L}_\xi F^h_i = a\delta^h_i + bF^h_i, \tag{12.61}$$

where ψ_i and φ_i are covectors, a and b are functions, δ^h_i is the Kronecker delta, and $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Proof. Let us consider an infinitesimal F-planar transformation of a manifold $\mathbb{A}_n$ determined by (12.60). Suppose that $F_i^h \neq \varrho\,\delta_i^h$. Let ℓ be an F-planar curve of the manifold $\mathbb{A}_n$ given by equations $x^h = x^h(t)$ and (12.59). The curve $\overline{\ell}$ corresponding to ℓ under transformation (12.60) has the equations

$$\overline{x}^h(t) = x^h(t) + \varepsilon\,\xi^h(x(t)). \tag{12.62}$$

The infinitesimal transformation (12.60) is F-planar if $\overline{\ell}$ is F-planar in the principal part. Hence, $\overline{x}^h(t)$ given by (12.62) satisfies in the principal part (12.59), which in this case take the form

$$\frac{d\overline{\lambda}^h(t)}{dt} + \Gamma^h_{\alpha\beta}(\overline{x}(t))\overline{\lambda}^\alpha(t)\overline{\lambda}^\beta(t) = \overline{\varrho}_1(t)\overline{\lambda}^h(t) + \overline{\varrho}_2(t)F^h_\alpha(\overline{x}(t))\overline{\lambda}^\alpha(t). \tag{12.63}$$

Let us find the objects involved in (12.63). From (12.62) we find the tangent vector $\overline{\lambda}^h(t)$ of the curve $\overline{\ell}$:

$$\overline{\lambda}^h(t) \equiv \frac{d\overline{x}^h(t)}{dt} = \frac{dx^h(t)}{dt} + \varepsilon\,\frac{\partial\xi^h(x(t))}{\partial x^\alpha}\,\frac{dx^\alpha(t)}{dt} = \lambda^h(t) + \varepsilon\lambda^\alpha(t)\partial_\alpha\xi^h(x(t)).$$

For the object of affine connection ∇ and the structure F, at the point $\overline{x}$ we have

$$\Gamma^h_{ij}(\overline{x}) = \Gamma^h_{ij}(x) + \varepsilon\,\frac{\partial\Gamma^h_{ij}(x)}{\partial x^\gamma}\xi^\gamma(x) + \boxed{\varepsilon^2} \quad \text{and} \quad F^h_i(\overline{x}) = F^h_i(x) + \varepsilon\,\frac{\partial F^h_i(x)}{\partial x^\gamma}\xi^\gamma(x) + \boxed{\varepsilon^2}\,.$$

Hereafter $\boxed{\varepsilon^2}$ stands for the terms containing higher powers of the parameter ε.

Let us expand $\overline{\varrho}_1(t)$ and $\overline{\varrho}_2(t)$ in power series with respect to ε:

$$\overline{\varrho}_1(t) = \overline{\varrho}_{1,0}(t) + \overline{\varrho}_{1,1}(t)\,\varepsilon + \boxed{\varepsilon^2} \quad \text{and} \quad \overline{\varrho}_2(t) = \overline{\varrho}_{2,0}(t) + \overline{\varrho}_{2,1}(t)\,\varepsilon + \boxed{\varepsilon^2}\,.$$

We substitute these expressions into (12.63) and obtain

$$\frac{d\lambda^h}{dt} + \varepsilon\,(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta + \frac{d\lambda^\alpha}{dt}\partial_\alpha\xi^h) +$$

$$+ (\Gamma^h_{\alpha\beta} + \varepsilon\,\xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \boxed{\varepsilon^2})\,(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)\,(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) =$$

$$= (\overline{\varrho}_{1,0} + \varepsilon\overline{\varrho}_{1,1} + \boxed{\varepsilon^2})\,(\lambda^h + \varepsilon\lambda^\gamma\partial_\gamma\xi^h) +$$

$$+ (\overline{\varrho}_{2,0} + \varepsilon\overline{\varrho}_{2,1} + \boxed{\varepsilon^2})\,(F^h_\alpha + \varepsilon\,\xi^\gamma\partial_\gamma F^h_\alpha + \boxed{\varepsilon^2})\,(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)\,.$$

Since the curve ℓ is F-planar, we can use (12.59) to eliminate $\frac{d\lambda^h}{dt}$ from the previous relation:

$$-\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + F^h_\alpha + \varepsilon(\partial_{\alpha\beta}\xi^h\lambda^\alpha\lambda^\beta - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\xi^h\lambda^\alpha\lambda^\beta + \varrho_1\lambda^\alpha\partial_\alpha\xi^h + \varrho_2\lambda^\alpha F^\beta_\alpha\partial_\beta\xi^h)$$

$$+\varrho_1\lambda^h + \varrho_2\lambda^\alpha + (\Gamma^h_{\alpha\beta} + \varepsilon\,\xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \boxed{\varepsilon^2})(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)(\lambda^\beta + \varepsilon\lambda^\gamma\partial_\gamma\xi^\beta) =$$

$$= (\overline{\varrho}_{1,0} + \varepsilon\overline{\varrho}_{1,1} + \boxed{\varepsilon^2})\,(\lambda^h + \varepsilon\lambda^\gamma\partial_\gamma\xi^h) + \tag{$*$}$$

$$+ (\overline{\varrho}_{2,0} + \varepsilon\overline{\varrho}_{2,1} + \boxed{\varepsilon^2})\,(F^h_\alpha + \varepsilon\,\xi^\gamma\partial_\gamma F^h_\alpha + \boxed{\varepsilon^2})\,(\lambda^\alpha + \varepsilon\lambda^\gamma\partial_\gamma\xi^\alpha)\,.$$

It is evident that $(*)$ holds true at each point $x \in A_n$. Therefore we can assume that ϱ_1, ϱ_2, $\overline{\varrho}_{1,0}$, $\overline{\varrho}_{1,1}$, $\overline{\varrho}_{2,0}$, and $\overline{\varrho}_{2,1}$ are functions of the point $x = (x^1, x^2, \dots, x^n)$ as well as of the tangent vector $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^n)$.

The constant term in $(*)$, i.e., the term which does not depend on ε, vanishes. Hence, after calculation, we get

$$(\overline{\varrho}_{1,0} - \varrho_1)\lambda^h + (\overline{\varrho}_{2,0} - \varrho_2)\lambda^\alpha F^h_\alpha = 0.$$

This relation holds true for all vectors λ^h at a given point x. Hence we have $\overline{\varrho}_{1,0} = \varrho_1$ and $\overline{\varrho}_{2,0} = \varrho_2$.

The linear (with respect to ε) term in $(*)$ can be rewritten as follows

$$\begin{aligned}(\partial_{\alpha\beta}\xi^h - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\xi^h + \xi^\gamma\partial_\gamma\Gamma^h_{\alpha\beta} + \Gamma^h_{\gamma\alpha}\partial_\beta\xi^\gamma + \Gamma^h_{\gamma\beta}\partial_\alpha\xi^\gamma)\lambda^\alpha\lambda^\beta + \\ + \varrho_2(F^\beta_\alpha\partial_\beta\xi^h - F^h_\gamma\partial_\alpha\xi^\gamma - \xi^\gamma\partial_\gamma F^h_\alpha)\lambda^\alpha - \overline{\varrho}_{1,1}\lambda^h - \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha.\end{aligned}$$

This term also vanishes, so, using the definition of Lie derivative, we get

$$\mathcal{L}_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_2\mathcal{L}_\xi F^h_\alpha\lambda^\alpha + \overline{\varrho}_{1,1}\lambda^h + \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha. \tag{12.64}$$

Here $\mathcal{L}_\xi$ stands for the Lie derivative with respect to ξ.

The transformation under consideration maps F-planar curves to F-planar curves up to the second order. Certainly, this is true also for geodesics, which are characterized by (12.59) with $\varrho_2(t) \equiv 0$. In this case, (12.64) turns into

$$\mathcal{L}_\xi\Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \overline{\varrho}_{1,1}\lambda^h + \overline{\varrho}_{2,1}F^h_\alpha\lambda^\alpha.$$

These equations hold true at any point and for any vector λ^h. By virtue of Lemma 12.5, from these equations we get (12.61a).

By (12.61a), under condition $\varrho_2 \equiv -1$ (this is possible because each F-planar curve is mapped onto an F-planar curve) the relations (12.64) can be rewritten as follows:

$$\mathcal{L}_\xi F^h_\alpha\lambda^\alpha = (\overline{\varrho}_{1,1} - 2\psi_\alpha\lambda^\alpha)\lambda^h + (\overline{\varrho}_{2,1} - 2\varphi_\beta\lambda^\beta)F^h_\alpha\lambda^\alpha.$$

From these relations (which hold true for any λ^h), the formula (12.61b) follows by Lemma 12.2.

Thus we have proved necessity. Sufficiency can be checked in a direct way. □

Note that in case $F^h_i = \varrho\,\delta^h_i$ or $\varphi_i = 0$, each infinitesimal F-planar transformation is an infinitesimal geodesic transformation. These transformations for Riemannian manifolds were studied by L.P. Einsenhart [50], see also [118, 149, 429], i.e. the following theorem holds in general.

Theorem 12.16 *A differential operator $X = \xi^\alpha(x)\partial_a$ $(\partial_a = \partial/\partial x^a)$ determines an infinitesimal projective (geodesic) transformation of a manifold $\mathbb{A}_n$ with affine connection if and only if*

$$(a)\quad \mathcal{L}_\xi\Gamma^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i \tag{12.65}$$

where ψ_i is a covector.

12.4 F-planar transformations

We will show that the infinitesimal F-planar transformations are closely related to the F-planar transformations [119, 632].

Let us consider an n-dimensional manifold $\mathbb{A}_n$ with symmetric affine connection ∇, where, along with the object of affine connection ∇, an affinor structure F is given.

Definition 12.8 An F-planar mapping of a manifold $\mathbb{A}_n$ onto itself is called an *F-planar transformation.*

The above definition generalized motions, projective and holomorphically projective transformations of manifolds with affine connection and Riemannian manifolds.

Thus, we have [489] the following

Theorem 12.17 *In a manifold $\mathbb{A}_n$ with affine connection, a one-parameter Lie group of F-planar transformations exists if and only if in $\mathbb{A}_n$, an infinitesimal F-planar transformation exists, and all such transformations have the same differential operator.*

This fact is a consequence of the following theorem ([632], see [119]).

Theorem 12.18 *An infinitesimal operator $X = \xi^\alpha(x)\partial_\alpha$ determines a one-parameter Lie group of F-planar transformations of a manifold $\mathbb{A}_n$ with affine connection if and only if (12.61) holds true.*

Proof. We choose a special coordinate system x in which the fundamental vector field $\xi^h(x)$ of a one-parameter Lie group has the form $\xi^h = \delta^h_1$. This one-parameter Lie group reads $\overline{x}^h = x^h + \delta^h_1\,\tau$, where τ is the canonical parameter.

For F-planar transformations we use the same methods as in the proof of the Theorem 12.1. We obtain easily

$$\begin{array}{lrcl} (a) & \Gamma^h_{ij}(\overline{x}) & = & \Gamma^h_{ij}(x) + \tilde{\psi}_{(i}\delta^h_{j)} + \tilde{\varphi}_{(i}F^h_{j)}(\overline{x}), \\ (b) & F^h_i(\overline{x}) & = & \tilde{a}\,\delta^h_i + \tilde{b}\,F^h_i(x), \end{array} \tag{12.66}$$

where $\tilde{\psi}$, $\tilde{\varphi}_i$, $\tilde{a}$, $\tilde{b}$ are objects which depend on x and τ.

In this coordinate system the following formulas are satisfied: $L_\xi\Gamma^h_{ij} = \partial_1\Gamma^h_{ij}$ and $L_\xi F^h_i = \partial_1 F^h_i$, hence from (12.66) we obtain the formulas (12.61). On the other hand, the formulas (12.61) generate a one-parameter group of F-planar transformations. □

We note that when the vector $\varphi_i = 0$ in (12.61), then the transformation is projective.

If the structure F satisfies the condition $F^h_\alpha F^\alpha_i = \pm\delta^h_i$, then the formula (12.61b) is simplified: $L_\xi F^h_i = 0$.

Using the fundamental equalities (12.61), we can see that any manifold $\mathbb{A}_n$ admits F-planar transformations (locally) if and only if there is a coordinate system (x) in which $\Gamma^h_{ij}(x)$ and the structure $F^h_i(x)$ have the following forms:

$$\Gamma^h_{ij} = \tilde{\Gamma}^h_{ij} + \alpha_{(i}\delta^h_{j)} + \beta_{(i}\tilde{F}^h_{j)}, \quad F^h_i = \alpha\delta^h_i + \beta\,\tilde{F}^h_i,$$

where $\alpha_i, \beta_i, \alpha, \beta$ are arbitrary functions depending on $x^1, x^2, \cdots, x^n$, and $\tilde{\Gamma}^h_{ij}, \tilde{F}^h_i$ are arbitrary functions depending on $x^2, x^3, \cdots, x^n$.

In [632] this statement is proved under the conditions $n > 3$ and $\operatorname{Rank} \|F - \alpha I\| > 1$. However, by a more detailed consideration, as, for example, in [478], one can prove that this statement holds also for $n > 2$.

As it is known, the Lie derivatives $\mathcal{L}_\xi \Gamma^h_{ij}$ and $\mathcal{L}_\xi F^h_i$ can be written as follows

$$\mathcal{L}_\xi \Gamma^h_{ij} = \xi^h_{,ij} - \xi^\alpha R^h_{ij\alpha} \quad \text{and} \quad \mathcal{L}_\xi F^h_i = \xi^\alpha F^h_{i,\alpha} + \xi^h_{,\alpha} F^\alpha_i - \xi^\alpha_{,i} F^h_\alpha .$$

Here the "," denotes the covariant derivative in $\mathbb{A}_n$ and R^h_{ijk} are components of the curvature tensor of $\mathbb{A}_n$.

Hence (12.61) can be written as

$$\begin{aligned} &(a) \qquad \xi^h_{,ij} = \xi^\alpha R^h_{ij\alpha} + \psi_i \delta^h_j + \psi_j \delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i \\ &(b) \qquad \xi^\alpha F^h_{i,\alpha} + \xi^h_{,\alpha} F^\alpha_i - \xi^\alpha_{,i} F^h_\alpha = a\delta^h_i + b F^h_i, \end{aligned} \tag{12.67}$$

In the space $\mathbb{A}_n$, under the conditions $(n > 3)$ and $\operatorname{Rank} \|F - \alpha I\| > 1$, the basic equations (12.61), which determine F-planar transformations and infinitesimal F-planar transformations, can be represented as a closed system of linear differential equations (written in terms of covariant derivatives) of Cauchy type in $n^2 + 3n$ unknown functions

$$\xi^h(x), \quad \xi^h_i(x), \quad \psi_i(x), \quad \varphi_i(x). \tag{12.68}$$

This system has at most one solution (12.68) for the initial values at a given point $x_o \in A_n$:

$$\xi^h(x_o) = \overset{\circ}{\xi}{}^h, \quad \xi^h_i(x_o) = \overset{\circ}{\xi}{}^h_i, \quad \psi_i(x_o) = \overset{\circ}{\psi}{}_i, \quad \varphi_i(x_o) = \overset{\circ}{\varphi}{}_i.$$

Hence in A_n, the dimension r of the group of F-planar transformations is less than or equal to $r_{_{Ft}} = n^2 + 3n$, and the dimension of the space of infinitesimal F-planar transformations is less than or equal to $r_{_{Ft}}$.

Let $\xi^h_1, \xi^h_2, \cdots, \xi^h_r$ be linearly independent solutions of equations (12.61) in $\mathbb{A}_n$, whereas any solution $\xi^h(x)$ is of the form

$$\xi^h = \sum_{\sigma=1}^{r} C^\sigma \xi^h_\sigma, \quad \text{where} \quad C^\sigma - \text{const} \ \ (\sigma = 1, 2, \cdots, r).$$

Because the commutator of any two vectors ξ^h_σ and ξ^h_ϱ $(\sigma, \varrho = 1, 2, \cdots, r)$ satisfies the conditions (12.61) on the manifold $\mathbb{A}_n$, then the system of vectors $\xi^h_1, \xi^h_2, \cdots, \xi^h_r$ generates the full Lie group of F-planar transformations of order r_{Ft}.

The above mentioned system can be written as follows

$$\begin{aligned} &\text{(a)} \ \xi^h_{,i} &&= \xi^h_i \\ &\text{(b)} \ \xi^h_{i,j} &&= \xi^\alpha R^h_{ij\alpha} + \psi_i \delta^h_j + \psi_c \delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i \\ &\text{(v)} \ \psi_{i,j} &&= {}^1Q_{ij\alpha}\xi^\alpha + {}^2Q^\beta_{ij\alpha}\xi^\alpha_\beta + {}^3Q^\beta_{ij}\varphi_\beta; \\ &\text{(d)} \ \varphi_{i,j} &&= {}^4Q_{ij\alpha}\xi^\alpha + {}^5Q^\beta_{ij\alpha}\xi^\alpha_\beta + {}^6Q^\beta_{ij}\varphi_\beta, \end{aligned} \tag{12.69}$$

where ${}^{\sigma}Q$ ($\sigma = \overline{1,6}$) are tensor objects composed from the geometric objects of the space $\mathbb{A}_n$, i.e., from the affine connection ∇ and the affinor F_i^h.

Setting $\xi_i^h = \xi^h_{,i}$ we obtain (12.69a). The equations (12.69b) are in fact (12.61a), written in the form (12.67).

The integrability conditions for (12.69b) are written as follows

$$\delta_i^h(\psi_{j,k} - \psi_{k,j}) + \delta_j^h\psi_{i,k} - \delta_k^h\psi_{i,j} + F_i^h(\varphi_{j,k} - \varphi_{k,j}) + F_j^h\varphi_{i,k} - F_k^h\varphi_{i,j} =$$

$$= \xi^\alpha R^h_{ijk,\alpha} + \xi_i^\alpha R^h_{\alpha jk} + \xi_j^\alpha R^h_{i\alpha k} + \xi_k^\alpha R^h_{ij\alpha} - \xi^h_\alpha R^\alpha_{ijk} \tag{12.70}$$

$$-\varphi_i(F^h_{j,k} - F^h_{k,j}) + F^h_{i,j}\varphi_k - F^h_{i,k}\varphi_j.$$

One can verify that equations (12.70) have a unique solution (as linear algebraic equations) with respect to the unknowns $\psi_{i,j}$ and $\varphi_{j,k}$. From this one can get (12.69c) and (12.69d), where the left-hand side is uniquely determined.

Equations (12.61b) are linear algebraic equations with respect to ξ^h and ξ_i^h with coefficients defined in $\mathbb{A}_n$ (one can show that a and b are certain linear functions in ξ^h and ξ_i^h). The integrability conditions for (12.69b) (these are (12.70)) are linear algebraic equations in ξ^h and ξ_i^h, ψ_i and φ_i with coefficients defined on $\mathbb{A}_n$.

Now assume that the affine connection ∇ of $\mathbb{A}_n$ and the structure F are analytic. Let us denote by (A_0) the integrability conditions for (12.69) combined with (12.61b). Then the system (A_1) of the first differential prolongations of the equations (A_0), the system (A_2) of the second prolongations, and so on, consist of linear algebraic equations with respect to the unknown tensors ξ^h and ξ_i^h, ψ_i and φ_i with coefficients defined in $\mathbb{A}_n$.

From the theory of partial differential equations of Cauchy type it follows

Theorem 12.19 *A manifold $\mathbb{A}_n$ ($n > 3$) with affine connection endowed with an affinor structure F such that* $\operatorname{Rank}\|F - \alpha I\| > 1$*, admits an F-planar transformation and an infinitesimal F-planar transformation if and only if the system of linear equations $(A_0), (A_1), (A_2), \ldots, (A_{N-1})$ has a non-trivial solution (12.68).*

The maximal number $r_{_{Ft}} \leq N \equiv n(n+3)$ of essential parameters on which the general solution of the system of equations (12.69) depends, is the dimension of the group of F-planar transformations of $\mathbb{A}_n$.

Using (12.61b) and their differential prolongations one can prove that the maximum $r = N$ cannot be achieved and, moreover, in fact $r_{_{Ft}} \leq N - 2(n-2) \equiv n(n+1) + 4$.

In a Riemannian manifold $\mathbb{V}_n$, according to the Theorem 4.11, the condition (12.61a) is equivalent to

$$(L_\xi g_{ij})_{,k} = 2\psi_k g_{ij} + \psi_{(i}g_{j)k} + \varphi_k F_{(ij)} + \varphi_{(i}F_{j)k}, \tag{12.71}$$

where $F_{ij} \overset{\text{def}}{=} g_{i\alpha}F_j^\alpha$.

If ψ is a gradient-like vector then an F-planar transformation is said to be F_2-*planar*, and if $\psi = \varphi_\alpha F_i^\alpha$ then it is said to be F_3-*planar*. In a Riemannian manifold, any F_3-planar transformations are F_2-planar.

If moreover, in a Riemannian space $\mathbb{V}_n$, the additional conditions $a_{i\alpha}F_j^\alpha + a_{j\alpha}F_i^\alpha = 0$ are satisfied, then the F_2- and F_3-planar transformation is said to be F_2^*- and F_3^*-planar, respectively.

The group of (homothetic) motions which preserve the affinor F_i^h, is said to be an F-*planar (homothetic) motion*, and its degree is denoted r_{Fmot} (r_{Fhom}). Obviously, F-planar motions and F-planar homothetic motions are subgroups of F-planar mappings, and also subgroups of the group of motions and homothetic motions, respectively.

We can prove:

Theorem 12.20 *The following inequalities hold:*

$$r_{F_3^* tr} \leq r_{F_3^*} + r_{F_3^* mot}; \quad r_{F_3^* tr} \leq r_{F_3^*} + r_{F_3^* hom} - 1,$$

where $r_{F_3^* p}$ is the degree of the full group of F_3^*-planar mappings, $r_{F_3^*}$ is the mobility degree of $\mathbb{V}_n$ under F_3^*-planar mappings, i.e. the number of parameters on which the general solution of the equations (12.57) depends.

Proof. Follows by examining the equations (12.57) and (12.71). Accounting $\overline{\xi}_i = \partial_i \overline{\xi}$, from (12.57) we get

$$b_{ij,k} \equiv (a_{ij} + 2\overline{\xi} g_{ij})_{,k} = 2\overline{\xi}_k g_{ij} + \overline{\xi}_{(i} g_{j)k} + \xi_{(i} F_{j)k}; \quad b_{\alpha(i} F_{j)}^\alpha = 0.$$

Hence according to Theorems 4.12 and 4.13, from (4.97) we get the inequalities (12.20). □

12.5 On F_2^ε-planar mappings of Riemannian manifolds

This section is devoted to special F-planar mappings between two n-dimensional (pseudo-) Riemannian manifolds. These results were obtained in papers [350, 487] by H. Chudá, I. Hinterleitner, J. Mikeš and P. Peška.

The PQ^ε-projective equivalence between n-dimensional Riemannian manifolds were introduced in 2003 by Topalov [896], P and Q are tensors of type $(1,1)$ for which $PQ = \varepsilon\,\mathrm{Id}$, $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 1, 1+n$. It immediately follows from their definition that PQ^ε-projective equivalence is the correspondence occurring in the earlier studied F-planar mappings (Mikeš, Sinyukov [690]) and $F = Q$. We prove that these mappings are F_2-planar mappings (see p. 398, Mikeš [644]), which generalize geodesic and holomorphically projective mappings, see [122,170,197].

In paper [896] by Topalov and paper [615] by Matveev and Rosemann, some properties of this equivalence were studied and among other things it was shown that if $\varepsilon = 0$ this equivalence is projective. This is the reason, why we study PQ^ε-projective equivalence where $\varepsilon \neq 0$ only. With a detailed analysis, we found that the tensor P, with all of its properties, is derived from the tensor Q and the number ε, so that $P = \varepsilon\, F^{-1}$. According to these facts, we renamed PQ^ε-projective equivalence as F_2^ε-planar mapping (for which $F \equiv Q$).

In this section we study F_2^ε-projective mappings between (pseudo-) Riemannian manifolds for $\varepsilon \neq 0$. We can suppose that ε is arbitrary function. For these mappings we find a fundamental system of closed linear equations in covariant derivatives and we obtain new results for initial conditions. We proved that a set of (pseudo-) Riemannian manifolds with $F^2 \neq \varepsilon\mathrm{Id}$, on which some (pseudo-) Riemannian manifold admits F_2^ε-projective mappings, depends on no more than $n(n-1)/2$ parameters.

12.5.1 PQ^ε-projective Riemannian manifolds

Let g and $\overline{g}$ be two Riemannian metrics on an n-dimensional manifold M. Consider the $(1,1)$-tensors P, Q which are satisfying the following conditions:

$$\begin{gathered} PQ = \varepsilon\,\mathrm{Id}, \\ g(X,PX) = 0,\ \overline{g}(X,PX) = 0, g(X,QX) = 0,\ \overline{g}(X,QX) = 0, \end{gathered} \tag{12.72}$$

for all X and where $\varepsilon \neq 1, n+1$ is a real number. These conditions are written in a different way in [615] (formula (1)).

Definition 12.9 [896] The metrics $g, \overline{g}$ are called *PQ^ε-projective* if for the 1-form Φ the Levi-Civita connections ∇ and $\overline{\nabla}$ of g and $\overline{g}$ satisfy

$$(\overline{\nabla} - \nabla)_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{12.73}$$

for all X, Y.

Remark. Two metrics g and $\overline{g}$ are denoted by the synonym PQ^ε-projective if they are *PQ^ε-projective equivalent*. On the other hand this notation can be seen from the point of view of mappings. Assume two Riemannian manifolds (M,g) and $(\overline{M},\overline{g})$. A diffeomorphism $f\colon M \to \overline{M}$ allows to identify the manifolds M

and $\overline{M}$. For this reason we can speak about PQ^ε-projective mappings (or more precisely diffeomorphisms) between (M,g) and $(\overline{M},\overline{g})$, when equations (12.72) and (12.73) hold. In these formulas $\overline{g}$ and $\overline{\nabla}$ mean in fact the pullbacks $f^*\overline{g}$ and $f^*\overline{\nabla}$.

Comparing formulas (12.5) and (12.73) we make sure that PQ^ε-projective equivalence is a special case of the F-planar mapping between Riemannian manifolds (M,g) and $(M,\overline{g})$. Evidently, this is if $\psi \equiv \Phi$, $F \equiv Q$ and $\varphi(\cdot) = -\Phi(P(\cdot))$.

Moreover, it follows elementary from (12.73) that ψ is a gradient-like form, see [896], thus a PQ^ε-projective equivalence is a special case of an F_2-planar mapping, p. 398.

12.5.2 Simplification of conditions (12.72) for $\varepsilon \neq 0$

Next, we will study PQ^ε-projective mappings for $\varepsilon \neq 0$. From the condition $PQ = \varepsilon\mathrm{Id}$, it follows

$$P = \varepsilon\, Q^{-1}\,. \tag{12.74}$$

This implies that P depends on Q and ε. Moreover two conditions in (12.72) depend on the other ones, i.e. in the definition of PQ^ε-projective mappings we can restrict on the conditions $g(X,QX) = 0$, $\overline{g}(X,QX) = 0$, $PQ = \varepsilon\,\mathrm{Id}$. This fact implies the following lemma:

Lemma 12.6 *If Q satisfies the conditions $g(X,QX) = 0$ and $\overline{g}(X,QX) = 0$ for $\varepsilon \neq 0$, then we obtain $g(X,PX) = 0$ and $\overline{g}(X,PX) = 0$.*

Proof. We can write the first conditions (12.72) for g in the local form as $g_{i\alpha}Q^\alpha_j + g_{j\alpha}Q^\alpha_i = 0$. These equations we contract with $\overline{Q}^i_k\overline{Q}^j_l$, where $\overline{Q} = Q^{-1}$, after some calculations we obtain

$$g_{li}\overline{Q}^i_k + g_{kj}\overline{Q}^j_l = 0\,,$$

i.e. $g(X,Q^{-1}X) = 0$ for all X. From that follows $g(X,PX) = 0$ for all X. Analogically it holds also for the metric $\overline{g}$. □

Remark. Next, we will suppose, that ε is nonzero function. This fact implies from above properties, which do not depend that $\varepsilon = \mathrm{const}\,$.

12.5.3 F_2^ε-projective mapping with $\varepsilon \neq 0$

Due to the above properties, from formula (12.73) and Lemma 12.6, we can simplify the Definition 12.9.

Let g and $\overline{g}$ be two (pseudo-) Riemannian metrics on an n-dimensional manifold M. Consider the regular $(1,1)$-tensors F which are satisfying the following conditions

$$g(X,FX) = 0 \qquad \text{and} \qquad \overline{g}(X,FX) = 0 \tag{12.75}$$

for all X.

Definition 12.10 The metrics g and $\overline{g}$ are called *F_2^ε-projective* if for a certain gradient-like form ψ the Levi-Civita connections ∇ and $\overline{\nabla}$ of g and $\overline{g}$ satisfy

$$(f^*\overline{\nabla} - \nabla)_X Y = \psi(X)Y + \psi(Y)X - \varepsilon\,\psi(F^{-1}X)FY - \varepsilon\,\psi(F^{-1}Y)FX, \tag{12.76}$$

for all vector fields X, Y and for all $x \in M$, ε is a non-zero function.

From the above discussion we obtain the following proposition:

Proposition 12.1 *A PQ^{ε}-projective metrics with gradient-like form ψ and non-zero function ε can be understood as an F_2^{ε}-planar mapping for which*

$$P = \varepsilon F^{-1} \qquad \text{and} \qquad Q = F. \tag{12.77}$$

Moreover, it is true following proposition.

Proposition 12.2 *Let ε $(\neq 0, n+1)$ be a constant. A PQ^{ε}-projective metrics can be understood as an F_2^{ε}-planar mapping with (12.77).*

Proof. We can rewrite formula (12.76) in the form

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} - \psi_\alpha P^\alpha_{(i} Q^h_{j)}\,. \tag{12.78}$$

Contracting h and j we get

$$\overline{\Gamma}^\alpha_{i\alpha} = \Gamma^\alpha_{i\alpha} + (n+1-\varepsilon)\cdot\psi_i\,.$$

Because $\varepsilon \neq n+1$ there is a function Ψ which is defined 1-form $\psi = \nabla\Psi$, i.e. $\psi_i = \partial\Psi/\partial x^i$, where $\Psi = \dfrac{1}{n+1-\varepsilon}\ \ln\sqrt{\left|\dfrac{\det\overline{g}}{\det g}\right|}$. □

On the base of Proposition 12.1 and the known formula (12.54) which valid for F_2-planar mappings, for the F_2^{ε}-projective equivalence we obtained, after lowering the indices i and j by the metric g, the following equations:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha P^\alpha_i g_{j\beta} F^\beta_k - \lambda_\alpha P^\alpha_j g_{i\beta} F^\beta_k\,, \tag{12.79}$$

where $P = F^{-1}$.

From conditions (12.75) and (12.54) we obtain $a(X, FX) = 0$ for all X, and equivalently in local form

$$a_{i\alpha}F^\alpha_j + a_{j\alpha}F^\alpha_i = 0. \tag{12.80}$$

We obtain the following theorem:

Theorem 12.21 *If a (pseudo-) Riemannian manifold (M, g, F) with regular structure F, for which $F^2 \neq \kappa\mathrm{Id}$ and $g(X, FX) = 0$ for all X, admits an F_2^{ε}-projective mapping onto a (pseudo-) Riemannian manifold $(\overline{M}, \overline{g})$, then the linear system of differential equations (12.79) and (12.80) with $\lambda_i = a_{\alpha\beta}T_i^{\alpha\beta}$ and $T_i^{\alpha\beta}$ is a certain tensor obtained from metric g and structure F.*

Proof. We will study the fundamental equations (12.79) and (12.80) of an F_2^{ε}-planar mapping $V_n \to \overline{V}_n$. Now we covariantly differentiate (12.80) and obtain

$$\nabla_k a_{i\alpha}F^\alpha_j + \nabla_k a_{j\alpha}F^\alpha_i = \overset{1}{T}_{ijk}\,, \tag{12.81}$$

where $\overset{1}{T}_{ijk} = -a_{i\alpha}\nabla_k F^\alpha_j - a_{j\alpha}\nabla_k F^\alpha_i$.

Using formula (12.79), we obtain

$$\begin{aligned}\lambda_i g_{\alpha k}F_j^{\alpha}+\lambda_{\alpha}F_j^{\alpha}g_{ik}-\lambda_{\beta}P_i^{\beta}g_{\alpha\gamma}F_j^{\alpha}F_k^{\gamma}-\varepsilon\lambda_j g_{i\alpha}F_k^{\alpha}+\lambda_j g_{\alpha k}F_i^{\alpha}\\ +\lambda_{\alpha}F_i^{\alpha}g_{jk}-\lambda_{\beta}P_j^{\beta}g_{\alpha\gamma}F_i^{\alpha}F_k^{\gamma}-\varepsilon\lambda_i g_{j\alpha}F_k^{\alpha}=\overset{1}{T}_{ijk}.\end{aligned} \tag{12.82}$$

It is known, that $g_{ij}, g_{\beta\gamma}F_i^{\beta}F_j^{\gamma}$ are symmetric and $g_{\alpha k}F_j^{\alpha}$ is antisymmetric tensors. After skew symmetrization formula (12.82) with respect to indices j,k and replacing indices i,k we added up the obtained formula with (12.82) and finally we get:

$$(\varepsilon+1)\cdot(g_{\alpha i}F_j^{\alpha}\lambda_k+g_{\alpha k}F_j^{\alpha}\lambda_i)+g_{ik}\lambda_{\alpha}F_j^{\alpha}-g_{\beta\gamma}F_i^{\beta}F_k^{\gamma}\lambda_{\alpha}P_j^{\alpha}=\overset{2}{T}_{ijk}, \tag{12.83}$$

where $\overset{2}{T}_{ijk}=\frac{1}{2}\,(\overset{1}{T}_{ijk}+\overset{1}{T}_{kji}-\overset{1}{T}_{kij})$.

Now, we create this homogeneous equation:

$$g_{\alpha i}F_j^{\alpha}A_k+g_{\alpha k}F_j^{\alpha}A_i+g_{ik}B_j-g_{\beta\gamma}F_i^{\beta}F_k^{\gamma}C_j=0 \tag{12.84}$$

with unknown variables A_i, B_i and C_i. Because rank $\|g_{\alpha i}F_j^{\alpha}\|>3$, from (12.84) we get $A_k=0$. Since $F^2\neq\varkappa Id$, then $g_{\beta\gamma}F_i^{\beta}F_k^{\gamma}\neq\varkappa g_{ik}$, and, finally, formula (12.84) implies $B_j=C_j=0$.

Therefore formula (12.83) implies $\lambda_{\alpha}F_i^{\alpha}=a_{\alpha\beta}\overset{3}{T}{}_i^{\alpha\beta}$ and $\lambda_{\alpha}P_i^{\alpha}=a_{\alpha\beta}\overset{4}{T}{}_i^{\alpha\beta}$, where $\overset{3}{T}$ and $\overset{4}{T}$ are a certain tensors obtained from g and F. Now, elementary, since F is regular, then $\lambda_i=T_i^{\alpha\beta}a_{\alpha\beta}$. □

12.5.4 F_2^{ε}-planar mappings with the $\overline{g}=k\cdot g$ condition

From the properties of equations (12.79) and (12.80) follows a new result for F_2^{ε}-planar mappings, for which $F^2\neq\kappa\mathrm{Id}$. These conditions we suppose for the whole studied (pseudo-) Riemannian manifolds (M,g,F). The system of equations (12.79) has the form of partial linear differential equations of Cauchy type in covariant derivative with respect to the unknown functions $a_{ij}(x)$. From the theory of this system (see [122, pp. 46–49]) follows that the system of equation (12.79) for initial condition at the point $x_0\in M$

$$a_{ij}(x_0)=\overset{0}{a}_{ij} \tag{12.85}$$

has only one unique solution.

Due to this, the general solution of (12.79) depends on the real parameters which can be, for example, the conditions (12.85). Because a_{ij} is symmetric, conditions can not be more then $n(n+1)/2$. Moreover, condition (12.80) implies further reduction of the parameters.

The structure F at the point x_0 can be written in Jordan's form as $F_i^i=\lambda_i$, $F_i^{i+1}=\mu_i=0,1$ and the other components are vanishing. Because $\det F\neq 0$, all $\lambda_i\neq 0$. We do not exclude that λ_i are complex numbers (in this case the transformation equations are complex at the point x_0).

Substituting $i = j$ to equation (12.80), we obtain $a_{ii}\lambda_i + a_{ii+1}\mu_{i+1} = 0$ (formally $\mu_{n+1} \equiv 0$), i.e. the diagonal components a_{ii} depend on the other components.

This implies that the maximum number of the independent components of $\overset{0}{a}_{ij}$, which is not greater than $n(n-1)/2 - n$, i.e. $n(n-1)/2$ parameters.

Therefore this theorem is valid.

Theorem 12.22 *A set of (pseudo-) Riemannian manifolds (M, g, F), $\det F \neq 0$ and $F^2 \neq \kappa\mathrm{Id}$, on which some (pseudo-) Riemannian manifold admits an F_2^{ε}-projective mapping, depends on not more than $n(n-1)/2$ parameters.*

We have the following theorem.

Theorem 12.23 *Let $\mathbb{V}_n = (M, g, F)$ and $\overline{\mathbb{V}}_n = (M, \overline{g}, F)$ be (pseudo-) Riemannian manifolds with $F^2 \neq \kappa\mathrm{Id}$ and $\mathbb{V}_n$, $\overline{\mathbb{V}}_n$ have in F_2^{ε}-planar correspondence.*

If the condition $\overline{g} = k \cdot g$ is valid for $x_0 \in M$, then g and $\overline{g}$ are homothetic in M, i.e.

$$\overline{g}(x) = k \cdot g(x)\,, \tag{12.86}$$

for all $x \in M$, with $k = \mathrm{const}$.

Proof. In the assumption of Theorem 12.23, Theorem 12.21 is valid. Then equation (12.79) holds. For the initial condition of (12.86) there is no more than one unique solution. On the other hand, a trivial solution of equations (12.79) is $\overline{g} = k \cdot g$, and it satisfies the initial condition (12.86). The given mapping is homothetic. □

13 HOLOMORPHICALLY PROJECTIVE MAPPINGS OF KÄHLER MANIFOLDS

As we have seen in Theorem 9.28, p. 340, there are no structure-preserving and non-trivial geodesic mappings of Kähler manifolds onto Kähler manifolds. That is why for Kähler manifolds we prefer to consider a wider class of mappings than geodesic ones, namely holomorphically projective mappings.

The topic of holomorphically projective mappings was introduced (for classical, elliptic) Kähler manifolds $\mathbb{K}_n^-$ by T. Otsuki and Y. Tashiro [737], for hyperbolic Kähler manifolds $\mathbb{K}_n^+$ by M. Prvanovic [754], and for parabolic Kähler manifolds $\mathbb{K}_n^o$ by V.V. Vishnevskij [190]. The theory of such mappings is treated e.g. in monographs by K. Yano [198] and V.V. Vishnevskij, A.P. Shirokov, V.V. Shurigin [191]. The monograph by N.S. Sinyukov [170] includes also results obtained by V.V. Domashev and J. Mikeš [381, 382], J. Mikeš [226, 628]. In the monograph by N.S. Sinyukov, I.N. Kurbatova, J. Mikeš [173], there are already contained the results [226, 381, 382, 628] and also the results of I.N. Kurbatova [569]. Here we cover all the results just mentioned, of course, in the light of new discoveries made in the field of F-planar mappings and transformations, see the previous Section. These results, and many others, can be found in the survey paper by J. Mikeš [119] and monography [122, pp. 239–262].

During the last years, a lot of papers were devoted to the theory of holomorphically projective mappings, let us mention e.g. A.V. Aminova, M. Chodorová (Škodová), H. Chudá (Vavříková), K.R. Esenov, N. Guseva, M. Haddad, I. Hinterleitner, A.V. Kalinin, Raad J. al Lamy, G. Markov, J. Mikeš, S.M. Minčić, P. Peška, O. Pokorná, M. Prvanović, Ž. Radulović, A. Sabykanov, T. Sakaguchi, M. Shiha, A.P. Shirokov, N.S. Sinyukov, E.N. Sinyukova, M.S. Stanković, G.A. Starko, A. Vanžurová, L.S. Velimirović, M. Zlatanović, etc. [166, 171, 172, 211, 214, 216, 233, 235, 262, 263, 340, 347, 350, 450, 487, 576, 688, 689, 693, 754–757, 760, 778, 799, 800, 802, 803, 827, 832, 850–852, 882].

In this part, we explain some classical results in this field as well as the recent and contemporary results, included the unicity theorems of the theory of holomorphically projective mappings of Kähler manifolds. We investigate the degree of mobility of Kähler manifolds relative to the holomorphically projective mappings.

13.1 Fundamental properties of holomorphically projective mappings

13.1.1 Definition of holomorphically projective mappings

In the present book, see Definition 3.8, p. 132, by a Kählerian space we mean a wide class of spaces defined as follows [118, 119, 173]:

A Riemannian space is called a *Kählerian space* K_n if, together with the metric tensor g, an affine structure F is defined that satisfies the relations (3.60):

$$F^2 = e\,Id; \quad g(X, FY) + g(Y, FX) = 0; \quad \nabla F = 0, \quad \text{where } e = \pm 1. \tag{13.1}$$

If $e = -1$, K_n is said to be a (classical) *elliptic Kählerian space* K_n^-. If $e = +1$, K_n is said to be a *hyperbolic Kählerian space* K_n^+. The spaces K_n must be of even dimension.

Definition 13.1 (T. Otsuki, Y. Tashiro [737] for $\mathbb{K}_n^-$) A curve ℓ in $\mathbb{K}_n$ which is given by the equations $\ell = \ell(t)$, $\lambda(t) = d\ell/dt$ $(\neq 0)$, $t \in I$, where t is a parameter, is called *analytically planar* , if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = \operatorname{Span}\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda, \tag{13.2}$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

Particularly, in the case $b(t) = 0$, an analytically planar curve is a geodesic.

The parameter t for an analytically planar curve can be chosen in such a way that $a(t) = 0$; such a parameter will be called *canonical* . Analytically planar curves parametrized by canonical parameter are solutions of the equations $\nabla_t \lambda = b(t)F\lambda$. We can check that for a prescribed function $b(t)$, this equation is uniquely solvable for the initial data $\ell(t_0) = x_0 \in \mathbb{K}_n$, $\lambda(t_0) = \lambda_0$. Consequently, the family of analytically planar curves in $\mathbb{K}_n$ depends on one function of one variable and $2n - 1$ parameters. This function plays the role of the first curvature of an analytically planar curve in $\mathbb{K}_n$. We mention the following definition which is connected with analytically planar curves.

Definition 13.2 A Kähler manifold $\mathbb{K}_n = (M, g, F)$ is said to be *H-complete* (shortly *complete*) if any analytically planar curve $\ell(t)$ is defined on M for all values of the canonical parameter $t \in \mathbb{R}$.

Definition 13.3 A mapping of a Kähler manifold $\mathbb{K}_n$ with the structure tensor F onto a Kähler manifold $\overline{\mathbb{K}}_n$ with the structure tensor $\overline{F}$ is called *holomorphically projective* (or *analytically planar*), if each analytically planar curve in $\mathbb{K}_n$ is mapped onto an analytically planar curve in $\overline{\mathbb{K}}_n$.

Obviously, analytically planar curves introduced above, and holomorphically projective mappings, belong in fact to F-planar curves (Def. 12.1, p. 385), and F-planar mappings (Def. 12.3, p. 386), respectively, such that the structure F ($\overline{F}$) is the structure tensor of the Kähler manifold $\mathbb{K}_n$($\overline{\mathbb{K}}_n$, respectively). From the results of Chapter 9, the following theorem follows as a consequence.

Theorem 13.1 *A holomorphically projective mapping f from $\mathbb{K}_n$ onto $\overline{\mathbb{K}}_n$ preserves the structures and is characterized by the following condition*

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + e\,\psi(FX)\,FY + e\,\psi(FY)\,FX \tag{13.3}$$

for any vector fields X, Y, where $P = \overline{\nabla} - \nabla$ is the deformation tensor field of f, ψ is a linear form.

In a common coordinate system (x) with respect to the mapping f, the components of the structure tensors coincide, $\overline{F}{}_i^h(x) = F_i^h(x)$, and the Christoffels are related by

$$\overline{\Gamma}{}_{ij}^h = \Gamma_{ij}^h + \psi_i\delta_j^h + \psi_j\delta_i^h + e\psi_{\bar{i}}\delta_{\bar{j}}^h + e\psi_{\bar{j}}\delta_{\bar{i}}^h \tag{13.4}$$

where $\psi_i(x)$ are components of the one-form ψ. By the definition of conjugation, $\psi_{\bar{i}} = \psi_\alpha F_i^\alpha$, $\quad \delta_{\bar{i}}^h = F_i^h$.

Proof. Let there exist a holomorphically projective mapping f of a Kähler manifold $\mathbb{K}_n$ with the structure tensor F onto a Kähler manifold $\overline{\mathbb{K}}_n$ with the structure tensor $\overline{F}$.

From Theorem 12.1 follows $\overline{F} = a\,F + b\,Id$. Since $F^2 = e\,Id$ and $\overline{F}^2 = \overline{e}\,Id$, $e, \overline{e} = \pm 1$, then $a = \pm 1$, $b = 0$, $\overline{e} = e$. We choose $\overline{F} = F$. Because $\nabla F = \overline{\nabla} F = 0$, from the Theorem 12.4 follows that $\psi(x) = \varphi(FX)$, and, evidently, $\varphi(X) = e\,\psi(FX)$. Finally from Theorem 12.1 we obtain the formula (13.3). □

According to Theorem 4.1 the formulae (13.4) are equivalent to

$$\overline{g}_{ij,k} = 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} - e\psi_{\bar{i}}\overline{g}_{\bar{j}k} - e\psi_{\bar{j}}\overline{g}_{\bar{i}k}, \tag{13.5}$$

where " , " is the covariant derivative in $\mathbb{K}_n$.

So the systems of equations (13.4) and (13.5) show two equivalent ways of presenting the fundamental equations of the theory of holomorphically projective mappings. Holomorphically projective mappings with $\psi \equiv 0$ will be called *trivial* or *affine*

Remark 13.1 Note that holomorphically projective mappings of elliptic Kähler manifolds were introduced and investigated by T. Otsuki and Y. Tashiro [737]. They supposed apriori that the complex structure is preserved.

In what follows let us consider holomorphically projective mappings preserving the stucture tensor without specifying it explicitely. That is, $\mathbb{K}_n$ admits a holomorphically projective mapping f onto $\overline{\mathbb{K}}_n$ if and only if in a common coordinate system (relative to f), (13.4) holds and the structure tensors of $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ coincide.

13.1.2 Equivalence classes of holomorphically projective mappings

It follows from the equations (13.4) that the class of Kähler manifolds is decomposed into equivalence subclasses with respect to holomorphically projective mappings.

A holomorphically projective mapping $f\colon \mathbb{K}_n \to \overline{\mathbb{K}}_n$, which is characterized by the equations (13.4), will be denoted by

$$\mathbb{K}_n \xrightarrow{HPM\ (\psi)} \overline{\mathbb{K}}_n.$$

The properties of equivalence can be easily checked:

1. The identity is holomorphically projective since we have $\mathbb{K}_n \xrightarrow{HPM\ (0)} \overline{\mathbb{K}}_n$.
2. If $\mathbb{K}_n \xrightarrow{HPM\ (\psi)} \overline{\mathbb{K}}_n$, then $\overline{\mathbb{K}}_n \xrightarrow{HPM\ (-\psi)} \mathbb{K}_n$.
3. If $\mathbb{K}_n \xrightarrow{HPM\ (\psi)} \overline{\mathbb{K}}_n$ and $\overline{\mathbb{K}}_n \xrightarrow{HPM\ (\overline{\psi})} \overline{\overline{\mathbb{K}}}_n$, then $\mathbb{K}_n \xrightarrow{HPM\ (\psi+\overline{\psi})} \overline{\overline{\mathbb{K}}}_n$.

Two manifolds $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ belong to the same *holomorphically projective class* iff there is a holomorphically projective mapping of $\mathbb{K}_n$ onto $\overline{\mathbb{K}}_n$. Spaces from the same holomorphically projective class are also called *holomorphically projectively equivalent*. We can apply this concept locally as well as globally.

13.1.3 Some geometric objects under HPM

Assume holomorphically projective mappings between the Kähler manifolds $\mathbb{K}_n \to \overline{\mathbb{K}}_n$. Then the equations (13.4) are satisfied.

Contracting (13.4) with respect h and j we find $\overline{\Gamma}^{\alpha}_{i\alpha} = \Gamma^{\alpha}_{i\alpha} + (n+2)\psi_i$. In any Riemannian manifold, the formula of Voss-Weyl holds, $\Gamma^{\alpha}_{i\alpha} = \frac{1}{2}\delta_i \ln|g|$. where $g = \det(g_{ij})$. In the case of holomorphically projective mapping of $\mathbb{K}_n$, it follows $2(n+2)\psi_i = \delta_i \ln|\overline{g}/g|$, where $\overline{g} = \det \overline{g}_{ij}$. It means that the form $\psi = d\Psi$ is exact, i.e. a gradient, namely, $\psi_i = \Psi_{,i}$ for the function

$$\Psi = \frac{1}{2(n+2)} \ln\left|\frac{\overline{g}}{g}\right|. \tag{13.6}$$

If we eliminate the components $\psi_i = \dfrac{1}{n+2}\left(\overline{\Gamma}^{\alpha}_{i\alpha} - \Gamma^{\alpha}_{i\alpha}\right)$ in (13.4) we get

$$\overline{T}^{h}_{ij}(x) = T^{h}_{ij}(x), \tag{13.7}$$

where

$$T^{h}_{ij}(x) = \Gamma^{h}_{ij} - \frac{1}{n+2}\left(\delta^{h}_{i}\Gamma^{\alpha}_{j\alpha} + \delta^{h}_{j}\Gamma^{\alpha}_{i\alpha} + e\delta^{h}_{\bar{i}}\Gamma^{\alpha}_{\bar{j}\alpha} + e\delta^{h}_{\bar{j}}\Gamma^{\alpha}_{\bar{i}\alpha}\right), \tag{13.8}$$

and $\overline{T}^{h}_{ij}$ have a similar form. In this way, we have proved invariance of geometric objects (13.8) under holomorphically projective mappings. These objects are similar to the Thomas parameters (6.13) of the theory of geodesic mappings.

The equality of Thomas-like objects (13.8) is a necessary and sufficient condition for existence of holomorphically projective mappings between $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$.

We can also deduce a relationship between components of the curvature tensors. In fact, from (13.4) and the well-known formulae (4.4) we have

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} + e\delta^h_{\bar{k}}\psi_{\bar{i}j} + e\delta^h_{\bar{j}}\psi_{\bar{i}k} + e\delta^h_{\bar{i}}(\psi_{\bar{k}j} - \psi_{\bar{j}k}), \qquad (13.9)$$

where we have introduced

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j - e\psi_{\bar{i}}\psi_{\bar{j}}. \qquad (13.10)$$

The type $(0,2)$ tensor ψ_{ij} is symmetric; we contract (13.9) in h,k and get for components the corresponding Ricci tensors on $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$:

$$\overline{R}_{ij} = R_{ij} - n\psi_{ij} + 2e\psi_{\bar{i}\bar{j}}.$$

Due to (3.90) and similar conditions for $\overline{\mathbb{K}}_n$, we deduce $\psi_{\bar{i}\bar{j}} = -e\psi_{ij}$. Hence the Ricci tensors are related by

$$\overline{R}_{ij} = R_{ij} - (n+2)\psi_{ij}. \qquad (13.11)$$

Now let us use (13.11) to eliminate ψ from (13.9). If we introduce

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left\{\delta^h_k R_{ij} - \delta^h_j R_{ik} + e\delta^h_{\bar{k}}R_{\bar{i}j} - e\delta^h_{\bar{j}}R_{\bar{i}k} + e\delta^h_{\bar{i}}(R_{\bar{k}j} - R_{\bar{j}k})\right\}$$

and define similarly $\overline{P}^h_{ijk}$ in $\mathbb{K}_n$ then the resulting formula is

$$\overline{P}^h_{ijk} = P^h_{ijk}. \qquad (13.12)$$

The last equation tells that the tensor P with components P^h_{ijk}, called the *tensor of holomorphically projective curvature*, is preserved under the mappings under consideration.

So we have proved:

Theorem 13.2 *The objects (13.8) and the tensor of holomorphically projective curvature are invariants under holomorphically projective mapings.*

Note that for hyperbolic Kähler manifods $\mathbb{K}^+_n$, this theorem was proved by M. Prvanović [754].

13.1.4 Holomorphically projectively flat Kähler manifolds

Let us introduce the following definitions.

Definition 13.4 A Kähler manifold K_n is called *holomorphically projective flat* if there exists a holomorphically projective mapping onto a flat space.

Obviously, for holomorphically projectively flat manifolds, the tensor of holomorphically projective curvature is zero.

Moreover, this condition fully characterizes such mappings as we show.

If $\mathbb{K}_n$ is a Kähler manifold with vanishing holomorphically projective curvature tensor, lowering the index h we get

$$(n+2)R_{hijk} = g_{hj}R_{ik} - g_{hk}R_{ij} + eg_{h\bar{k}}R_{i\bar{j}} - eg_{h\bar{j}}R_{i\bar{k}} - eg_{h\bar{i}}(R_{\bar{k}j} - R_{\bar{j}k}).$$

Contracting both sides with g^{hk} (the matrix inverse to g_{ij}) and making use of (3.82), (3.84), (3.91) and (13.12), we find for the Ricci tensor $Ric = \frac{R}{n}g$ where $R = R_{\alpha\beta}g^{\alpha\beta}$ is the scalar curvature. Whence the above formulas read

$$R_{hijk} = \frac{R}{n(n+2)}(g_{hj}g_{ik} - g_{hk}g_{ij} + eg_{h\bar{k}}g_{i\bar{j}} - eg_{h\bar{j}}g_{i\bar{k}} - 2eg_{h\bar{i}}g_{j\bar{k}}). \qquad (13.13)$$

Comparing (13.13) and (3.96) we conclude that the manifold $\mathbb{K}_n$ under consideration is a space of constant holomorphic curvature equal to $k = \frac{4R}{n(n+2)}$. On the other hand, from (13.13) we get $P^h_{ijk} = 0$. That is, for $n > 2$, the Kähler manifold is of constant holomorphic curvature if and only if the corresponding tensor of holomorphically projective curvature is zero identically. Since the tensor of holomorphically projective curvature is preserved under holomorphically projective mappings we have proved:

Theorem 13.3 *If a Kähler manifold $\mathbb{K}_n$ admits a holomorphically projective mapping onto a Kähler manifold $\overline{\mathbb{K}}_n$ of constant holomorphic curvature then $\mathbb{K}_n$ is also a space of constant holomorphic curvature.*

The first proof of the Theorem 13.3 for Kähler manifolds is due to Otsuki and Tashiro. We can establish a stronger result:

Theorem 13.4 *For any pair $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ of the spaces of constant holomorphic curvature there exists (locally) a non-trivial holomorphically projective mapping that sends one to the other.*

Proof. Suppose that a Kähler manifold $\mathbb{K}_n$ with constant holomorphic curvature k, metric tensor g and structure tensor F are given. Let us consider the system of equations (13.5)

$$\begin{aligned} \overline{g}_{ij,k} &= 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik} - e\psi_{\bar{i}}\,\overline{g}_{\bar{j}k} - e\psi_{\bar{j}}\,\overline{g}_{\bar{i}k}, \\ \psi_{i,j} &= \psi_i\psi_j + e\psi_{\bar{i}}\psi_{\bar{j}} - 4e(\overline{k}\,\overline{g}_{ij} - k\,g_{ij}), \end{aligned} \qquad (13.14)$$

where $\overline{k}$ is some (arbitrary) real constant.

Our equations form a system of Cauchy type in first order covariant derivatives in the variables $\overline{g}_{ij}(x)$ and $\psi_i(x)$. We complete the system by $\overline{g}_{\bar{i}\bar{j}} = -e\overline{g}_{ij}$ to a completely integrable system, that is, the integrability conditions for (13.14) are identically satisfied in $\mathbb{K}_n$. The differential prolongations of $\overline{g}_{\bar{i}\bar{j}} = -e\overline{g}_{ij}$ are identically satisfied as well. As a consequence, for any initial data

$$\overline{g}_{ij}(x_0) = \overset{0}{\overline{g}}_{ij}, \quad \psi_i(x_0) = \overset{0}{\psi}_i, \quad (\overset{0}{\overline{g}}_{ij} = \overset{0}{\overline{g}}_{ji} = -e\overset{0}{\overline{g}}_{\bar{i}\bar{j}}, |\overset{0}{\overline{g}}_{ij}| \neq 0, \overset{0}{\psi}_i \neq 0)$$

there exist a solution $\overline{g}_{ij}(x)$ and $\psi_i(x)$. Accounting (13.9), (13.10) and (13.14) we find easily that the metric tensor $\overline{g} = (\overline{g}_{ij}(x))$ defines a Kähler manifold $\overline{\mathbb{K}}_n$

of constant holomorphic curvature $\overline{k}$. Since the space of constant holomorphic curvature is determined by its holomorphic curvature (and by the signature of the metric) uniquely up to isometry (see Theorem 4.9, p. 195) it is guaranteed that any space $\mathbb{K}_n$ of constant holomorphic curvature k admits non-trivial holomorphically projective mapping onto another Kähler manifold of arbitrary constant holomorphic curvature $\overline{k}$. □

13.2 Fundamental theorems of the theory of HP mappings

The equations (13.4) and (13.5) are fundamental equations that control the existence of holomorphically projective mappings of Kähler manifolds. In the given Kähler manifold these equations represent a first order system of non-linear partial differential equations in the components $\overline{g}_{ij}(x)$ of the metric tensor and the components $\psi_i(x)$ of a one-form.

For examining the conditions of existence and uniqueness of the solution, the methods explained above cannot be directly used. Using the methods developed in the theory of geodesic mappings of Riemannian manifolds, we show how our methods can be modified, presented in a principially new way, that allows us to discover new general aspects of the theory.

13.2.1 Linear fundamental equations of the theory of HPM

Assume $\mathbb{K}_n$ admits a holomorphically projective mapping onto $\overline{\mathbb{K}}_n$. Then the equations (13.5) hold relative to a common coordinate system with respect to the mapping. Let us set

$$\tilde{g}_{ij} = \exp(-2\psi)\overline{g}_{ij}. \tag{13.15}$$

Naturally, the tensor $\tilde{g}$ satisfies

$$\tilde{g}_{ij} = \tilde{g}_{ji} = -e\tilde{g}_{\bar{i}\bar{j}}, \quad |\tilde{g}_{ij}| \neq 0. \tag{13.16}$$

Covariant differentiation of (13.15) in $\mathbb{K}_n$ and (13.5) gives

$$\tilde{g}_{ij,k} = \psi_{(i}\tilde{g}_{j)k} - e\psi_{(\bar{i}}\tilde{g}_{\bar{j})k}. \tag{13.17}$$

The components of the dual tensor $(\tilde{g}^{ij})$ to $(\tilde{g}_{ij})$,

$$\tilde{g}^{i\alpha}\tilde{g}_{\alpha j} = \delta^i_j, \tag{13.18}$$

satisfy the equations similar to (13.16), namely

$$\tilde{g}^{ij} = \tilde{g}^{ji} = -e\tilde{g}^{\bar{i}\bar{j}}, \quad |\tilde{g}^{ij}| \neq 0. \tag{13.19}$$

Covariant differentiation of (13.18) in $\mathbb{K}_n$ gives $\tilde{g}^{i\alpha}{}_{,k}\tilde{g}_{\alpha j} + \tilde{g}^{i\alpha}\tilde{g}_{\alpha j,k} = 0$ or, by (13.18), $\tilde{g}^{ij}{}_{,k} = -\tilde{g}_{\alpha\beta,k}\tilde{g}^{\alpha i}\tilde{g}^{\beta j}$. According to (13.17) we get

$$\tilde{g}^{ij}{}_{,k} = \lambda^{(i}\delta^{j)}_k - e\lambda^{(\bar{i}}\delta^{\bar{j})}_k, \tag{13.20}$$

where

$$\lambda^i = -\tilde{g}^{i\alpha}\psi_\alpha. \tag{13.21}$$

If we lower (in $\mathbb{K}_n$) the indices i, j in (13.20) and set

$$a_{ij} = \tilde{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}, \quad \lambda_i = \lambda^\alpha g_{\alpha i} \tag{13.22}$$

we get

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}}g_{\bar{j}k} - e\lambda_{\bar{j}}g_{\bar{i}k}. \tag{13.23}$$

Obviously, by (13.19) and (13.22) the tensor $a(a_{ij})$ satisfies

$$a_{ij} = a_{ji} = -ea_{\bar{i}\bar{j}}, \quad |a_{ij}| \neq 0. \tag{13.24}$$

Hence if (13.5) holds then the equations (13.23) and (13.24) are necessarily satisfied.

The other implication also holds. In fact, suppose that a_{ij} and ψ_i satisfy (13.23) and (13.24). Then (13.20) and (13.19) hold with $\tilde{g}^{ij} = a_{\alpha\beta}g^{\alpha i}g^{\beta j}$ and $\lambda^i = \lambda_\alpha g^{\alpha i}$. Since the tensor $\tilde{g}^{ij}$ satisfies (13.19) then its dual tensor $\tilde{g}_{ij}$ has analogous properties (13.16). The covariant derivative of (13.18) together with (13.20) gives (13.17) where $\psi_i = -\lambda^\alpha \tilde{g}_{\alpha i}$. Let $\tilde{\Gamma}^h_{ij}$ be components of the metric connection corresponding to the metric tensor $\tilde{g}_{ij}$. According to

$$\tilde{\Gamma}^\alpha_{i\alpha} = \frac{1}{2}\tilde{g}^{\alpha\beta}\partial_i\tilde{g}_{\alpha\beta} = \frac{1}{2}\tilde{g}^{\alpha\beta}(\tilde{g}_{\alpha\beta,i} + 2\tilde{g}_{\alpha\gamma}\Gamma^\gamma_{\beta i}),$$

(13.17) and (13.18), we find $\psi_i = \tilde{\Gamma}^\alpha_{\alpha i} - \Gamma^\alpha_{\alpha i}$. Now by the Voss-Weyl formulas we have $\psi_i = \frac{1}{2}\,\partial_i \ln\left|\frac{\tilde{g}}{g}\right|$ where $g = |g_{ij}|$ and $\tilde{g} = |\tilde{g}_{ij}|$, that is, ψ_i is a gradient, i.e. $\psi_i = \Psi_{,i}$ with $\Psi = \frac{1}{2}\ln\left|\frac{\tilde{g}}{g}\right|$. We find $\overline{g}_{ij} = e^{2\Psi}\tilde{g}_{ij}$, and check that (13.5) follows from (13.17). Obviously, $\overline{g}_{ij}$ is a metric tensor of the Kähler manifold $\mathbb{K}_n$ with the structure tensor F. Hence we have proved

Theorem 13.5 *The Kähler manifold $\mathbb{K}_n$ admits holomorphically projective mapping onto the $\overline{\mathbb{K}}_n$ if and only if in $\mathbb{K}_n$, there exists a solution of (13.23) and (13.24).*

Remark. In the Theorem 13.5, the relations (13.24) may be omitted. Let $\tilde{a}_{ij}$ fulfils the equations (13.23) for certain $\lambda_i \neq 0$. Then a tensor $a_{ij} = \frac{1}{4}\,(\tilde{a}_{(ij)} - e\,\tilde{a}_{(\bar{i}\bar{j})}) + c\,g_{ij}$, where c is a suitable constant, fulfils (13.24) and (13.23), simultaneously.

It also follows from this theorem:

Theorem 13.6 *If a Kähler space $\mathbb{K}_n$ admits a nontrivial geodesic mapping, then $\mathbb{K}_n$ admits a nontrivial holomorphically projective mapping.*

Proof. Let us suppose that the Kähler space $\mathbb{K}_n$ admits a nontrivial geodesic mapping, i.e. in $\mathbb{K}_n$ there exists a tensor $\tilde{a}_{ij}$ which fulfils the Sinyukov equations (8.6): $\tilde{a}_{ij,k} = \lambda_i\,g_{jk} + \lambda_j\,g_{ik}$. Putting $a_{ij} = \tilde{a}_{ij} - e\,\tilde{a}_{\bar{i}\bar{j}}$ we clearly find out, that the equations (13.23) hold. □

Therefore, equidistant Kähler manifolds constructed in Section 2.3, p. 153, admit also a nontrivial holomorphically projective mappings.

13.2.2 The first quadratic integral of geodesics and HP mappings

Contracting (13.23) with g^{ij} gives $4\lambda_i = (a_{\alpha\beta}g^{\alpha\beta})_{,i}$. Thus the one-form is exact, $\lambda = d\Lambda$, i.e. $\lambda_i = \partial_i\Lambda$ for some function Λ that depends on derivatives of $a_{\alpha\beta}g^{\alpha\beta}$.

If a Kähler manifold $\mathbb{K}_n$ admits non trivial holomorphically projective mappings onto $\overline{\mathbb{K}}_n$, then in $\mathbb{K}_n$, there exist two indepent nontrivial first quadratic integrals of geodesics:

$$\omega = \exp(-4\,\Psi)\;\overline{g}, \quad \text{and} \quad \omega = a - 2\Lambda\,g. \tag{13.25}$$

Here g and $\overline{g}$ are metrics of $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$, a is a symmetric form from the solution of the Domashev-Mikeš-Kurbatova equation (13.23).

13.2.3 Fundamental equations of HPM in Cauchy form

If a Kähler manifold $\mathbb{K}_n$ is given we know of course its metric tensor g as well as its structure tensor F. To be able to decide the question on existence or non-existence of non-trivial holomorphically projective mappings of the Kähler manifold $\mathbb{K}_n$, we must investigate the system (13.23) with respect to the components a_{ij} of the tensor a. But the system is not closed since on the right hand side of (13.23), there are also components λ_i of the unknown one-form λ; these components can be expressed by means of the tensors a, g: $\lambda_i = \frac{1}{4}\left(a_{\alpha\beta}g^{\alpha\beta}\right)_{,i} = \partial_i\Lambda$ and some function Λ that depends on the derivatives of $a_{\alpha\beta}g^{\alpha\beta}$. But it means that the system (13.23) turns out not to be solvable with respect to the derivatives of the unknown functions, that is, it cannot be examined by regular methods. Hence keeping in mind that this new form of the system (13.23) is not convenient for us, we try to overcome the difficulties by studying the integrability conditions of the system. For this purpose, let us express the covariant derivative of (13.23) with respect to x^ℓ, alternate in k, ℓ, account skew symmetry of $g_{j\bar{k}}$, the Ricci identity and the fact that F is parallel, $F^h_{i,j} = 0$. We get

$$a_{\alpha i}R^\alpha_{jkl} + a_{\alpha j}R^\alpha_{ikl} = g_{k(i}\lambda_{j),l} - eg_{k(\bar{i}}\lambda_{\bar{j}),l} - g_{l(i}\lambda_{j),k} + eg_{l(\bar{i}}\lambda_{\bar{j}),k}. \tag{13.26}$$

Contracting (13.26) with g^{jk} we find

$$n\lambda_{l,i} = \mu g_{li} - a_{\alpha i}R^\alpha_l - a_{\alpha\beta}R^{\alpha..\beta}_{.il.} + e\lambda_{\bar{i},\bar{l}} + \lambda_{i,l}, \tag{13.27}$$

where

$$\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}, \quad R^{h..k}_{.ij.} = R^h_{.ij\alpha}g^{\alpha k}, \quad R^h_i = R_{i\alpha}g^{\alpha h}.$$

From (3.85), (3.88), (3.90) and (13.24) it is not difficult to show that the 1-form λ is analytic, that is, it satisfies $\lambda_{\bar{i},\bar{j}} = -e\lambda_{i,j}$. Thus the expression (13.27) reads

$$n\lambda_{i,j} = \mu g_{ij} - a_{\alpha j}R^\alpha_i - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{13.28}$$

Alternating the last formula in i, j we arrive at

$$a_{i\alpha}R^\alpha_j - a_{j\alpha}R^\alpha_i = 0. \tag{13.29}$$

The integrability conditions of (13.28) take the form

$$\begin{aligned}(n+2)\lambda_\alpha R^\alpha_{ijk} &= g_{ij}(\mu_{,k} + \lambda_\alpha R^\alpha_k) - g_{ik}(\mu_{,j} + \lambda_\alpha R^\alpha_j)\\ &+a_{\alpha i}R^{\alpha\beta}_{..kj,\beta} - a_{\alpha\beta}R^{\alpha..\beta}_{.ijk,.} + eg_{i[\bar{j}}R^\alpha_{\bar{k}]}\lambda_\alpha + 2e\lambda_{\bar{i}}R_{\bar{j}k}.\end{aligned} \tag{13.30}$$

Using (3.89) and conjugation in (13.30) we obtain $g_{i[j}M_{k]} + eg_{i[\bar{j}}M_{\bar{k}]} = 0$, where $M_i = \mu_{,i} + 2\lambda_\alpha R^\alpha_i$. Contracting with g^{ij} we get $M_i = 0$, thus we calculate

$$\mu_{,i} = -2\lambda_\alpha R^\alpha_i. \tag{13.31}$$

Now (13.23), (13.28) and (13.31) is a closed system of differential equations of Cauchy type in the unknown functions $a_{ij}(x)$, $\lambda_i(x)$ and $\mu(x)$.

We proved (for $\mathbb{K}_n^-$: Mikeš [628], for $\mathbb{K}_n^+$: Kurbatova [569]; see [119, 122, 171, 173]):

Theorem 13.7 *A Kähler manifold $\mathbb{K}_n$ admits holomorphically projective mappings if and only if the system of differential equations*

$$\begin{aligned} (a) \quad a_{ij,k} &= \lambda_i g_{jk} + \lambda_j g_{ik} - e\,\lambda_{\bar{i}} g_{\bar{j}k} - e\,\lambda_{\bar{j}} g_{\bar{i}k}, \\ (b) \quad n\,\lambda_{i,j} &= \mu g_{ij} - a_{i\alpha} R^{\alpha}_{j} - a_{\alpha\beta} R^{\alpha..\beta}_{.ij.} \\ (c) \quad \mu_{,i} &= -2\,\lambda_\alpha R^{\alpha}_{i}, \end{aligned} \tag{13.32}$$

has a solution a_{ij}, λ_i and μ satisfying the conditions (13.24).

Remark The proof of this Theorem is in the case that $\mathbb{K}_n, \overline{\mathbb{K}}_n \in C^3$. In [473] the equations (13.32) are proved in the case $\mathbb{K}_n^- \in C^2$ and $\overline{\mathbb{K}}_n^- \in C^1$. It also valides for the case $\mathbb{K}_n^+ \in C^2$ and $\overline{\mathbb{K}}_n^+ \in C^1$.

Now, we simplify the proof of Theorem 13.7 for the case $\mathbb{K}_n, \overline{\mathbb{K}}_n \in C^2$:

Proof. For these conditions equations (13.32a), (13.32b) and conditions (13.26) – (13.29) are valid. Because λ_i is gradient-like, from $\lambda_{\bar{i},\bar{j}} = -e\lambda_{i,j}$ it follows $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$, this means that $\lambda_{\bar{i}}$ is the Killing vector (for $\mathbb{K}_n^-$ this fact is in [262]). From this it follows that $\lambda_{\bar{i}}$, $\lambda_i \in C^2$ and that $\lambda_{\bar{i}}$ is also affine vector for which (4.77): $\lambda_{\bar{i},jk} = \lambda_{\bar{\alpha}} R^{\alpha}_{kji}$ holds. After contraction with $g^{\bar{i}j}$ we obtain equation (13.32c). □

From the above mentioned it follows:

Theorem 13.8 *If a holomorphically projective mapping $f\colon \mathbb{K}_n \to \overline{\mathbb{K}}_n$ is locally trivial, then f is globally trivial.*

Proof. If in a neighbourhood U at point x_0: $\lambda_i|_U = 0$, then $\lambda_{\bar{i}}(x_0) = 0$ and $\partial_j \lambda_{\bar{i}}(x_0) = 0$. For these initial conditions the equations: $\lambda_{\bar{i},jk} = \lambda_{\bar{\alpha}} R^{\alpha}_{kji}$ has only trivial solution: $\lambda_{\bar{i}} = 0$. This implies global trivial solution $\lambda_i = 0$. □

Remark For many special Kähler spaces it is proved local existence of only trivial holomorphically projective mapping, see 13.2.6. It follows from this Theorem that there does not exist any global holomorphically projective mappings.

The family of differential equations (13.32) is linear with coefficients of intrinsic character in $\mathbb{K}_n$ and independent of the choice of coordinates. If the metric tensor g and the structure tensor F of the Kähler manifold $\mathbb{K}_n$ are real then for the initial data

$$a_{ij}(x_0) = \overset{o}{a}{}_{ij}, \quad \lambda_i(x_0) = \overset{o}{\lambda}{}_i, \quad \mu(x_0) = \overset{o}{\mu} \tag{13.33}$$

the system (13.32) has at most one solution. Accounting that the initial data

must satisfy (13.24), which in read, canonical coordinates,

$$\begin{aligned} &a_{a+mb+m} = a_{ab}, \quad a_{a+mb} = -a_{ab+m} \quad (e=-1),\\ &a_{ab} = a_{a+mb+m} = 0 \quad (e=+1), \end{aligned} \tag{13.34}$$

it follows that the general solution of (13.32) depends on r_{hpm} significant parameters where

$$r_{hpm} \leq (m+1)^2, \qquad m = \frac{n}{2}. \tag{13.35}$$

The solution of (13.32) satisfying the given initial conditions (13.33) can be given as a Taylor series, and if necessary, enumerated in a neighborhood of a given point x_0 of the space.

13.2.4 Integrability conditions of fundamental equations of HPM

Of course, the system (13.32) might not be consistent. The solution of (13.32) exists if and only if the integrability conditions are satisfied. As far as the first part of (13.23) is concerned, the integrability conditions are obtained from (13.26) (if we substitute derivations of λ_i by expressions from (13.28)) in the form

$$a_{\alpha\beta} T^{\alpha\beta}_{ijkl} = 0, \tag{13.36}$$

where

$$T^{\alpha\beta}_{ijkl} = n\delta^{\alpha}_{(i} R^{\beta}_{j)kl} + g_{k(i} M^{\alpha\beta}_{j)l} - g_{l(i} M^{\alpha\beta}_{j)k} - e g_{k(\bar{i}} M^{\alpha\beta}_{\bar{j})l} + e g_{l(\bar{i}} M^{\alpha\beta}_{\bar{j})k}, \tag{13.37}$$

where $M^{\alpha\beta}_{jl} = -\delta^{\alpha}_{j} R^{\beta}_{l} - R^{\alpha..\beta}_{.jl.}$. For the second part of (13.32), namely (13.28), the integrability conditions are obtained from (13.30) if we substitute $\mu_{,i}$ according to (13.31):

$$(n+2)\lambda_\alpha P^{.\alpha}_{i.jk} - a_{\alpha i} R^{\alpha\beta}_{..jk,\beta} - a_{\alpha\beta} R^{\alpha}_{.ijk,.}{}^{\beta} = 0 \tag{13.38}$$

where $P_{i.jk}^{\ \ h} = g^{h\beta} g_{i\alpha} P^{\alpha}_{\beta jk}$.

Finally, the integrability conditions of (13.31) are

$$a_{\alpha\beta}(R^{\gamma}_{l} R^{\alpha}_{.\gamma k.}{}^{\beta} - R^{\gamma}_{k} R^{\alpha}_{.\gamma l.}{}^{\beta}) + n\lambda_\alpha R^{\alpha}_{[k,l]} = 0. \tag{13.39}$$

Together, the integrability conditions of the system (13.32) are given by (13.36), (13.38) and (13.39); let us add also (13.24), and denote the new system by (B). Obviously, (B) is a system of linear homogeneous algebraic equations in a_{ij} and λ_i with coefficients in $\mathbb{K}_n$. They must be satisfied identically for any solution of the system (13.32) whenever it exists.

If we differentiate (B) covariantly and use (13.32) we obtain the first prolongation of (B) which we denote by (B_1). Obviously, (B_1) is also a linear homogeneous algebraic system of equations in a_{ij}, λ_i and μ with coefficients in $\mathbb{K}_n$. The differential prolongation of (B_1) will be denoted by (B_2) etc. Since the number of unknown functions is finite there is a natural number N, $N \leq (m+1)^2$ for which the prolongations (B_N) of order N are consequences of (B), (B_1), …, (B_{N-1}). As well known from the analytic theory of differential equations, the system (13.32) has a non-trivial solution in a neighborhood of a fixed point x_0 if and only if the equations (B), (B_1), …, (B_{N-1}) have a non-trivial solution in this point. Hence the following holds:

Theorem 13.9 *A Kähler manifold $\mathbb{K}_n$ admits non-trivial holomorphically projective mappings onto a Kähler manifold $\overline{\mathbb{K}}_n$ if and only if the system (B), (B_1), $\ldots$, (B_{N-1}) has in $\mathbb{K}_n$ a non-trivial solution $a_{ij}(x)$, $\lambda_i(x)$, $\mu(x)$ such that* $\det(a_{ij}) \neq 0$, $\lambda_i \neq 0$.

Using regular methods, for a given Kähler manifold $\mathbb{K}_n$ we can (from the given components $g_{ij}(x)$ of the metric and $F_i^h(x)$ of the complex structure) set up the system of equations (B), (B_1), $\ldots$, (B_{N-1}) and check whether the system has a solution or not. Since the system has a tensorial character (particularly, is independent of the choice of local coordinates) the Theorem 13.9 includes necessary and sufficient intrinsic tensorial properties of the prescribed Kähler manifold as far as the existence or non-existence of non-trivial holomorphically projective mappings is concerned, of course, in an implicit form. The Theorems 13.7 and 13.9 together supply us with a regular method that anables us to decide effectively whether a Kähler manifold admits non-trivial holomorphically projective mappings or not, and in the affirmative case, we are in principle able to find all Kähler manifolds that can serve as images of $\mathbb{K}_n$ under the mappings considered. Hence Theorems 13.7 and 13.9 turn out to be the fundamental theorems of the theory of holomorphically projective mappings.

If we wish to solve the system (B), (B_1), $\ldots$, (B_{N-1}), or even (13.32), in practice for large n, technical complications might arise that cannot be overcome "by power". It seems to be necessary to pay attention to qualitative investigations of the above equations for the sake of specification of those Kähler manifolds that admit solutions of the system, even with a large freedom, on the other hand distinguishing such manifolds for which there is no solution. In what follows, we intend to show some steps in this direction.

13.2.5 Reduction of fundamental equations of HP mappings

Let us calculate the covariant derivative of (13.36) in $\mathbb{K}_n$ with respect to x^h. By (13.23) we have

$$(\lambda_{(\alpha}g_{\beta)h} - e\lambda_{(\overline{\alpha}}g_{\overline{\beta})h})\,T^{\alpha\beta}_{ijkl} + a_{\alpha\beta}T^{\alpha\beta}_{ijkl,h} = 0. \tag{13.40}$$

Let us ivestigate the linear algebraic homogeneous system of equations in the form

$$(\lambda^*_{(\alpha}g_{\beta)h} - e\lambda^*_{(\overline{\alpha}}g_{\overline{\beta})h})\,T^{\alpha\beta}_{ijkl} = 0$$

with the unknowns λ_i^*. From this it follows that $\lambda_i^* = 0$ or the tensor of holomorphically projective curvature is vanishing. Evidently from (13.40) follows

Theorem 13.10 *Let $\mathbb{K}_n$ be a Kähler manifold which is not (locally) holomorphically projectively flat around a point $p \in \mathbb{K}_n$. There is a neighorhood of p at $\mathbb{K}_n$ which admits a holomorphically projective mapping onto $\overline{\mathbb{K}}_n$, if and only if the complete set of linear differential equations of Cauchy type in the covariant derivatives in $\mathbb{K}_n$*

$$a_{ij,k} = \mathcal{Z}_i(a)\,g_{jk} + \mathcal{Z}_j(a)\,g_{ik} - e\mathcal{Z}_{\bar{i}}(a)\,g_{\bar{j}k} - e\mathcal{Z}_{\bar{j}}(a)\,g_{\bar{i}k} \tag{13.41}$$

has a solution with respect to an unknown symmetric regular tensor a_{ij} $(= -e\,a_{\bar{i}\bar{j}}, \det(a_{ij}) \neq 0)$. Here $\mathcal{Z}_i(a) = a_{\alpha\beta}G_i^{\alpha\beta}$ and $G_i^{\alpha\beta}$ are determined by the objects of $\mathbb{K}_n$.

13.2.6 HP mappings of generalized recurrent Kähler manifolds

It is known from the theory of geodesic mappings that for symmetric, recurrent and some types of generalized recurrent Riemannian manifolds $\mathbb{V}_n$ $(n > 2)$ of non-constant curvature, there are no non-trivial geodesic mappings. Analogous results can be obtained also for holomorphically projective mappings of Kähler manifolds.

Suppose that under a non-trivial holomorphically projective mappings of $\mathbb{K}_n$ onto $\overline{\mathbb{K}}_n$, for the tensor a satisfying (13.23), the following equations hold:

$$a_{\alpha\beta}T^{\alpha\beta}_{ijkl,h} = 0, \tag{13.42}$$

where T is given by the formulae (13.37). Therefore from (13.40) we get

Lemma 13.1 *If a Kähler manifold admits a non-trivial holomorphically projective mapping onto $\overline{\mathbb{K}}_n$ such that the tensor a defined by the formulae (13.22) satisfies the conditions (13.42) then $\mathbb{K}_n$ is a space of constant holomorphic curvature.*

Theorem 13.11 *Kähler manifolds $\mathbb{K}_n$ which are not space of constant holomorphic curvature and satisfying*

$$T^{(\alpha\beta)}_{ijkl,h} + b\,T^{(\overline{\alpha}\overline{\beta})}_{ijkl,h} = N^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}_{ijklh}T^{(\alpha\beta)}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} + \tilde{N}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}_{ijklh}T^{(\overline{\alpha}\overline{\beta})}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}, \tag{13.43}$$

where b, $b \neq e$, is some function, N, $\tilde{N}$ are some tensors of type $(4,5)$, do not admit non-trivial holomorphically projective mappings.

Proof. Suppose $\mathbb{K}_n$ satisfying the assumptions admits a non-trivial holomorphically projective mapping. Then the tensor a defined by (13.22) satisfies (13.36). Contract (13.43) with $a_{\alpha\beta}$. By (13.36) and (13.24), since $1 - eb \neq 0$, it follows (13.42). Hence it means by Lemma 13.1 that $\mathbb{K}_n$ is a space of constant holomorphic curvature, a contradiction. □

From the structure of the tensor $T(T^{\alpha\beta}_{ijkl})$ we can deduce, using (13.37), that symmetric and recurrent Kähler manifolds satisfy $R^h_{ijk,l} = 0$ and $R^h_{ijk,l} = \varphi_i\,R^h_{ijk}$. We get

Theorem 13.12 *Kähler manifolds $\mathbb{K}_n$ with nonconstant holomorphic curvature do not admit nontrivial holomorphically projective mappings onto symmetric and recurrent Kähler manifolds.*

Sakaguchi [778] proved in 1974 this Theorem for classical Kähler manifolds $\mathbb{K}^-_n$. For $\mathbb{K}^-_n$, whose metric has an arbitrary signature, Sakaguchi' results were generalized by Domashev and Mikeš [381, 382] and for $\mathbb{K}^+_n$ by Kurbatova [569] (see also [119, 122, 170, 171, 173]).

13.2.7 HP mappings with certain initial conditions

Here we study holomorphically projective mappings$f$ between Kähler spaces $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ with the initial condition $\overline{g}(f(x_o)) = k \cdot g(x_o)$, where g and $\overline{g}$ are the metrics of $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$.

This condition is equivalent to $a(x_o) = \varkappa \cdot g(x_o)$. We proved: if at the point $x_o \in \mathbb{K}_n$ the tensor of holomorphically projective curvature does not vanish, then f is homothetic.

The system (13.32) has no more than one solution for initial conditions at the point $x_o \in U$: $a_{ij}(x_o) = a^o_{ij}$, $\lambda_i(x_o) = \lambda^o_i$, $\mu(x_o) = \mu^o$. Evidently, if

$$a_{ij}(x_o) = \varkappa \cdot g_{ij}(x_o),\ \varkappa \neq 0,\ \lambda_i(x_o) = 0,\ \mu(x_o) = 0, \tag{13.44}$$

then the initial condition correspond to a trivial solution $a = \varkappa \cdot g$, i.e. $\overline{g} = k \cdot g$.

Elementary, this solution is unique on the whole manifold. It means, that $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ are homothetic.

Further we formulate a theorem, which is analogous to Theorem 10.15 in the theory of geodesic mappings. The proof, too, is completely analogous, so we omit it here.

Theorem 13.13 (Chodorová, Chudá, Mikeš [342]) *Let f be a holomorphically projective mapping between Kähler spaces $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$ with the condition $\overline{g}(f(\overline{x}_o)) = k \cdot g(x_o)$, where g and $\overline{g}$ are metrics of $\mathbb{K}_n$ and $\overline{\mathbb{K}}_n$. If the tensor of holomorphically projective curvature does not vanish in the point $x_o \in \mathbb{K}_n$ then f is a homothetic mapping.*

13.3 Manifolds $\mathbb{K}_n[B]$

13.3.1 Holomorphically projective mappings of $\mathbb{K}_n[B]$ spaces

It will be mentioned later that manifolds admitting holomorphically projective mappings play an important role in the theory of holomorphically projective mappings. Let us deal with the following definition (Mikeš, see [119, 226, 227, 632]), which anables us to make our formulations more precise. These results were proved for $\mathbb{K}_n^-$, evidently they are true also for $\mathbb{K}_n^+$.

Definition 13.5 A Kähler manifold $\mathbb{K}_n$ will be called a $\mathbb{K}_n[B]$ manifold if it admits a nonhomothetic holomorphically projective mapping with

$$\begin{aligned} &\text{(a)}\ \ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}k} - e\lambda_{\bar{j}} g_{\bar{i}k}, \\ &\text{(b)}\ \ \lambda_{i,j} = \mu\, g_{ij} + B\, a_{ij}, \end{aligned} \tag{13.45}$$

where μ and B are some functions.

Note that if a manifold $\mathbb{K}_n[B]$ admits any holomorphically projective mapping then the corresponding conditions (13.45) are satisfied, just with the same B. Since $\lambda_i \in C^2$ it follows from formula (13.45b) that $\mu, B \in C^1$.

The formulae (13.45b) are equivalent to

$$\psi_{ij} = \overline{B}\,\overline{g}_{ij} - B\, g_{ij}, \tag{13.46}$$

where $\overline{B}$ is a differentiable function; the proof of this fact follows by covariant differentiation of the formula (13.21), see [119, 226, 227, 632].

These conditions are especially fulfilled for holomorphically projective mappings of manifolds of constant holomorphical curvature; see p. 422. Clearly, the following lemma follows from "symmetry" of formulas (13.46).

Lemma 13.2 *If a manifold $\mathbb{K}_n[B]$ admits a nontrivial holomorphically projective mapping onto $\overline{\mathbb{K}}_n$, then $\overline{\mathbb{K}}_n$ is a $\overline{\mathbb{K}}_n[\overline{B}]$ manifold where $\overline{B}$ is some function, i.e. the class of spaces $\mathbb{K}_n[B]$ is closed with respect to holomorphically projective mappings; moreover*

$$B = const \ \Leftrightarrow\ \overline{B} = const.$$

For $n > 4$ it holds that $B, \overline{B}$ are const.

Note that last sentece is in [226] presented for $n = 4$, but it is not true. The proof of this sentence follows from integrability condition of equations (13.45b).

The manifolds $\mathbb{K}_n$ admitting nontrivial holomorphically projective mappings and having a concircular field ξ are $\mathbb{K}_n[B]$ manifolds. In this case, ξ is convergent and $B = 0$.

Under holomorphically projective mappings of $\mathbb{K}_n[B]$ onto $\overline{\mathbb{K}}_n[\overline{B}]$, the tensors $\mathcal{Z}^h_{ijk}$ and $\mathcal{Z}_{ij}$ are invariant:

$$\overline{\mathcal{Z}}^h_{ijk} = \mathcal{Z}^h_{ijk} \quad \text{and} \quad \overline{\mathcal{Z}}_{ij} = \mathcal{Z}_{ij}, \tag{13.47}$$

where

$$\begin{aligned} &(a)\ \mathcal{Z}^h_{ijk} = R^h_{ijk} - B\,(\delta^h_k g_{ij} - \delta^h_j g_{ik} - e\,\delta^h_{\bar{k}} g_{i\bar{j}} + e\,\delta^h_{\bar{j}} g_{i\bar{k}} + 2e\,\delta^h_{\bar{i}} g_{j\bar{k}}),\\ &(b)\ \mathcal{Z}_{ij} = \mathcal{Z}^\alpha_{ij\alpha}. \end{aligned} \tag{13.48}$$

The above immediately follows from (13.9), (13.10) and (13.46).

The integrability conditions of (13.45a) can be written in the form

$$a_{i\alpha}\mathcal{Z}^\alpha_{jkl} + a_{j\alpha}\mathcal{Z}^\alpha_{ikl} = 0. \tag{13.49}$$

1) Basic equations of the manifolds $\mathbb{K}_n[B]$, $B = \text{const}$, admitting holomorphically projective mappings, have the following form:

$$(a)\ a_{ij,k} = \lambda_{(i}\,g_{j)k} - e\lambda_{(\bar{i}}\,g_{\bar{j})k};\ (b)\ \lambda_{i,j} = \mu g_{ij} + B\,a_{ij};\ (c)\ \mu_{,i} = 2B\,\lambda_i. \tag{13.50}$$

It follows from analysis of the integrability conditions (13.45). The integrability conditions of the equations (13.50b) take the following form:

$$\lambda_\alpha \mathcal{Z}^\alpha_{ijk} = 0. \tag{13.51}$$

2) For the manifolds $\mathbb{K}_n[0]$, the equations (13.50) may be simplified to the form

$$(a)\ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}k} - e\lambda_{\bar{j}} g_{\bar{i}k};\quad (b)\ \lambda_{i,j} = \mu g_{ij},\ \mu = \text{const}. \tag{13.52}$$

The integrability conditions may be written as: $a_{i\alpha}R^\alpha_{jkl} + a_{j\alpha}R^\alpha_{ikl} = 0$ and $\lambda_\alpha R^\alpha_{ijk} = 0$. It follows from (13.52b) that in $\mathbb{K}_n[0]$ there exist convergent vector fields. Consequently, Theorems 8.6 and 13.15 imply that the manifolds $\mathbb{K}_n[0]$ admitting nontrivial holomorphically projective and also geodesic mappings are equidistant manifolds of basic type [630, 631].

The set of solutions of the system (13.45) forms, for $B = \text{const}$, a special Jordan algebra relative to the multiplication operation (see [119]):

$$\left(\overset{1}{a}, \overset{1}{\lambda}, \overset{1}{\mu}\right) \times \left(\overset{2}{a}, \overset{2}{\lambda}, \overset{2}{\mu}\right) = \left(\overset{3}{a}, \overset{3}{\lambda}, \overset{3}{\mu}\right),$$

with

$$\begin{aligned} 2\overset{3}{a}_{ij} &= B\overset{1}{a}{}^s_{\cdot(i}\overset{2}{a}_{j)s} - \overset{1}{\lambda}_{(i}\overset{2}{\lambda}_{j)} + e\overset{1}{\lambda}_\alpha\overset{2}{\lambda}_\beta F^\alpha_{(i}F^\beta_{j)};\\ 2\overset{3}{\lambda}_i &= B\,(\overset{1}{\lambda}{}^\alpha\overset{2}{a}_{i\alpha} + \overset{2}{\lambda}{}^\alpha\overset{1}{a}_{i\alpha}) - \overset{1}{\mu}\overset{2}{\lambda}_i - \overset{2}{\mu}\overset{1}{\lambda}_i;\\ \overset{3}{\mu} &= B\overset{1}{\lambda}{}^\alpha\overset{2}{\lambda}_\alpha - \overset{1}{\mu}\overset{2}{\mu}. \end{aligned}$$

In [119] was established that every solution a_{ij} of (13.45) in $K_n[B]$, $B = \text{const} \neq 0$, is associated with a covariantly constant field A_{ab} in the Riemannian space $\overline{V}_{n+2}$ whose metric tensor has the structure

$$G_{ab} = \frac{1}{B}e^{2Bx^0}\begin{pmatrix} -B & 0 & 0\\ 0 & g_{ij} + eB\tau_i\tau_j & eB\tau_i\\ 0 & eB\tau_j & eB \end{pmatrix},$$

where $g_{ij}(x^1,\dots,x^n)$ is the metric tensor of $K_n[B]$, $B = \text{const} \neq 0$, and $\tau_i(x^1,\dots,x^n)$ is a covector potencial, i.e., $F_{ij} = \partial_{[j}\tau_{i]}$ (the form $F_{ij} \equiv g_{i\alpha}F_j^\alpha$ is exact), $a,\ b = 0,1,\dots,n,n+1$, with

$$A_{ab} = \begin{pmatrix} \mu & \lambda_i & 0 \\ \lambda_j & a_{ij} + \tau_{(i}F_{j)}^\alpha\lambda_\alpha + \mu\tau_i\tau_j & \lambda_\alpha F_i^\alpha - \mu\tau_i \\ 0 & \lambda_\alpha F_j^\alpha - \mu\tau_j & \mu \end{pmatrix}.$$

More general results were obtained by I. Shandra [784], see also [399].

13.3.2 Properties of the spaces $\mathbb{K}_n[B]$

Theorem 13.14 *Any holomorphically projective mapping of a Kähler space $\mathbb{K}_n[B]$, $B \neq 0$, is either nontrivial or homothetic.*

Proof. Let us assume that $\mathbb{K}_n[B]$, $B \neq 0$, admits an affine (i.e. a trivial holomorphically projective) mapping. Then, for $\lambda_i = 0$, there exists a solution of (13.45). From (13.45b) it follows that $\mu g_{ij} + B\,a_{ij} = 0$. If $B \neq 0$, we get that $a_{ij} = -\frac{\mu}{B}\,g_{ij}$. Hence, by [170, p. 121], it follows that any holomorphically projective map is homothetic. □

Theorem 13.15 *In a Kähler space $\mathbb{K}_n[B]$, if there exists a nonzero isotropic vector among the vectors λ_i satisfying (13.45), then $B = 0$.*

Proof is analogous to that of Theorem 8.6. In a similar way one can prove the following theorem.

Theorem 13.16 *Assume that in a Kähler space $\mathbb{K}_n[B]$, there are mutually orthogonal vectors among the nonzero vectors λ_i which satisfy (13.45). Then $B = 0$.*

The contraposition of Theorem 13.15 is the following Lemma:

Lemma 13.3 *In a Kähler space $\mathbb{K}_n[B]$, $B \neq 0$, a vector λ_i is non isotropic.*

Lemma 13.4 *In a Kähler space $\mathbb{K}_n[B]$, $B \neq 0$, a vector ψ_i is non isotropic.*

13.3.3 On the degree of mobility of $\mathbb{K}_n$ relative to HPM

Recall that the number of essential parameters r_{hpm} from (13.32) was called the *degree of mobility of K_n relative to* HPM. The analysis of the integrability conditions for (13.32) and their differential prolongations gives the following results.

The Kähler manifolds $K_n^\pm$ of constant holomorphic curvature and only these spaces have $r_{hpm} = (\frac{n}{2}+1)^2$ (for $\mathbb{K}_n^-$, see [119, 170, 628]; for K_n^+, see [119, 122, 173, 569]).

Mikeš established [632] that there was no $\mathbb{K}_n^\pm$ for which

$$r_{hpm} \in \left((\tfrac{n}{2}+1)^2, \tfrac{1}{4}\,n^2 - n + 2\right).$$

If $r_{hpm} > 2$ for $\mathbb{K}_n^{\pm}$, then equations (13.32) take the form [632]: (13.50), and $\mathbb{K}_n^{\pm}$ is manifold $\mathbb{K}_n[B]$, where B is a constant.

The Kähler manifold $K_n^{\pm}$ that admits $r_{hpm} = \frac{1}{4}n^2 - n + 2$ is characterized by the following necessary and sufficient conditions: the Riemannian tensor satisfies

$$R_{hijk} = B\left(g_{h[k}g_{j]i} + eF_{h[k}F_{j]i} - 2eF_{hi}F_{jk}\right) + e \cdot a_{[h}a_{\bar{i}]}a_{[i}a_{\bar{k}]},$$

where $\mathrm{e} = \pm 1$, and the vector a_i satisfies the relations

$$a_{i,j} = c_i a_j - ec_{\bar{i}}a_{\bar{j}} + a_i{}^1\xi_j + a_{\bar{i}}{}^2\xi_j; \qquad c_{i,j} = c_i c_j - ec_{\bar{i}}c_{\bar{j}} + a_i{}^3\xi_j + a_{\bar{i}}{}^4\xi_j - Bg_{ij},$$

where ${}^{\sigma}\xi_i$ are vectors and B is a constant. There exist examples of spaces in which the indicated conditions are satisfied.

13.3.4 HP transformations and manifolds $\mathbb{K}_n[B]$

The topic of holomorphically projective transformations was introduced (for classical, elliptic) Kähler manifolds $\mathbb{K}_n^-$ by S. Ishihara and S. Tachibana [502, 502, 504].

Let us mention the main concepts concerning the holomorphically projective transformations (or holomorphically projective motions) of Kähler manifolds.

Definition 13.6 A holomorphically projective mapping from a manifold $\mathbb{K}_n$ onto itself is called a *holomorphically projective transformation* or *holomorphically projective motion*.

The above definition generalizes motions, affine, homothetic, conformal and projective motions of manifolds with affine connection and Riemannian manifolds.

Evidently a holomorphically projective transformation is a special F-planar transformation. Therefore from Theorem 6.10 follows:

Theorem 13.17 *An infinitesimal operator $X = \xi^{\alpha}(x)\partial_{\alpha}$ determines a one-parameter Lie group of holomorphically projective transformations of a Kähler manifold $\mathbb{K}_n$ if and only if it satisfies the conditions*

$$(a)\ \ \mathcal{L}_{\xi}\Gamma_{ij}^{h} = \psi_i\delta_j^h + \psi_j\delta_i^h + \varphi_i F_j^h + \varphi_j F_i^h; \quad (b)\ \ \mathcal{L}_{\xi}F_i^h = 0, \tag{13.53}$$

where ψ_i is a covector, $\varphi_i = e\,\psi_{\alpha}F_i^{\alpha}$ and $\mathcal{L}_{\xi}$ is the Lie derivative in the direction ξ.

Note that if the one form $\psi = 0$ in (6.51), the transformation is affine.

The transformation $\overline{x}^h = x^h + \varepsilon\,\xi^h(x)$, where ε is a *small parameter*, is an *infinitesimal holomorphically projective transformation* if the equations of the analytically planar curves are preserved within an accuracy up to the second order of smallness; the necessary and sufficient condition for the existence of an infinitesimal holomorphically projective transformation is the relation (13.53).

It was proved in [502, 502, 504] that the equation (13.53) of holomorphically projective transformations can be written in the form of a closed linear system of PDEs of Cauchy type in the unknown functions $\xi^h(x)$, $\psi_i(x)$ and $\xi^h_i(x)$.

Hence the dimension r_{hpt} of the Lie group of holomorphically projective transformations does not exceed the number $n^2 + 2n$. The maximum can be reached only for holomorphically projective flat manifolds.

The group of holomorphically projective transfromations involves the subgroup of affine, homothetic and isometric motions.

Theorem 13.18 (for $\mathbb{K}^-_n$: S. Ishihara [502]) *An infinitesimal operator $X = \xi^\alpha(x)\partial_\alpha$ determines a one-parameter Lie group of holomorphically projective transformations (and infinitesimal holomorphically projective transformation) of a Kähler manifold $\mathbb{K}_n$ if and only if it satisfies the following condition*

$$h_{ij,k} = 2\,g_{ij}\psi_k + \psi_{(i}g_{j)k} - e\,\psi_{(\bar{i}}g_{\bar{j})k} \quad \textit{and} \quad h_{ij} = \mathcal{L}_\xi g_{ij} \equiv \xi_{(i,j)}. \tag{13.54}$$

Proof. Let us compare the formulas (4.90) and (13.3). It follows from Theorem 4.11, p. 200, that the formula (13.54) is true. □

Contracting (13.54) with g^{ij} we obtain, that $\psi_k = \frac{1}{n+1}\,(\xi_{\alpha,\beta}g^{\alpha\beta})_{,k} = \Psi_{,k}$. Evidently if $\psi_i = 0$ then X determines an affine transformation (or trivial holomorphically projective transformation).

Putting
$$a_{ij} = \xi_{(i,j)} - 2\,\Psi\,g_{ij} \tag{13.55}$$
we obtain the Domashev-Mikeš-Kurbatova equations (13.23) of holomorphically projective mappings.

Clearly, (13.55) and (13.23) are necessary and sufficient conditions for the existence of the holomorphically projective transformations.

Analysing (13.54) and (13.55) we have that a nontrivial holomorphically projective transformation generates a nontrivial holomorphically projective mapping. Converse of this proposition is not true.

Hereafter we shall assume that the structure of the tensor P admits its null (see Theorem 4.13, p. 200). In other words $a_{ij} = \text{const}\ g_{ij}$ is a partial solution of the equation (13.54). In this case, we can prove the following statement [633, 635].

Theorem 13.19 *The inequality $r_{hpt} \le r_{phm} + r_{hom} - 1$ holds, where r_{hpt} and r_{hom} are orders of holomorphically projective and homothetic groups and r_{hpm} is the degree of holomorphically projective mapping on the manifold $\mathbb{K}_n$.*

For manifolds $\mathbb{K}_n[B]$, the following theorem is true.

Theorem 13.20 *Let a manifold $\mathbb{K}_n[B]$, $B = \text{const}$, admits holomorphically projective transformations with a holomorphically projective vector λ_i. If $B \neq 0$, it is non trivial, and $r_{hpt} = r_{hpm} + r_{hom} - 1$. If $B = 0$, it is affine.*

Proof. On a manifold $\mathbb{K}_n[B]$, the equations (13.50) are satisfied. Differentiating (13.50b) we get $\lambda_{i,jk} = 2B\,g_{ij}\lambda_k + B\,g_{jk}\lambda_i + B\,g_{jk}\lambda_j - e\,B\,g_{\bar{j}k}\lambda_{\bar{i}} - eB\,g_{\bar{j}k}\lambda_{\bar{j}}$.

Because λ_i is gradient-like we may write:

$$\lambda_{(i,j)k} = 4B\, g_{ij}\lambda_k + 2B\, g_{jk}\lambda_i + 2B\, g_{jk}\lambda_j - 2e\, B\, g_{\bar{j}k}\lambda_{\bar{i}} - 2eB\, g_{\bar{j}k}\lambda_{\bar{j}}.$$

Comparing it with (13.54) we have that λ_i is a projective vector, which generates a projective transformation. This transformation is nontrivial for $B \neq 0$ and it is affine for $B = 0$. □

13.3.5 K-concircular vector fields and HPM

Yamaguchi [947] calls the vector field ξ^h in the Kähler manifold $\mathbb{K}_n^-$ a *K-torse-forming vector*, if it satifies the condition $\xi^h_{,i} = a\delta^h_i + bF^h_i + \alpha_i\xi^h + \beta_i\xi^\alpha F^h_\alpha$. Esenov's works [214, 396] are devoted to the study of $K_n^\pm$ in which there exist vector fields of this kind. He shows that K-torse-forming vector fields are HPM-invariant. In his works, he studies $K_n^\pm$ in which the conditions $\lambda_{i,j} = ag_{ij} + c(\lambda_i\lambda_j - e\lambda_{\bar{i}}\lambda_{\bar{j}})$, where a, c are invariants, are satisfied.

By analogy, we call the vector fields λ_i *K-concircular*. Esenov [214, 396] tried to find a metric of $K_n^\pm$ that would admit K-concircular vector fields.

By the methods described in [630, 631], we prove: if a Riemannian space is defined by the relations

$$g_{ab} = g_{a+m\,b+m} = \partial_{ab}G + \partial_{a+m\,b+m}G; \qquad g_{a\,b+m} = \partial_{a\,b+m}G - \partial_{a+m\,b}G,$$

where $G = G\left(x^1 + s(x^2, x^3, \ldots, x^{m+2}, x^{m+3}, \ldots, x^n)\right)$; $G'G'' \neq 0$, where $G,\ s \in C^3$, are functions of the indicated arguments, $a,\ b = \overline{1,m}$, $m = n/2$, then it is, for $|g_{ij}| \neq 0$, the space K_n^- whose structure is defined by the relations $F_b^{a+m} = -F^a_{b+m} = \delta^a_b$, $F^a_b = F^{a+m}_{b+m} = 0$, the vector $\lambda^h = \delta^h_1$ is K-concircular, and the invariants a and c are defined by the relations $a = \frac{1}{2}(\ln g')'$, $c\frac{1}{2}(\ln a)'/g''$. It is obvious that always $a \neq 0$. In K_n^- $(n > 4)$, in which there exists a K-concircular vector field for $a \neq 0$, there is a coordinate system in which the metric has the indicated form.

The Riemannian space whose metric is defined by the relations

$$g_{a\,b+m} = \partial_{a\,b+m}G; \quad g_{ab} = g_{a+m\,b+m} = 0$$

where $G = G\left(x^1 + x^{1+m} + s(x^2 + x^{2+m}, x^3 + x^{3+m}, \ldots, x^m + x^n)\right)$, $G'G'' \neq 0$, $G,\ s \in C^3$ are functions of the indicated arguments, is, for $|g_{ij}| \neq 0$, the space K_n^+ whose structure is defined by the relations $F^a_b = -F^{a+m}_{b+m} = \delta^a_b$, $F^{a+m}_b = F^a_{b+m} = 0$, the vector $\lambda^h = \delta^h_1 + \delta^h_{1+m}$, and the invariants a and c have the same structure.

These spaces admit NHPM [105]. The metrics of the holomorphically projectively corresponding spaces $K_n^\pm$, that contain K-concircular fields, have been found in explicit form. These fields exist in spaces of constant holomorphic curvature.

Muto [715] investigated the spaces K_4^- which contain Killing fields satisfying the condition $\xi_{i,j} = F_{ij}$. The vector $\lambda_i \equiv \xi_\alpha F_i^\alpha$ defines the convergent vector field (see [167]). It turns out that if $K_n^\pm$ contains a vector field ξ_i for which $\xi_{i,j} = ag_{ij} + bF_{ij}$, then this space contains a convergent vector field.

13.3.6 Holomorphically complete manifolds $\mathbb{K}_n[B]$

I. Hasegawa and K. Yamauchi in [459] proved that an infinitesimal holomorphically projective transformation is infinitesimal isometry on a compact classic Kähler manifold $\mathbb{K}_n^-$ with non-positive constant scalar curvature and a compact classic Kähler manifold $\mathbb{K}_n^-$ with constant scalar curvature is holomorphically isometric to a complex projective space with Fubini-Studi metric (i.e. manifold with constant holomorphic curvature) provided $\mathbb{K}_n^-$ admits a non-isometric infinitesimal holomorphically projective transformation.

The investigation of the holomorphically projective mappings of complete Einstein Kähler manifold $\mathbb{K}_n^-$ was carried out by H. Akbar-Zadeh and R. Couty in [248, 249].

We prove the following theorems.

Theorem 13.21 *Let a Kähler manifold $\mathbb{K}_n[B]$, $B = \text{const}$, admit a holomorphically projective mapping f onto a complete manifold $\overline{\mathbb{K}}_n$.*

1. *If $\mathbb{K}_n[B]$ has an indefinite metric then f is affine.*
2. *If $B \geq 0$ then f is affine.*

Proof. Let us suppose that $f\colon \mathbb{K}_n[B] \to \overline{\mathbb{K}}_n$ $(= \overline{\mathbb{K}}_n[\overline{B}])$ is a holomorphically projective mapping and the equation (13.4) holds with $\psi_i = \partial_i \Psi$; B and $\overline{B}$ are constants. Let γ be a geodesic on $\mathbb{K}_n[B]$ and an analytically planar curve on $\overline{\mathbb{K}}_n[\overline{B}]$ with natural parameter t and with canonical parameter τ, respectively. Assume $\dot{\tau} = d\tau(t)/dt > 0$ for the parameter transformation $\tau = \tau(t)$. Then the following holds:

$$\psi_\alpha \dot{\gamma}^\alpha = \frac{1}{2}\,\frac{d}{dt}\,(\ln \dot{\tau}) \quad \text{where} \quad \dot{\gamma}^h = \frac{d}{dt}\,\gamma. \tag{13.56}$$

Along the geodesic γ, we can write

$$\Psi(t) = \Psi(\gamma(t)) = \frac{1}{2}\,\ln(\dot{\tau}(t)) + c_o, \quad c_o = \text{const}\,. \tag{13.57}$$

Because g and $\mathrm{e}^{-4\Psi}\,\overline{g}$ are first integrals of geodesics (see (13.25)), the following holds: $g_{ij}\dot{\gamma}^i\dot{\gamma}^j = \varepsilon = \pm 1, 0$ and $\overline{g}_{ij}\dot{\gamma}^i\dot{\gamma}^j = c\,\mathrm{e}^{4\Psi(t)}$, $c = \text{const}\,.$

On the manifold $\mathbb{K}_n[B]$ the equation (13.46) is satisfied, i.e. in expanded form, it may be expressed by $\psi_{i,j} = \psi_i\psi_j + e\,\psi_{\overline{i}}\psi_{\overline{j}} + \overline{B}\,\overline{g}_{ij} - B\,g_{ij}$. For a geodesic γ, after contraction with $\dot{\gamma}^i\dot{\gamma}^j$, we get

$$\ddot{\Psi} = (\dot{\Psi})^2 + c\,\overline{B}\mathrm{e}^{4\,\Psi} - (\varepsilon\,B - e\,\chi^2).$$

Hence $\chi = \psi_{\overline{\alpha}}\dot{\gamma}^\alpha = \text{const}$, because $d\chi/dt = \psi_{\overline{\alpha},\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta = 0$.

For the inverse mapping $f^{-1}\colon \overline{\mathbb{K}}_n[\overline{B}] \to \mathbb{K}_n[B]$ it holds $\overline{\Psi}(\tau) = -\Psi(t(\tau))$. We obtained

$$\frac{\delta^2\overline{\Psi}}{d\tau^2} = \left(\frac{\overline{\Psi}}{d\tau}\right)^2 + (\varepsilon\,B - e\,\chi^2)\,\mathrm{e}^{4\,\overline{\Psi}} - c\,\overline{B}. \tag{13.58}$$

From the equation (13.58), it follows the equation (8.43), where $a = c\overline{B}$, $b = \varepsilon\,B - e\,\chi^2$.

For a complete $\overline{\mathbb{K}}_n[\overline{B}]$ the following part of this proof is completely analogical to that of the Theorem 8.14. Interchanging of $\mathbb{K}_n[B]$ and $\overline{\mathbb{K}}_n[\overline{B}]$ we prove the theorem. □

13.4 HPM of special Kähler manifolds

13.4.1 HPM of T-k-pseudosymmetric Kähler manifolds

By means of $\mathcal{Z}^h_{ijk}$ we introduce into consideration the operation $\langle\langle lm\rangle\rangle$ as follows:

$$T^{h_1\dots h_p}_{i_1\dots i_q\langle\langle lm\rangle\rangle} \equiv \sum_{s=1}^{q} T^{h_1\dots h_p}_{i_1\dots i_{s-1}\alpha i_{s+1}\dots i_q}\mathcal{Z}^{\alpha}_{i_s lm} - T^{h_1\dots h_{s-1}\alpha h_{s+1}\dots h_p}_{i_1\dots i_q}\mathcal{Z}^{h_s}_{\alpha lm},$$

where T is a tensor of the type (p, q). When $B = 0$, then $T_{\langle\langle lm\rangle\rangle} = T_{,[lm]}$. This operation possesses the properties

$$(u \pm v)_{\langle\langle lm\rangle\rangle} = u_{\langle\langle lm\rangle\rangle} \pm v_{\langle\langle lm\rangle\rangle};\ (uv)_{\langle\langle lm\rangle\rangle} = u_{\langle\langle lm\rangle\rangle}v + uv_{\langle\langle lm\rangle\rangle};$$

$$g_{ij\langle\langle lm\rangle\rangle} = 0;\ g^{ij}_{\langle\langle lm\rangle\rangle} = 0;\ \delta^i_{j\langle\langle lm\rangle\rangle} = 0.$$

The analogy of the Walker identities is valid:

$$R_{hijk\langle\langle lm\rangle\rangle} + R_{jklm\langle\langle hi\rangle\rangle} + R_{lmhi\langle\langle jk\rangle\rangle} = 0.$$

We say that the Kähler space $K^{\pm}_n$ is *T-k-pseudosymmetric* ($\mathrm{TPs}_n[\mathrm{B}]$) if the condition $T_{\langle\langle lm\rangle\rangle} = 0$ is fulfilled in it.

We say that $\mathrm{TPs}_n[\mathrm{B}]$, whose tensor T cannot be represented as the tensor sum of the products of invariants, Kronecker deltas, the F structure, the metric tensor and its inverse, is an *principal T-k-pseudosymmetric space* $\mathrm{TPs}^*_n[\mathrm{B}]$.

The spaces $\mathrm{TPs}_n[0]$ are *T-semisymmetric* (TPs_n). These spaces generalize the *semisymmetric* spaces (Ps_n), for which $R^h_{ijk,[lm]} = 0$, and the *Ricci semisymmetric* spaces (RicPs_n), for which $R_{ij,[lm]} = 0$. By analogy, $\mathrm{Ps}_n[\mathrm{B}]$ is a *k-pseudosymmetric* space and $\mathrm{RicPs}_n[\mathrm{B}]$ is a *Ricci k-pseudosymmetric* space. For these spaces, $R^h_{ijk\langle\langle lm\rangle\rangle} = 0$ and $R_{ij\langle\langle lm\rangle\rangle} = 0$ respectively.

The mentioned notions were in a similar form introduced by Mikeš in [226, 227, 688]. These manifolds were studied by Luczyszyn and Olszak [597, 598, 736].

Holomorphically projective mappings of semisymmetric Kähler spaces K^-_n were studied by Sakaguchi [142]. The result obtained by Mikeš [119, 226, 227, 688] has been strengthened as follows:

Theorem 13.22 (Mikeš [119, 226, 227]) *Let a tensor fild c_{ij} on $\mathbb{K}_n$ be given and let $c_{ij} \neq \varrho_1\, g_{ij} + \varrho_2\, g_{\bar{i}\,j}$ where ϱ_1, ϱ_2 are functions, and*

$$c_{ij\langle\langle lm\rangle\rangle} = 0.$$

If $\mathbb{K}_n$ admit a holomorphically projective mapping onto $\overline{\mathbb{K}}_n$, then $\mathbb{K}_n$ is $\mathbb{K}_n[B]$ space.

Theorem 13.23 (Mikeš [119, 226, 227]) *Let a vector field c_i on $\mathbb{K}_n$ be given and let $c_i \neq 0$ and*

$$c_{i\langle\langle lm\rangle\rangle} = 0.$$

If $\mathbb{K}_n$ admit a holomorphically projective mapping onto $\overline{\mathbb{K}}_n$, then $\mathbb{K}_n$ is $\mathbb{K}_n[B]$ space.

Theorem 13.24 *If a semisymmetric Kähler space Ps_n with non constant holomorphic curvature admits nontrivial holomorphically projective mapping onto $\overline{\mathbb{K}}_n$, then $\overline{\mathbb{K}}_n$ is an equidistant k-pseudosymmetric space, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

Theorem 13.25 *If a Ricci semisymmetric space $RicPs_n \not\equiv \mathcal{E}_n$ admits nontrivial holomorphically projective mappings onto $\overline{\mathbb{K}}_n$, then this space is an equidistant Ricci k-pseudosymmetric space, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

Corollary 13.1 *If a reducible Kähler space $\mathbb{K}_n$ admits a nontrivial holomorphically projective mapping then it is equidistant, and $\lambda_{i,j} = \mu\, g_{ij}$, $\mu = \text{const}$.*

Theorem 13.26 *If a k-pseudosymmetric space $Ps_n[B]$ (resp. a Ricci k-pseudosymmetric space $RicPs_n[B]$) admits nontrivial holomorphically projective mappings onto $\overline{\mathbb{K}}_n$, then the space $\overline{\mathbb{K}}_n$ is a space $\mathbb{K}_n[B]$, and $B, \overline{B}$ are constants.*

Theorem 13.27 *If a k-pseudosymmetric space $Ps_n[B]$ (resp. a Ricci k-pseudosymmetric space $RicPs_n[B]$) admits geodesic mapping onto $\overline{\mathbb{K}}_n$, then the space $\overline{\mathbb{K}}_n$ is a k-pseudosymmetric space $\overline{P}s_n[\overline{B}]$ (resp. a Ricci k-pseudosymmetric space $Ric\overline{P}s_n[\overline{B}]$), and $B, \overline{B}$ are constants.*

It as established that the spaces $\mathrm{Ps}_n[\mathrm{B}]$ and $\mathrm{RicPs}_n[\mathrm{B}]$ form the closed (relative to holomorphically projective mappings) class of Kähler spaces.

Theorem 13.28 (Haddad [216, 450, 688]) *The principal T-k-pseudosymmetric space $\mathrm{TPs}^*_n[\mathrm{B}]$ ($n > 4(s-1)$, where s is the valence of the tensor T) admit nontrivial holomorphically projective mappings if and only if they are the $\mathbb{K}_n[B]$ spaces.*

In the works by Mikeš [119, 634] and Sinyukov, Sinyukova [827, 832], a series of results for global geodesic mappings of compact semisymmetric and Ricci-semisymmetric Kähler manifolds with additional conditions is obtained.

From Theorem 13.21 for (Ricci) semisymmetric, (Ricci) k-pseudosymmetric we obtained following facts which are more general.

Theorem 13.29 *Let a semisymmetric manifold (resp. Ricci semisymmetric manifold) $\mathbb{K}_n$ admit a non-affine holomorphically projective mapping f onto a complete manifold $\overline{\mathbb{K}}_n$ or a complete $\mathbb{K}_n$ admit a non-affine holomorphically projective transformation. Then $\mathbb{K}_n$ is a space with non vanishing constant holomorphic curvature (resp. an Einstein space).*

Theorem 13.30 *Let a k-pseudosymmetric manifold (resp. Ricci k-pseudosymmetric) Kähler manifold $\mathbb{K}_n$ admits a non-affine holomorphically projective mapping f onto a complete manifold $\overline{\mathbb{K}}_n$. If $\mathbb{K}_n$ has an indefinite metric then f is affine. If $B \geq 0$ then f is affine.*

Theorem 13.31 *Let a k-pseudosymmetric manifold (resp. Ricci k-pseudosymmetric) Kähler complete manifold $\mathbb{K}_n$ admits a non-affine holomorphically projective transformation mapping f. If $\mathbb{K}_n$ has an indefinite metric then f is affine. If $B \geq 0$ then f is affine.*

13.4.2 HPM of Einstein spaces and of their generalizations

As was mentioned above the spaces of constant holomorphic curvature, which are necessarily Einstein spaces, admit non trivial holomorphically projective mappings (NHPM), and the analogy of the Beltrami theorem is valid in the theory of GM. As far as we know, no analogy of Theorem 9.6, p. 320, has been established for Einsteinian spaces. However, if the Einsteinian space $\mathbb{K}_n$ admits NHPM onto the Einsteinian space $\mathbb{K}_n$, then these spaces are the $\mathbb{K}_n\,[R/(n(n+2))]$ and $\overline{\mathbb{K}}_n\,[R/(n(n+2))]$ spaces, respectively. Haddad [216,450] proved that the 4-dimesional Einsteinian $\mathbb{K}_n^-$ spaces do not admit NHPM onto the Einsteinian spaces of nonconstant holomorphic curvature and do not admit nontrivial holomorphically projective transformations.

The investigation of the NHPM of complete Einsteinian $\mathbb{K}_n^-$ was carried out in [248].

We say that L_n^* is the Kähler space $\mathbb{K}_n$ in which the condition

$$R_{ij,k} = a_k g_{ij} + b_{(i}g_{j)k} + e\bar{b}_{(i}F_{j)k} \not\equiv 0,$$

where $a_i \equiv R_{,i}/(n+2)$, $b_i \equiv R_{,i}/(2(n+2))$, $\bar{b}_i \equiv b_\alpha F_i^\alpha$, is satisfied. This class of spaces generalizes Einsteinian spaces and the spaces L_n.

Haddad's works [216, 536] are devoted to the study of the HPM of the L_n^* spaces. He established that the spaces L_n^* are characterized by the condition $B_{ijk,\alpha}^\alpha = 0$, where B is the Bochner tensor, and $R \not\equiv \text{const}$. These spaces admit NHPM and the general solution of the fundamental equations of HPM has the following representation: $a_{ij} = c_1 g_{ij} + c_2 \left(R_{ij} - \frac{R}{(n+2)} g_{ij}\right)$, c_1, c_2 are constants.

13.4.3 Spaces that locally do not admit nontrivial HPM

Many authors isolated Kähler spaces which do not admit either nontrivial holomorphically projective mappings (NHPM) or nontrivial holomorphically projective transformations (NHPT).

Note that the Kähler spaces $\mathbb{K}_n^{\pm}$, which do not admit NHPM, do not admit NHPT either, as well as nontrivial geodesic mappings or nontrivial projective transformations. In these spaces there are no nonconstant concircular and K-concircular vector fields. In this section this is not especially stipulated.

Sakaguchi [778] proved in 1974 that proper Kählerian symmetric spaces $\mathbb{K}_n^-$ of nonconstant holomorphic curvature do not admit NHPM. For symmetric $\mathbb{K}_n^-$ with a metric of arbitrary signature, Sakaguchi's result was proved by Domashev and Mikeš [226, 227, 382] and for $\mathbb{K}_n^+$ by Kurbatova [569] (se also [119, 170, 171]). In [119, 227, 639] Mikeš indicated more general conditions for recurence under which $\mathbb{K}_n^{\pm}$ do not admit NHPM. In particular, recurrent, m-recurrent, 2-symmetric, and generalized recurrent D_n^2 Kähler spaces $K_n^{\pm}$ do not admit NHPM.

There are no spaces $\mathbb{K}_n[0]$ in which one of the following conditions would be satisfied (Mikeš [119, 227, 639], Haddad [216]):

$$\begin{aligned}
&\text{(a)} \quad R_{hijk,l} + R_{hi\alpha\beta,\gamma} S^{\alpha\beta\gamma}_{jk} S_l + R_{hi\alpha\beta} S^{\alpha\beta}_{jkl} = 0;\\
&\text{(b)} \quad R_{hijk,lm} = R_{hi\alpha\beta} S^{\alpha\beta}_{jklm} + R_{hijk,\alpha} S^{\alpha}_{lm};\\
&\text{(c)} \quad R_{hijk,l_1\ldots l_m} = R_{hi\alpha\beta} S^{\alpha\beta}_{jkl_1\ldots l_m};\\
&\text{(d)} \quad \left(R_{ij,k} - R_{ik,j}\right)_{,l_1\ldots l_m} = R_{jk\alpha\beta} S^{\alpha\beta}_{il_1\ldots l_m},
\end{aligned} \tag{13.59}$$

where S are tensors and $R_{hijk} \neq 0$.

It follows that Ps_n, RicPs_n, or reducible Kähler spaces $\mathbb{K}_n^{\pm}$, in which conditions (13.59) are satisfied do not admit NHPM.

In particular, non-Einsteinian Ricci-symmetric ($R_{ij,k} = 0$), Ricci-2-symmetric ($R_{ij,kl} = 0$), Ricci-recurrent ($R_{ij,k} = \varphi_k R_{ij}$), and Ricci-2-recurrent ($R_{ij,kl} = \varphi_{kl} R_{ij}$) Kähler spaces $\mathbb{K}_n^{\pm}$ do not admit NHPM.

In many works, the authors investigated the spaces $\mathbb{K}_n^{\pm}$, in which $R_{ij,k} = R_{ik,j} \neq 0$. It was proved that spaces of this kind do not exist.

Remark 13.2 The results listed above were previously formulated only locally, see [119, 170]. Hence according to Theorem 13.8 the results we may extend to the whole space, and we can formulate these theorems even globally.

Above-mentioned results about on holomorphically projective mappings of semisymmetric and generalized recurrent manifolds with affine connection were generalized in papers [119, 577, 702, 802, 803, 882] by al Lamy, Mikeš, Shiha, Škodová, etc.

13.5 Holomorphically-projective mappings of parabolic Kähler spaces

In this section we study fundamental equations of holomorphically projective mappings of parabolic Kähler spaces (which are generalized classical, pseudo- and hyperbolic Kähler spaces) with respect to the smoothness class of metrics. We show that holomorphically projective mappings preserve the smoothness class of metrics.

First we study the general dependence of holomorphically-projective mappings of parabolic Kähler manifolds in dependence on the smoothness class of the metric. We present well known facts, which were proved by M. Shiha, J. Mikeš et al, see [235, 676, 689, 799, 800, 802].

I. Hinterleitner [470] has solved the analogically problems for classical, pseudo- and hyperbolic Kähler manifolds. In this section were generalized results which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro, see [19, 119, 122, 170, 173, 198, 226, 382, 569, 628, 737, 754, 882, 884]. In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects. This result was inspired in [482, 483] of geodesic mappings.

13.5.1 HP mappings theory for $\mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$

In the following definition we introduce generalizations of Kähler manifolds. A basis on this definition see monography by V.V. Vishnevskii, A.P. Shirokov and V.V. Shurigin [191].

Definition 13.7 An n-dimensional (pseudo-) Riemannian manifold (M, g) is called an *m-parabolic Kähler manifold* $\mathbb{K}_n^{o(m)}$, if beside the metric tensor g, a tensor field F of a rank $m \geq 2$ of type $(1,1)$ is given on the manifold M_n, called a *structure* F, such that the following conditions hold:

$$F^2 = 0, \quad g(X, FX) = 0, \quad \nabla F = 0, \tag{13.60}$$

where X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in $\mathbb{K}_n^{o(m)}$.

We remind, that (pseudo-) and hyperbolic Kähler spaces, were characterized by conditions $F^2 = \pm\mathrm{Id},\ g(X, FX) = 0,\ \nabla F = 0$, and defined on p. 132.

Definition 13.8 A curve ℓ in $\mathbb{K}_n$ which is given by the equation $\lambda = \ell(t),\ \ell = d\ell/dt\ (\neq 0), t \in I$, where t is a parameter is called *analytical planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(\lambda) F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t. Particularly, in the case $b(t) = 0$, an analytical planar curve is a geodesic.

Assume the parabolic Kähler manifolds $\mathbb{K}_n^{o(m)} = (M, g, F)$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} = (\overline{M}, \overline{g}, \overline{F})$ with metrics g and $\overline{g}$, structures F and $\overline{F}$, Levi-Civita connections ∇ and $\overline{\nabla}$, respectively. Here $\mathbb{K}_n$, $\overline{\mathbb{K}}_n \in C^1$, i.e. $g, \overline{g} \in C^1$ which means that their components g_{ij}, $\overline{g}_{ij} \in C^1$. Likewise, as in [235, 799, 801] we introduce the following notations, this is an analogy by [737], see [122, p. 240].

Definition 13.9 A diffeomorphism $f\colon \mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$ is called a *holomorphically-projective mapping* of $\mathbb{K}_n^{o(m)}$ onto $\overline{\mathbb{K}}_n^{o(\overline{m})}$ if f maps any analytical planar curve in $\mathbb{K}_n^{o(m)}$ onto an analytical planar curve in $\overline{\mathbb{K}}_n^{o(\overline{m})}$.

Assume a holomorphically-projective mapping $f\colon \mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$. Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\overline{M}$, respectively, such that locally, $f : \mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$ maps points onto points with the same coordinates, and $\overline{M} = M$. A manifold $\mathbb{K}_n^{o(m)}$ admits a holomorphically-projective mapping onto $\overline{\mathbb{K}}_n^{o(\overline{m})}$ if and only if the following equations [235, 801]:

$$\overline{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + \varphi(FX)FY + \varphi(FY)FX \tag{13.61}$$

hold for any tangent fields X, Y and where ψ is a gradient-like form and $\psi(X) = \varphi(FX)$. If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically-projective* . Moreover, structures F and $\overline{F}$ are preserved, i.e. $\overline{F} = F$, and $\overline{m} = m$. This fact implies from the theory of F-planar mappings, see pp. 385-393. In local form:

$$\overline{\Gamma}_{ij}^h = \Gamma_{ij}^h + \psi_i \delta_j^h + \psi_j \delta_i^h + \varphi_i F_j^h + \varphi_j F_i^h, \quad \psi_i = \varphi_j F_i^j,$$

where Γ_{ij}^h and $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of K_n and $\overline{K}_n$, ψ_i, F_i^h are components of ψ, F and δ_i^h is the Kronecker delta,

$$\psi_i = \frac{\partial \Psi}{\partial x^i}, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \overline{g}}{\det g} \right|.$$

Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots \bar{i} \dots} = A_{\dots k \dots} F_i^k.$$

Equations (13.61) are equivalent to the following equations

$$\begin{aligned} \nabla_Z \overline{g}(X, Y) = 2\psi(Z)\overline{g}(X, Y) + \psi(X)\overline{g}(Y, Z) + \psi(Y)\overline{g}(X, Z) \\ + \varphi(FX)\overline{g}(FY, Z) + \varphi(FY)\overline{g}(FX, Z). \end{aligned} \tag{13.62}$$

In local form:

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{jk} + \psi \overline{g}_{ik} + \varphi_i \overline{g}_{\bar{j}k} + \varphi_j \overline{g}_{\bar{i}k},$$

where " , " denotes the covariant derivative on $\mathbb{K}_n^{o(m)}$.

M. Shiha and J. Mikeš [799, 801] proved that equations (13.61) and (13.62) are equivalent to

$$\begin{gathered}\nabla_Z a(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) + \\ \theta(X)g(FY,Z) + \theta(Y)g(FX,Z).\end{gathered} \tag{13.63}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \theta_i g_{\bar{j}k} + \theta_j g_{\bar{i}k},$$

where

$$\text{(a)}\ \ a_{ij} = \mathrm{e}^{2\Psi}\overline{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}, \quad \text{(b)}\ \ \lambda_i = \theta_{\bar{i}}, \quad \text{(c)}\ \ \theta_i = -\mathrm{e}^{2\Psi}\overline{g}^{\alpha\beta}g_{\beta i}\varphi_\alpha. \tag{13.64}$$

From (13.63) follows that λ_i is gradient-like vector and it holds

$$\lambda_i = \partial_i \Lambda, \qquad \Lambda = 1/4\ a_{\alpha\beta}g^{\alpha\beta}. \tag{13.65}$$

On the other hand [122]:

$$\overline{g}_{ij} = \mathrm{e}^{2\Psi}\tilde{g}_{ij}, \quad \Psi = \frac{1}{2}\ln\left|\frac{\det\tilde{g}}{\det g}\right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha}g^{j\beta}a_{\alpha\beta}\|^{-1}. \tag{13.66}$$

The above formulas are the criterion for holomorphically-projective mappings $\mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(m)}$, globally as well as locally.

Theorem 13.32 *A diffeomorphism $f : \mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$ is a holomorphically-projective mapping if and only if there exist a solution of the following linear Cauchy-like system*

$$\begin{aligned} &a)\ a_{ij,k} = \lambda_{(i}g_{j)k} + \theta_{(i}F_{j)k};\\ &b)\ \theta_{i,j} = \tau F_{ij} + a_{\alpha\beta}M^{\alpha\beta}_{1|ij};\\ &c)\ \tau_{,i} = \theta_\alpha M^{\alpha}_{2|i} + a_{\alpha\beta}M^{\alpha\beta}_{3|i}\end{aligned} \tag{13.67}$$

on unknown tensor a_{ij} ($a_{ij} = a_{ji}$, $a_{\bar{i}j} + a_{i\bar{j}} = 0$, $\det a_{ij} \neq 0$), a vector λ_i, and a function τ. Here $M^{\alpha\beta}_{1|ij}$, $M^{\alpha}_{2|i}$, $M^{\alpha\beta}_{3|i}$ are tensors determined from metric and structure tensors g_{ij} and F^h_i of the space $\mathbb{K}_n^{o(m)}$.

Remark. This theorem was proved with assuming that $\mathbb{K}_n^{o(m)}$ and $\overline{\mathbb{K}}_n^{o(m)}$ belong to C^3 class. We will prove, that the Theorem 13.32 valides too if $\mathbb{K}_n^{o(m)} \in C^3$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$. The system (13.67) has at most one solution for the initial values in a point $x_0 : a_{ij}(x_0)$, $\lambda_i(x_0)$ and $\tau(x_0)$. Hence, the general solution of this system depends on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters.

The integrability of conditions (13.67) and their differential prolongations are linear algebraic equations on the components of the unknown tensors a_{ij}, λ_{ij} and τ with coefficients from $\mathbb{K}_n^{o(m)}$.

13.5.2 HP mappings of parabolic Kähler space of class C^2

The direct substitution of (13.61) implies that Riemannian tensors of spaces $\mathbb{K}_n^{o(m)}$ and $\overline{\mathbb{K}}_n^{o(m)}$, which are holomorphically-projectively corresponding of m-parabolic Kähler space are connected by the following relations:

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} + F^h_k\varphi_{ij} - F^h_j\varphi_{ik} + F^h_i\varphi_{[kj]}, \tag{13.68}$$

where $\overline{R}^h_{ijk}$, R^h_{ijk} are Riemannian tensors of $\mathbb{K}_n^{o(m)}$ and $\overline{\mathbb{K}}_n^{o(m)}$,

$$\varphi_{ik} \equiv \varphi_{i,j} - \psi_i\varphi_j - \varphi_i\psi_j; \quad \psi_{ij} \equiv \varphi_{\bar{i}\bar{j}}. \tag{13.69}$$

Tensor $\overline{\varphi}_{ij}$ has this form:

$$\psi_{ij} = \psi_{i,j} - \psi_i\psi_j. \tag{13.70}$$

As ψ_i is a gradient, that this tensor is symmetrical, that is

$$\psi_{ij} = \psi_{ji}. \tag{13.71}$$

Contracting (13.68) with respect to indices h and k, we obtain connection between Ricci tensors

$$\overline{R}_{ij} = R_{ij} + n\psi_{ij}, \tag{13.72}$$

where R_{ij} and $\overline{R}_{ij}$ are Ricci tensors of $\mathbb{K}_n^{o(m)}$ and $\overline{\mathbb{K}}_n^{o(m)}$.

Next, we will proceed similarly as in work of M. Shiha [235, 799]. Let we consider the integrability condition of equation (13.63):

$$a_{\alpha(i}R^\alpha_{j)kl} = g_{k(i}\lambda_{j),l} - g_{l(}\lambda_{j),k} - F_{k(i}\theta_{j),l} + F_{l(i}\theta_{j),k}. \tag{13.73}$$

Contracting (13.73) with $F^k_{k'}$ and also with $F^l_{l'}$, we obtain two expressions. After removing primes, we sum them up. Since $\mathbb{K}_n^{o(m)}$ it holds $R^h_{i\alpha k} + R^h_{ij\alpha}F^\alpha_k = 0$, we get $F_{k(i}\Lambda_{j)l} - F_{l(i}\Lambda_{j)k} = 0$, where $\Lambda_{ij} \equiv \lambda_{i,j} + \theta_{i,\alpha}F^\alpha_j$. It follows that $\Lambda_{ij} = \Lambda F_{ij}$, i.e. $\lambda_{i,j} + \theta_{i,\alpha}F^\alpha_j = \Lambda F_{ij}$ where Λ is a function. Contracting (13.73) with g^{jk}, we obtain

$$n\lambda_{i,j} = \mu g_{ij} + \nu F_{ij} - a_{\alpha i}R^\alpha_j - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{13.74}$$

where R^h_{ijk} and R_{ij} are Riemann and Ricci tensors, respectively, the operation of lifting and lowering indices are induced by the metric tensor, and μ, ν are certain functions. After symmetrizing (13.74) we get

$$n\lambda_{i,j} = \mu g_{ij} - \frac{1}{2}\, a_{\alpha(i}R^\alpha_{j)} - a_{\alpha\beta}R^{\alpha..\beta}_{.ij.}. \tag{13.75}$$

Substituting (13.75) to (13.73) we obtain

$$a_{\alpha\beta}M^{\alpha\beta}_{jkl} = F_{li}\theta_{j,k} + F_{lj}\theta_{i,k} - F_{ki}\theta_{j,l} + F_{kj}\theta_{i,l}. \tag{13.76}$$

From this implies that M are tensors determined by g_{ij} and F^h_i on $\mathbb{K}_n^{o(m)}$. More precisely

$$M^{\alpha\beta}_{ijkl} \equiv \delta^\alpha_{(i}R^\beta_{j)kl} + M^{\alpha\beta}_{4|k(i}g_{j)l} - M^{\alpha\beta}_{4|l(i}g_{j)k}; \qquad nM^{\alpha\beta}_{4|li} \equiv \frac{1}{2}\,\delta^\alpha_{(i}R^\beta_{j)} - R^{\alpha..\beta}_{.ij.}.$$

Let ε^j and ν^k be vectors such that $\varepsilon^j\nu^k F_{jk} = 1$. Denote $M_i \equiv \varepsilon^\alpha F_{\alpha i}$. Contracting (13.76) with $\varepsilon^i\varepsilon^j\nu^k$ we get

$$\varepsilon^\alpha\theta_{\alpha,l} = \tau M_l + a_{\alpha\beta}M^{\alpha\beta}_{5|l}, \tag{13.77}$$

where $\tau \equiv \lambda_{\alpha,\beta}\varepsilon^\alpha\nu^\beta$. Contracting (13.76) with $\varepsilon^j\nu^k$ and using (13.77) we have following formula

$$\theta_{i,j} = \tau F_{ij} + \Lambda_i M_j + a_{\alpha\beta}M^{\alpha\beta}_{6|ij}, \tag{13.78}$$

where λ_i is a vector. Substituting (13.78) into (13.76) we have

$$\Lambda_i(M_kF_{lj} - M_lF_{kj}) + \Lambda_j(M_kF_{li} - M_lF_{ki}) = a_{\alpha\beta}M^{\alpha\beta}_{7|ijkl}. \tag{13.79}$$

So we proved following lemma:

Lemma 13.5 *Let $\mathbb{K}_n^{o(m)}$ and $\overline{\mathbb{K}}_n^{o(\overline{m})}$ belong to class C^2. If $\mathbb{K}_n^{o(m)}$ admit a holomorphically-projective mapping onto $\overline{\mathbb{K}}_n^{o(\overline{m})}$ than formulae (13.67a) and (13.67b) hold.*

13.5.3 HP mappings $\mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$ for $\mathbb{K}_n^{o(m)} \in C^r$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$

The Theorem 13.32 was proved by M. Shiha [235, 799] with arbitrary differentiability of metrics. Now, we shall prove this Theorem for holomorphically-projective mapping $\mathbb{K}_n^{o(m)} \to \overline{\mathbb{K}}_n^{o(\overline{m})}$ for $\mathbb{K}_n^{o(m)} \in C^r$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$, by J. Mikeš, P. Peška and M. Shiha see [676] .

Theorem 13.33 *Let $\mathbb{K}_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$. If $\mathbb{K}_n^{o(m)}$ admits holomorphically-projective mapping onto $\overline{\mathbb{K}}_n^{o(\overline{m})}$ then $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^r$.*

Proof of Theorem 13.33 is based on the proof of Theorem 13.34.

Theorem 13.34 *Let $\mathbb{K}_n^{o(m)} \in C^3$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$. If $\mathbb{K}_n^{o(m)}$ admits holomorphically-projective mapping onto $\overline{\mathbb{K}}_n^{o(\overline{m})}$ then $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^3$.*

For first, we prove that metric g and structure F have the same differentiation.

Theorem 13.35 *If $\mathbb{K}_n^{o(m)} = (M, g, F) \in C^r$, i.e. $g(x) \in C^r$, then $F(x) \in C^r$, for $r \in \mathbb{N}$ and $r = \infty, \omega$.*

Proof. Let $\mathbb{K}_n^{o(m)} \in C^r$, i.e. the components of metric $g_{ij}(x) \in C^r$ in a coordinate chart x. It is a priori valid, that $F_i^h \in C^1$. The formula $\nabla F = 0$ can be written $\partial_k F_i^h = F_\alpha^h\Gamma_{ik}^\alpha - F_i^\alpha\Gamma_{\alpha k}^h$, where $\Gamma_{ijk} = 1/2(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\partial_k = \partial/\partial x^k$, and $\Gamma_{ij}^h = g^{hk}\Gamma_{ijk}$ are the Christoffel symbols of the first and second kind, respectively. It holds, that Γ_{ijk} and $\Gamma_{ij}^h \in C^{r-1}$. From this equation immediately follows $F_i^h(x) \in C^r$, i.e. $F \in C^r$. □

For proving theorem 13.34, we need following lemma:

Lemma 13.6 *Let $\lambda^h(x) \in C^1$ be a vector field, $F_i^h(x) \in C^2$ is a tensor field of* $\operatorname{rank} F \geq 2$. *If $\partial_i\lambda^h - \varrho F_i^h \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.*

Proof. Let

$$\partial_i\lambda^h - \varrho F_i^h(x) = f_i^h(x) \in C^1.$$

Because $\operatorname{rank} F_i^h \geq r$ and $F_i^h(x) \in C^2$, then exist a regular tensor field $\Omega_i^h(x) \in C^2$ that $F_i^h\Omega_j^i = \Phi_i^h \equiv \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. We put $v^h = \lambda^\alpha\Omega_\alpha^h$. Then we have

$$\partial_i v^h - \varrho\Phi_\alpha^h = \partial_i(\lambda^\alpha\Omega_\alpha^h) - \varrho\Phi_\alpha^h = f_i^\alpha\Omega_\alpha^h + \lambda^\alpha\partial_i\Omega_\alpha^h.$$

Because $f_i^\alpha\Omega_\alpha^h + \lambda^\alpha\partial_i\Omega_\alpha^h \in C^1$, from Lemma 3.5, p. 142, it implies $v^h \in C^2$ and $\varrho \in C^1$. Since $\varrho \in C^1$, then $\lambda^h(x) \in C^2$. □

It follows prooving of Theorem 13.34:

Proof. Equations (13.67b) can be written in the following form:

$$\partial_j\theta^h = \tau F_j^h - \theta^\alpha\Gamma_{\alpha j}^h + a_{\alpha\beta}M_{1|j}^{\alpha\beta h},$$

i.e. $\partial_j\theta^h - \tau F_j^h = f_j^h$, where $f_j^h = -\theta^\alpha\Gamma_{\alpha j}^h + a_{\alpha\beta}M_{1|j}^{\alpha\beta h}$. Because $f_j^i \in C^1$ and $F_i^h \in C^3$ then from Lemma 13.6 follows $\theta^i \in C^2$ and $\tau \in C^1$. Using (13.67a) and (13.67b) we obtain $a_{ij}(x) \in C^3$ and $\Psi \in C^3$. Finally, from (13.66) we have $\overline{g}_{ij}(x) \in C^3$. □

In to this moment, we proved formulae (13.67a) and (13.67b). For complete proof of Theorem 13.32, we have to prove the latest formula (13.67c).

Proof. Proof of Theorem 13.34 allows us differentiation of equation (13.67b). Application to this, we have:

$$\theta_{i,jk} = \tau_{,k}F_{ij} + a_{\alpha\beta,k}M_{1|ij}^{\alpha\beta} + a_{\alpha\beta}M_{1|ij,k}^{\alpha\beta}. \tag{13.80}$$

After alternation with respect to indices j and k and using Ricci identity, we have

$$\tau_{,j}F_{ik} - \tau_{,k}F_{ij} = a_{\alpha\beta}M_{8|ijk}^{\alpha\beta} + \theta_\alpha M_{9|ijk}^{\alpha\beta}. \tag{13.81}$$

Contracting (13.81) with $\varepsilon^i\varepsilon^j\nu^k$, we obtain $\tau_{,a}\varepsilon^\alpha = a_{\alpha\beta}M_{9|}^{\alpha\beta} + \lambda_\alpha M_{10|}^{\alpha\beta}$. Finally, contracting (13.81) with $\varepsilon^j\nu^i$, we have (13.67c) and Theorem 13.32 is proved. □

At the end, we prove Theorem 13.33.

Proof. If $\mathbb{K}_n^{o(m)} \in C^r$ $(r \geq 3)$ and $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^2$, then by Theorem 13.34, $\overline{\mathbb{K}}_n^{o(\overline{m})} \in C^3$ and formulas (13.67) hold. Because the system of equations (13.67) is closed, we can differentiate equation (13.61) $(r-1)$ times. So we can convince ourselves that $a_{ij} \in C^r$, and also $\overline{g}_{ij} \in C^r$ $(\equiv \overline{\mathbb{K}}_n^{o(\overline{m})} \in C^r)$. □

13.5.4 Holomorphically projective flat parabolic Kähler spaces

We continue in studying of holomorphically-projective mappings between parabolic Kähler spaces and define holomorphically-projective flat parabolic Kähler spaces. We found the tensor characteristic of these spaces and obtained their metric tensors.

Definition 13.10 A parabolic Kähler space $\mathbb{K}_n^{o(m)}$ is said to be *holomorphically-projective flat*, if it admits a holomorphically-projective mapping onto a flat space, i.e. the space with the vanishing Riemannian tensor.

We have the following theorem.

Theorem 13.36 *The parabolic Kähler spaces $\mathbb{K}_n^{o(m)}$ is holomorphically-projective flat if and only if the following conditions are true for the Riemannian tensor*

$$R_{hijk} = c\,(2\,F_{hi}F_{jk} + F_{hj}F_{ik} - F_{hk}F_{ij}), \tag{13.82}$$

where $c = \text{const}$, $F_{ij} = g_{i\alpha}F_j^{\alpha}$.

Proof. Let a parabolic Kähler space $\mathbb{K}_n^{o(m)}$ admit a holomorphically-projective mapping onto a flat space $\overline{\mathbb{V}}_n$ ($\overline{R}_{ijk}^h = 0$), which should be a parabolic Kähler space $\overline{\mathbb{K}}_n^{o(m)}$ same.

If $\overline{R}_{ijk}^h = 0$ then after omitting the index h (13.68) takes the form

$$R_{hijk} = -\psi_{ij}g_{kh} + \psi_{ik}g_{jh} - \varphi_{ij}F_{hk} + \varphi_{ik}F_{hj} + (\varphi_{jk} - \varphi_{kj})F_{hi}. \tag{13.83}$$

Let us symmetrize (13.83) at indices h and i. Then, using the properties of the Riemannian tensor we get:

$$-\psi_{ij}g_{kh} + \psi_{ik}g_{jh} - \varphi_{ij}F_{hk} + \varphi_{ik}F_{hj} - \psi_{hj}g_{ki} + \psi_{hk}g_{ji} - \varphi_{hj}F_{ik} + \varphi_{hk}F_{ij} = 0.$$

Analyzing of this formula, we obtain $\psi_{ij} = 0$ and

$$\varphi_{ij} = c\,F_{ij}, \tag{13.84}$$

where c is a certain function. Thus (13.84) takes the form (13.82).

On the basis (13.69), formula (13.84) takes the form

$$\varphi_{i,j} = \psi_i\varphi_j + \varphi_i\psi_j + c\,F_{ij}. \tag{13.85}$$

The condition of integrability takes the form: $c_{,k}F_{ij} - c_{,j}F_{ik} = 0$. From foregoing one it is implied, that $c_{,i} = 0$ and $c = \text{const}$.

So, we have shown that the Riemannian tensor at all holomorphically-projective flat parabolic Kähler spaces $\mathbb{K}_n^{o(m)}$ satisfies (13.82).

It is easy to check that any parabolic Kähler space $\mathbb{K}_n^{o(m)}$, in which the Riemannian tensor satisfies (13.82), admits holomorphically-projective mapping onto a flat space $\overline{\mathbb{K}}_n^{o(m)}$.

Make sure that the system of equations (13.62) and (13.85) is completely integrable in this $\mathbb{K}_n^{o(m)}$ and has the solution $\overline{g}_{ij}(x)$, $\varphi_i(x)$ for any initial conditions

$$\overline{g}_{ij}(x_o) = \overset{o}{\overline{g}}_{ij} \quad \text{and} \quad \varphi_i(x_o) = \overset{o}{\varphi}_i(x)(10) \tag{13.86}$$

for which $\det\|\overset{o}{\overline{g}}_{ij}\| \neq 0$, $\overset{o}{\overline{g}}_{ij} = \overset{o}{\overline{g}}_{ji}$ and $\overset{o}{\overline{g}}_{i\alpha}F_j^\alpha(x_o) + \overset{o}{\overline{g}}_{j\alpha}F_i^\alpha(x_o) = 0$.

Consequently, the space $\mathbb{K}_n^{o(m)}$ admits a holomorphically-projective mapping onto a space $\overline{\mathbb{K}}_n^{o(m)}$ with the metric tensor $\overline{g}_{ij}(x)$ and the structure $F_i^h(x)$. Using (13.68) we can see, that $\overline{R}_{ijk}^h = 0$, hence $\overline{\mathbb{K}}_n^{o(m)}$ is a flat space. This completes the proof. □

The direct analysis of (13.82) leads us to the following

Lemma 13.7 *A holomorphically-projective flat parabolic Kähler space $\mathbb{K}_n^{o(m)}$ is a Ricci flat symmetric space, i.e. a Ricci tensor is vanishing and the Riemannian tensor is covariantly constant in this $\mathbb{K}_n^{o(m)}$.*

13.5.5 On isometries between holomorphically-projective flat $\mathbb{K}_n^{o(m)}$

We denote $K_n^{o(m,c)}$ a holomorphically-projective flat parabolic Kähler space, which determined by (13.82), and prove the following theorem.

Theorem 13.37 *Two holomorphically-projective flat parabolic Kähler spaces $K_n^{o(m,c)}$ and $\overline{K}_n^{o(\overline{m},\overline{c})}$ are locally isometric if and only if $\overline{m} = m$, the metric signatures are coincident, and the constants c and $\overline{c}$ have the same sign.*

Proof. Let us consider the given spaces $K_n^{o(m,c)}$ and $\overline{K}_n^{o(\overline{m},\overline{c})}$ which are related to the coordinate systems x and $\overline{x}$ respectively. It is natural to consider the case, when the constants c and $\overline{c}$ are not equal to zero.

We will search an isometric mapping $f\colon K_n^{o(m,c)} \to \overline{K}_n^{o(\overline{m},\overline{c})}$. As it is known, the mapping $f\colon \overline{x}^h = \overline{x}^h(x^1, x^2, \dots, x^n)$ is an isometric mapping if and only if

$$g_{ij}(x) = \overline{g}_{\alpha\beta}(\overline{x}(x))\partial_i\overline{x}^\alpha\partial_j\overline{x}^\beta. \tag{13.87}$$

Denote $\overline{x}_i^h \equiv \partial_i\overline{x}^h$. From (13.87) it follows that

$$\partial_i\overline{x}^h = \overline{x}_i^h, \quad \partial_j\overline{x}_i^h = \overline{\Gamma}_{\alpha\beta}^h\overline{x}_i^\alpha\overline{x}_j^\beta - \Gamma_{ij}^\alpha\overline{x}_\alpha^h, \tag{13.88}$$

where Γ_{ij}^h and $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $K_n^{o(m,c)}$ and $\overline{K}_n^{o(\overline{m},\overline{c})}$.

The system (13.88) for the unknown functions $\overline{x}^h(x)$, $\overline{x}_i^h(x)$ has a solution for initial conditions $\overline{x}^h(x_o) = \overline{x}_o^h$ and $\overline{x}_i^h(x_o) = y_i^h$, where the following properties are satisfied

$$\overline{g}_{\alpha\beta}(\overline{x}_o)y_i^\alpha y_j^\beta = g_{ij}(x_o), \quad F_i^\alpha(x_o)\, y_\alpha^h = \sqrt{\overline{c}/c}\, \overline{F}_\alpha^h(\overline{x}_o)\, y_i^\alpha, \tag{13.89}$$

where F_i^h and $\overline{F}_i^h$ are the structures of $K_n^{o(m,c)}$ and $\overline{K}_n^{o(m,c)}$, respectively.

Initial conditions y_i^h from (13.89) exist if only if $\overline{m} = m$, the signatures of the metric g and $\overline{g}$ are coincident, and the constants c and $\overline{c}$ have the same sign. Conditions (13.89) follow from (13.87) and from an integrability condition of system (13.88): $R_{hijk} = \overline{R}_{\alpha\beta\gamma\delta}\overline{x}_h^\alpha \overline{x}_i^\beta \overline{x}_j^\gamma \overline{x}_k^\delta$. □

13.5.6 HP mappings of holomorphically-projective flat $\mathbb{K}_n^{o(m)}$

We can prove the next theorem in the similar way as Theorem 13.36.

Theorem 13.38 *If the holomorphically-projective flat parabolic Kähler space $\mathbb{K}_n^{o(m,c)}$ admits a holomorphically-projective mapping onto some parabolic Kähler space $\overline{\mathbb{K}}_n^{o(m)}$, then $\overline{\mathbb{K}}_n^{o(m)}$ is a holomorphically-projective flat parabolic Kähler space $\overline{\mathbb{K}}_n^{o(m,\overline{c})}$ too.*

In addition the next theorem holds

Theorem 13.39 *Any holomorphically-projective flat parabolic Kähler space $\mathbb{K}_n^{o(m,c)}$ admits a nontrivial holomorphically-projective mapping onto some holomorphically-projective flat parabolic Kähler space $\overline{\mathbb{K}}_n^{o(m,\overline{c})}$ with a given constant $\overline{c}$ and a given signature of the metric $\overline{g}_{ij}$.*

Proof. The availability of this theorem follows from the existence of the solutions $\overline{g}_{ij}(x)$ and $\varphi_i(x)$ of equations (13.62) and

$$\varphi_{i,j} = \psi_i\varphi_j + \varphi_i\psi_j + c\,F_{ij} - \overline{c}\,\overline{F}_{ij},$$

where $\overline{F}_{ij} = \overline{g}_{i\alpha}F_j^\alpha$, for any initial conditions (13.86) for which $\det\|\overset{o}{\overline{g}}_{ij}\| \neq 0$, $\overset{o}{\overline{g}}_{ij} = \overset{o}{\overline{g}}_{ji}$ and $\overset{o}{\overline{g}}_{i\alpha}F_j^\alpha(x_o) + \overset{o}{\overline{g}}_{j\alpha}F_i^\alpha(x_o) = 0$, in the space $\mathbb{K}_n^{o(m,c)}$. □

Theorem 13.40 *Between any holomorphically-projective flat parabolic Kähler spaces it is possible to establish a nontrivial holomorphically-projective mapping.*

Proof. Let us have two arbitrary holomorphically-projective flat parabolic Kähler spaces $\mathbb{K}_n^{o(m,c)}$ and $\overline{\mathbb{K}}_n^{o(m,\overline{c})}$. By Theorem 13.39, there exists some space $\tilde{K}_n^{o(m,c)}$ with a signature of a metric of $\overline{\mathbb{K}}_n^{o(m,\overline{c})}$, on which $\mathbb{K}_n^{o(m,c)}$ admits nontrivial holomorphically-projective mapping. By Theorem 13.37, the spaces $\overline{\mathbb{K}}_n^{o(m,\overline{c})}$ and $\tilde{K}_n^{o(m,c)}$ are isometric, which prove the theorem. □

13.5.7 Metric of holomorphically-projective flat $\mathbb{K}_n^{o(m)}$

In a symmetric space its a metric tensor may be rebuilt in some Riemannian coordinate system $(y^1, y^2, \dots y^n)$ at a point x_o by the known formulas [139]

$$g_{ij} = \overset{o}{g}_{ij} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\, 2^{2k+2}}{(2k+2)!} m_i^{\sigma_1} m_{\sigma_1}^{\sigma_2} \cdots m_{\sigma_{k-1}j}, \tag{13.90}$$

where $m_{ij} = \overset{o}{R}_{i\alpha j\beta} y^\alpha y^\beta$, $m_j^i = m_{i\alpha}\overset{o}{g}^{\alpha i}$ and $\overset{o}{g}_{ij}$ $\overset{o}{g}^{ij}$ and $\overset{o}{R}_{hijk}$ are the components of the metric, its inverse and Riemannian tensors at the point x_o.

Taking into acount the representation of Riemannian tensor (13.82) and properties of structures F_i^h the formulas (13.90) take the form:

$$g_{ij} = \overset{o}{g}_{ij} - c\, F_i F_j, \tag{13.91}$$

where $F_i = \overset{o}{F}_{i\alpha} y^\alpha$, $\overset{o}{F}_{ij}$ are the components of tensor F_{ij} at x_o.

Note, that for a given point x_o of holomorphically-projective flat parabolic Kähler space $\mathbb{K}_n^{o(m,c)}$ the metric and structure tensors may be simultaneously reduced to the form:

$$\overset{o}{g}_{ij} = \begin{pmatrix} 0 & 0 & b_{ab} \\ 0 & \overset{*}{e} & 0 \\ b_{ab}^T & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overset{o}{F}_i^h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_m & 0 & 0 \end{pmatrix},$$

where b_{ab}^T is a transposed matrix b_{ab}, $a, b = \overline{1, m}$, E_m is the identity matrix,

$$b_{ab} = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & 0 & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & 0 & & & & -1 & 0 \end{pmatrix} \text{ and } \overset{*}{e} = \begin{pmatrix} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_{n-2m} \end{pmatrix},\ e_a = \pm 1.$$

Thus, we proved the following theorem.

Theorem 13.41 *In the holomorphically-projective flat parabolic Kähler space* $\mathbb{K}_n^{o(m,c)}$ *there exists a coordinate system* y *in which the metric tensor has the form (13.91).*

Not neglecting generality of reasons, on basic of Theorem 13.37 we can consider $c = 0, \pm 1$ that is, spaces $K_n^{o(m,0)}$, $K_n^{o(m,+1)}$ and $K_n^{o(m,-1)}$.

14 ALMOST GEODESIC MAPPINGS

The theory of geodesic mappings of Riemannian spaces and manifolds with affine connections is a wider and simultaneously a geometrically natural generalizations of the theory of projective spaces. It subsumes the theory of $(n-2)$-projective spaces (Kagan [79], Vrançeanu [933]), the theory of holomorphically projective mappings of Kähler and hyperbolic Kähler spaces, concircular geometry (Yano [949]), the theory of spaces admitting of convergent (P.A. Shirokov [167]) and geodesic (Shapiro [787, 788]) vector fields; it has interesting intersections with the theory of quasigeodesic mappings (A.Z. Petrov [743]).

For modelling of various physical processes, geodesic lines and almost geodesic curves serve as a useful tool. Trasformations or mappings between spaces (endowed with the metric or connection) which preserve such curves play an important role in physics, particularly in mechanics, and in geometry as well. Our aim is to continue investigations concerning existence of almost geodesic mappings of manifolds with linear (affine) connection without torsion.

In the present chapter we first set forth the fundamental facts in the theory of almost geodesic mappings of manifolds with affine connection without torsion.

14.1 Almost geodesic mappings

Geodesic and almost geodesic lines serve as a useful tool for modelling of various physical processes, and mappings between spaces (endowed with the metric or connecion) and trasformations which preserve such curves, play an important role in geometry as well as in physics, particularly in mechanics, optics and the theory of relativity, [139, 140, 743].

Many geometric problems connected with the topic of differential geometry are solved by means of differential equations, particularly, the problems are often answered by solving systems of partial differential equations (PDE's) for components of some geometric objects (e.g. tensors), [50–52, 56, 118, 119, 121, 122, 139, 140, 149, 170, 198, 200]. We intend to study here the existence problem of canonical almost geodesic mappings, and as we shall see, our main tool will be to construct and solve a suitable system of PDE's of Cauchy type that controlles the situation. One of characteristic properties of a system of PDE's of Cauchy type is that the solution of such a system depends on a finite number of real (or complex) parameters. Moreover, solutions of such systems can be effectively enumerated, eventually some approximation can be found.

Unless otherwise specified, all objects under consideration are supposed to be differentiable of a sufficiently high class (mostly, differentiability of the class C^3 is sufficient).

14.1.1 Almost geodesic curves

Let $A_n = (M, \nabla)$ be an n-dimensional manifold endowed with a linear connection ∇. Let $\ell\colon\ I \to M$, $t \mapsto \ell(t)$ defined on an open interval $I \subset \mathbb{R}$ be a curve on M satisfying the regularity condition

$$\ell'(t) = d\ell(t)/dt \neq 0 \quad \text{for all} \quad t \in I.$$

Denote by λ the corresponding tangent vector field along ℓ ("velocity field"), $\lambda(t) = (\ell(t), \ell'(t))\,,\ t \in I$, and let

$$\lambda_1 = \nabla(\lambda;\lambda) = \nabla_\lambda\lambda, \qquad \lambda_2 = \nabla^2(\lambda;\lambda,\lambda) = \nabla_\lambda\lambda_1. \tag{14.1}$$

Geodesics $\ell(s)$, parametrized by canonical affine parameter (given up to the affine transformations $s \mapsto as + b$), are characterized by $\nabla_\lambda\lambda = 0$. Geodesic curves (i.e. arbitrarily parametrized) can be characterized by the formula $\nabla_\lambda\lambda = \varrho\lambda$ where $\varrho(t) : I \to \mathbb{R}$ is a real function.

Let $D = \operatorname{span}(X_1, X_2)$ (i.e. the vector fields X_1, X_2 along ℓ form a basis of D). Recall that D is parallel (along c) if and only if the covariant derivatives along ℓ of basis vector fields belong to the distribution (the property is independent of reparametrization of the curve) [170, 171, 816].

As a generalization of geodesic, let us introduce an *almost geodesic curve* as a curve ℓ satisfying:

Definition 14.1 There exists a two-dimensional (differentiable) distribution D parallel along ℓ (relative to ∇) such that for any tangent vector of ℓ, its parallel translation along ℓ (to any other point) belongs to the distribution D.

Equivalently, ℓ is almost geodesic if and only if there exist vector fields X_1, X_2 parallel along ℓ (i.e. satisfying $\nabla_\lambda X_i = a^j X_j$ for some differentiable functions $a_i^j(t)\colon\ I \to \mathbb{R}$) and differentiable real functions $b^i(t)$, $t \in I$ along ℓ, such that $\lambda = b^1 X_1 + b^2 X_2$ holds. For almost geodesic curves, the vector fields λ_1 and λ_2 belong to the corresponding distribution D. If the vector fields λ and λ_1 are independent at any point (and hence the (local) curve ℓ is not a geodesic one), we can write $D = \operatorname{span}(\lambda, \lambda_1)$. So we get another equivalent characterization: *a curve is almost geodesic if and only if* $\lambda_2 \in \operatorname{span}(\lambda, \lambda_1)$.

There exists two functions $\varrho_1(t)$ and $\varrho_2(t)$ for which

$$\lambda_2 = \varrho_1(t)\,\lambda + \varrho_2(t)\,\lambda_1. \tag{14.2}$$

For a priori defined functions $\varrho_1(t)$ and $\varrho_2(t)$, and initial conditions: $\ell(t_0) = x_0$, $\ell'(t_0) = \lambda_0 \neq 0$ and $\ell''(t_0) = \lambda_0^*$ exist unique solution of equation (14.2).

The parameter t can be found, so that or function $\varrho_1(t) = 0$ or $\varrho_2(t) = 0$.

14.1.2 Almost geodesic mappings, basic definitions

Geodesic mappings of manifolds with linear connection are diffeomorphisms characterized by the property that all geodesics are send onto (unparametrized in general) geodesic curves. The classification of geodesic mappings is more or less known. Recall that even for Riemannian spaces, there is a lack of a nice simple criterion for decision when a given Riemannian space admits non-trivial geodesic mappings.

Let $A_n = (M, \nabla)$, $\overline{A}_n = (\overline{M}, \overline{\nabla})$ be n-dimensional manifolds ($n > 2$) each endowed with a torsion-free linear connection.

We may ask which diffeomorphisms of manifolds send almost geodesic curves onto almost geodesic again. The answer is: such mappings reduce to geodesic ones, since there are "too many" almost geodesic curves. It appears that the following definition is more acceptable.

Definition 14.2 We say that a diffeomorphism $f\colon M \to \overline{M}$ is *almost geodesic* if any geodesic curve of (M, ∇) is mapped under f onto an almost geodesic curve in $(\overline{M}, \overline{\nabla})$.

This concept of an almost geodesic mapping was introduced by N.S. Sinyukov [816] and before by V.M. Chernyshenko [337], from a rather different point of view. The theory of almost geodesic mappings was treated in [170, 171].

Due to the fact that f is a diffeomorphism we can accept the useful convention that both linear connections ∇ and $\overline{\nabla}$ are in fact defined on the same underlying manifold M, so that we can consider their difference tensor field of type $(1,2)$, $P = \overline{\nabla} - \nabla$, called sometimes a *deformation tensor* of the given connections under f [170], given by $\overline{\nabla}(X,Y) = \nabla(X,Y) + P(X,Y)$ for $X, Y \in \mathcal{X}(M)$. Since the connections are symmetric, P is also symmetric in X, Y. Of course, we identify objects on M with their corresponding objects on $\overline{M}$: a curve c on M identifies with its image $\overline{c} = f \circ c$, its tangent vector field $\lambda(t)$ with the corresponding vector field $\overline{\lambda}(t) = Tf(\lambda(t))$ etc.

Besides the deformation tensor, we will use the tensor field of type $(1,3)$, denoted by the same symbol P, introduced by

$$P(X,Y,Z) = \sum_{CS(X,Y,Z)} \nabla_Z P(X,Y) + P(P(X,Y),Z), \quad X,Y,Z \in \mathcal{X}(M),$$

where $\sum_{CS(\,,\,,\,)}$ means the cyclic sum on arguments in brackets (i.e. symmetrization without coefficients).

Almost geodesic diffeomorphisms $f\colon (M,\nabla) \to (M,\overline{\nabla})$ are characterized by the following condition on the type $(1,3)$ tensor P:

$$P(X_1,X_2,X_3) \wedge P(X_4,X_5) \wedge X_6 = 0, \quad X_i \in \mathcal{X}(M), \quad i = 1,\dots,6; \qquad (14.3)$$

$X \wedge Y$ means the decomposable bivector, the exterior product of X and Y.

N.S. Sinyukov [170, 171, 816] distinguished three kinds of almost geodesic mappings, namely π_1, π_2, and π_3, characterized, respectively, by the conditions for the deformation tensor:

$$\pi_1\colon \nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X + b(X)\cdot P(X,X),\ X \in \mathcal{X}(M),$$

where $a \in S^2(M)$ is a symmetric tensor field of type $(0,2)$ and b is a 1-form;

$$\pi_2\colon P(X,X) = \psi(X)\cdot X + \varphi(X)\cdot F(X), \quad X \in \mathcal{X}(M),$$

where ψ and φ are 1-forms, and F is a type $(1,1)$ tensor field satisfying

$$\nabla_X F(X) + \varphi(X)\cdot F(F(X)) = \mu(X)\cdot X + \varrho(X)\cdot F(X), \quad X \in \mathcal{X}(M)$$

for some 1-forms μ, ϱ;

$$\pi_3\colon P(X,X) = \psi(X)\cdot X + a(X,X)\cdot Z, \quad X \in \mathcal{X}(M)$$

where ψ is a 1-form, $a \in S^2(M)$ is a symmetric bilinear form and $Z \in \mathcal{X}(M)$ is a vector field satisfying

$$\nabla_X Z = h\cdot X + \theta(X)\cdot Z$$

for some scalar function $h\colon M \to \mathbb{R}$ and some 1-form θ.

In coordinate these conditions have following forms [170, 171]:

1. The mapping is the almost geodesic of π_1 type, if

$$P^h_{(ij,k)} + P^\alpha_{(ij} P^h_{k)\alpha} = \delta^h_{(i} a_{jk)} + b_{(i} P^h_{jk)} \tag{14.4}$$

where a_{ij} and b_i are tensors.

2. The mapping is the almost geodesic of π_2 type, if

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \tag{14.5}$$

$$F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\sigma_{j)}; \tag{14.6}$$

where F^h_i is a tensor of type $(1,1)$ and $\psi_i, \varphi_i, \mu_i, \sigma_i$ are covectors.

3. The mapping is the almost geodesic of π_3 type, if

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + \varphi^h \omega_{ij}, \tag{14.7}$$

$$\varphi^h_{,i} = \varrho\delta^h_i + \varphi^h a_i, \tag{14.8}$$

where ψ_i, φ^h, a_i are vectors, ω_{ij} is a symmetric tensor and ϱ is a function.

Remark. One should note that these types of almost geodesic mappings can intersect.

14.1.3 On a classification of almost geodesic mappings

Berezovski and Mikeš [208, 650, 651] takes the following result.

Theorem 14.1 *Only three types, π_1, π_2 and π_3, of almost geodesic mappings of spaces with affine connection $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$ $(n > 5)$ can exist.*

Proof. Let us consider an almost geodesic mapping of spaces with affine connection $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$. With these mappings, conditions of necessity and sufficiency (14.3):

$$P^{[h}_{\alpha\beta\gamma}\, P^{i}_{\delta\varepsilon}\, \delta^{j]}_{\eta}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}\lambda^{\delta}\lambda^{\varepsilon}\lambda^{\eta} = 0, \tag{14.9}$$

where $[i, j, k]$ denote the alternation, must be satisfied for all λ.

This relation is a homogeneous six-order polynomial with respect to a component of an arbitrary vector λ^h and tensors $P^h_{\alpha\beta\gamma}$, $P^h_{\alpha\beta}$ are dependent only on coordinates of a point x^h. By virtue of the arbitrariness of λ^h condition (14.9) is equal to the relation

$$P^{[h}_{(\alpha\beta\gamma}\, P^{i}_{\delta\varepsilon}\, \delta^{j]}_{\eta)} = 0. \tag{14.10}$$

Conditions (14.9) and (14.10) are the basic equations of the almost geodesic mappings, as well. In particular, a mapping is said to be the almost geodesic if and only if the deformation tensor P^h_{ij} satisfies equation (14.10).

In the following we shall exclude from consideration the situation when the almost geodesic mapping is the geodesic one. The latter are the partial, but well studied case of almost geodesic mappings. Hence, we shall suppose that the deformation tensor P^h_{ij} satisfies the condition

$$P^h_{ij} \neq \delta^h_i\psi_j + \delta^h_j\psi_i, \tag{14.11}$$

where ψ_i is a covector.

Condition (14.11) provides the existence of such a vector ε^h that ε^h and $P^h = P^h_{\alpha\beta}\varepsilon^{\alpha}\varepsilon^{\beta}$ are not collinear. Then, contracting (14.10) by $\varepsilon^{\alpha}\varepsilon^{\beta}\varepsilon^{\gamma}\varepsilon^{\delta}\varepsilon^{\varepsilon}\varepsilon^{\eta}$ and by virtue of the non-collinearity of the vectors ε^h and P^h, we get

$$A^h_{\alpha\beta\gamma}\varepsilon^{\alpha}\varepsilon^{\beta}\varepsilon^{\gamma} = w_1\,\varepsilon^h + w_2\,P^h, \tag{14.12}$$

where w_1 and w_2 are functions on x^h. Contracting (14.10) by $\varepsilon^{\beta}\varepsilon^{\gamma}\varepsilon^{\delta}\varepsilon^{\varepsilon}\varepsilon^{\eta}$, $\varepsilon^{\gamma}\varepsilon^{\delta}\varepsilon^{\varepsilon}\varepsilon^{\eta}$ and $\varepsilon^{\delta}\varepsilon^{\varepsilon}\varepsilon^{\eta}$, in turns, and taking into account (14.12) and the intermediate results we have obtained, in total we are convinced that the tensor P^h_{ijk} can be represented as

$$A^h_{ijk} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)} + F^h_{(i}\overset{1}{w}{}_{jk)} + \varepsilon^h\overset{2}{w}{}_{ijk} + P^h\overset{3}{w}{}_{ijk}, \tag{14.13}$$

where F^h_i, a_{jk}, $\overset{1}{w}{}_{jk}$, $\overset{2}{w}{}_{ijk}$, $\overset{3}{w}{}_{ijk}$, b_i are tensors.

Thus, the following assertion is proved.

Lemma 14.1 *If $\mathbb{A}_n$ admits an almost geodesic mapping onto $\bar{\mathbb{A}}_n$, different from the geodesic one, then condition (14.13) is satisfied.*

Obviously, if the almost geodesic mapping is different from mapping π_1, then the tensor

$$\tilde{A}^h_{ijk} = F^h_{(i}\overset{1}{w}_{jk)} + \varepsilon^h \overset{2}{w}_{ijk} + P^h \overset{3}{w}_{ijk} \neq 0, \tag{14.14}$$

In the following we shall consider the almost geodesic mapping, different from π_1, for which (14.14) is true. Then, using (14.13) we shall exclude from (14.10) the tensor, and after transformations we get

$$\tilde{A}^{[h}_{(\alpha\beta\gamma} P^i_{\delta\varepsilon} \delta^{j]}_{\eta)} = 0. \tag{14.15}$$

It can be shown that condition (14.14) provides the existence of vector $\overline{\varepsilon}^h$, such that

$$(\tilde{A}^h_{\alpha\beta\gamma}\delta^i_\varepsilon - \tilde{A}^i_{\alpha\beta\gamma}\delta^h_\varepsilon)\,\overline{\varepsilon}^\alpha\overline{\varepsilon}^\beta\overline{\varepsilon}^\gamma\overline{\varepsilon}^\varepsilon \neq 0.$$

Using then, an analogous contraction of (14.15), by turn, we get

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)} + \varepsilon^h\psi^1_{ij} + P^h\psi^2_{ij} + \overline{\varepsilon}^h\psi^3_{ij} + \overline{P}^h\psi^4_{ij}, \tag{14.16}$$

where ψ^1_{ij}, ψ^2_{ij}, ψ^3_{ij}, ψ^4_{ij}, $\overline{P}^h$ are tensors.

Let us denote

$$\overline{P}^h_{ij} = \varepsilon^h\psi^1_{ij} + P^h\psi^2_{ij} + \overline{\varepsilon}^h\psi^3_{ij} + \overline{P}^h\psi^4_{ij}. \tag{14.17}$$

If $\overline{P}^h_{ij} = 0$, then from (14.16), there follows (14.5). Naturally, we suppose that $\varphi_i \neq 0$, then condition (14.9) can be written in the form

$$(F^h_{\alpha,\beta} + F^h_\varepsilon F^\varepsilon_\alpha\varphi_\beta)\lambda^\alpha\lambda^\beta\ \varphi_g\lambda^\gamma = \tilde{a}\,\lambda^h + \tilde{b}\,F^h_\alpha\lambda^\alpha,$$

where $\tilde{a}$ and $\tilde{b}$ are functions dependent of x^h and λ^h. Since $\psi_i \neq 0$, it is not difficult to obtain from the latest

$$(F^h_{\alpha,\beta} + F^h_\varepsilon F^\varepsilon_\alpha\varphi_\beta)\lambda^\alpha\lambda^\beta = \tilde{\tilde{a}}\,\lambda^h + \tilde{\tilde{b}}\,F^h_\alpha\lambda^\alpha, \tag{14.18}$$

Assuming that $F^h_i \neq \varrho\,\delta^h_i + \eta_i\Theta^h$ and using the methods described in [2], the correctness of (14.6) follows from (14.18), i.e. on these conditions the mapping is on almost geodesic mapping of the π_2 type.

When $F^h_i = \varrho\,\delta^h_i + \eta_i\Theta^h$, in this case the deformation tensor P^h_{ij} take the form (14.7). Substituting them into (14.9), we get

$$a_{\alpha,\beta}\lambda^\alpha\lambda^\beta\ \Theta^h_{,\gamma}\lambda^\gamma = \tilde{a}\,\lambda^h + \tilde{b}\,\Theta^h,$$

where $\tilde{a}, \tilde{b}$ are functions dependent on x^h and λ^h.

Since $a_{ij} \neq 0$, it is easy to obtain from these relations

$$\Theta^h_{,\gamma}\lambda^\gamma = \tilde{\tilde{a}}\,\lambda^h + \tilde{\tilde{b}}\,\Theta^h.$$

From this relation it is easy to obtain (14.8), i.e. the mapping is on almost geodesic of the π_3 type.

Assuming $\overline{P}^h_{ij} \neq 0$ we get

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + \Theta^h\omega_{ij} + \varepsilon^h\omega^1_{ij} + P^h\omega^2_{ij} + \bar{\varepsilon}^h\omega^3_{ij} + \overline{P}^h\omega^4_{ij}, \tag{14.19}$$

Using equations (14.9) and (14.19) and taking into account that the space dimension $n > 5$, i t is easy to show that the deformation tensor P^h_{ij} takes either form (14.4) or (14.5) or (14.7). Thus, the theorem has been proved. □

Analogous conceptions were introduced for almost geodesic mappings of affine-connected spaces with a torsion [943], and the theorem is true in this case too.

In [818] infinitesimal almost geodesic transformations are considered. It is true for them that other infinitesimal almost geodesic transformation, different from π_1, π_2 and π_3 types, do not exist.

14.1.4 On a completeness classification of almost geodesic mappings

In the subsection a classification of almost geodesic mappings is specified. It is proved that if an almost geodesic mapping f is simultaneously π_1 and π_2 (or π_3) then f is a mapping of affine connection spaces with preserved linear (or quadratic) complex of geodesic lines.

The subsection is devoted to an investigation of completeness of a classification of almost geodesic mappings of affine connection spaces $\mathbb{A}_n$ without the torsion.

In [170, 171] the almost geodesic mappings of an affine connection space A_n were introduced and three types of them were distinguished, π_1, π_2 and π_3. Berezovsky and Mikeš [296] proved that for $n > 5$ other types of almost geodesic mappings do not exist. However, one can not exclude the case when a mapping π_τ $(\tau = 1, 2, 3)$ is simultaneously a mapping π_σ $(\sigma \neq \tau)$.

In this subsection we characterize non-overlapping types of almost geodesic mappings. We receive the complete classification of these mappings for $n > 5$.

Under an almost geodesic mapping, only the mappings π_1, π_2 and π_3 act in the neighborhood of every point of the space $\mathbb{A}_n$ $(n > 5)$, exept, maybe, the set of points of measure zero [296].

It is natural to presume that the affinor F^h_i of the mapping π_2 satisfies $F^h_i \not\equiv \varrho\delta^h_i + \varphi^h a_i$ and $\varphi^h\omega_{ij} \not\equiv 0$ for the mapping π_3. Then $\pi_2 \cap \pi_3 = \varnothing$. Indeed, let us suppose, that a mapping is simultaneously π_2 and π_3. Then (14.5) and (14.7) imply

$$\delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)} = \delta^h_{(i}\overset{*}{\psi}_{j)} + \varphi^h\omega_{ij}. \tag{14.20}$$

Since $\varphi_i \not\equiv 0$ then there exists a vector ε^i such that $\varepsilon^\alpha\varphi_\alpha = 1$. Contracting (14.20) with $\varepsilon^i\varepsilon^j$ we get

$$F^h_\alpha\varepsilon^\alpha = \alpha\varepsilon^h + \beta\varphi^h,$$

where α, β are functions.

By the help of the above formula and after contracting (14.20) with ε^j we have

$$F^h_i = \varrho\delta^h_i + \varphi^h a_i$$

which was required to prove.

Theorem 14.2 *If an almost geodesic mapping f is simultaneously π_1 and π_2 then f is a mapping of an affine connection space with preserving a linear complex of geodesic lines.*

Proof. Let a mapping f be an almost geodesic mapping of types π_1 and π_2 simultaneously. After substituting (14.5) in (14.4) and taking into account (14.6) one finds

$$\delta^h_{(i}A_{jk)} + F^h_{(i}B_{jk)} = 0 \tag{14.21}$$

where $B_{jk} \equiv \varphi_{(j,k)} - \varphi_{(j}\theta_{k)}$, A_{jk}, θ_k are tensors. Equation (14.21) implies $A_{jk} \equiv 0$ and $B_{jk} \equiv 0$.

The construction of these tensors shows that relation

$$\varphi_{(i,j)} = \varphi_{(i}\theta_{j)} \tag{14.22}$$

is correct.

A mapping π_2 such that (14.5), (14.6) and (14.22) hold, is, evidently, a mapping π_1. On the other hand, equations (14.5) and (14.22) characterize mappings preserving a linear complex of geodesic lines [337]. The theorem is proved. □

Theorem 14.3 *If an almost geodesic mapping f is simultaneously π_1 and π_3 then f is a mapping of an affine connection space which preserves a quadratic complex of geodesic lines.*

Proof. Let a mapping f be an almost geodesic mapping of types π_1 and π_3 simultaneously. After substituting (14.7) in (14.4) and taking into account (14.8) we obtain

$$\delta^h_{(i}A_{jk)} + \varphi^h B_{ijk} = 0, \tag{14.23}$$

where $B_{ijk} \equiv \omega_{(ij,k)} - a_{(i}\omega_{jk)}$, A_{jk}, a_i are tensors. From (14.23) we have $A_{jk} \equiv 0$ and $B_{ijk} \equiv 0$.

From here we get

$$\omega_{(ij,k)} = a_{(i}\omega_{jk)}. \tag{14.24}$$

Mappings π_3 given by (14.7), (14.8) and satisfying conditions (14.24) are π_1 mappings.

On the other hand, equations (14.7) and (14.24) characterize mappings preserving a quadratic complex of geodesic lines [337]. The theorem is proved. □

In a natural way, there are distiguished mappings $\pi_{12} = \pi_1 \cap \pi_2$ and $\pi_{13} = \pi_1 \cap \pi_3$.

As we have already noted, mappings π_{12} preserve a linear complex of geodesic lines and these mappings are characterized by equations (14.5), (14.6) and (14.22).

Mappings π_{13} preserve a quadratic complex of geodesic lines and are characterized by equations (14.7), (14.8) and (14.24).

Theorem 14.4 *The space $\mathbb{A}_n$ ($n > 5$), except, maybe, the set of measure zero, is divided into open domains. In each of them one of the following six mappings acts: geodesic, π_{12}, π_{13}, $\pi_1 \setminus \{\pi_2 \cup \pi_3\}$, $\pi_2 \setminus \pi_1$, $\pi_3 \setminus \pi_1$.*

14.2 Almost geodesic mappings of type π_1

This section is devoted to the study of almost geodesic mappings of first type π_1 of manifolds with linear (affine) connection. Mappings of the type π_1 are not studied very much because their equations are too difficult.

14.2.1 Canonical almost geodesic mappings $\tilde{\pi}_1$

Our aim is to continue investigations concerning existence of almost geodesic mappings of manifolds with linear (affine) connection, particularly of the so-called $\tilde{\pi}_1$ mappings, i.e. canonical almost geodesic mappings of type π_1 according to Sinyukov.

First we give necessary and sufficient conditions for existence of $\tilde{\pi}_1$ mappings of a manifold endowed with a linear connection onto pseudo-Riemannian manifolds. The conditions take the form of a closed system of PDE's of first order of Cauchy type. Further we deduce necessary and sufficient conditions for existence of $\tilde{\pi}_1$ mappings onto generalized Ricci-symmetric spaces. Our results are generalizations of some previous theorems obtained by N.S. Sinyukov.

We are interested here in a particular subclass of π_1-mappings, the so-called $\tilde{\pi}_1$-*mappings*, or *canonical* almost geodesic mappings, distinguished by the condition $b = 0$. That is, $\tilde{\pi}_1$-mappings are just morphisms satisfying

$$\nabla_X P(X,X) + P(P(X,X),X) = a(X,X)\cdot X, \quad a \in S^2(M),\ X \in \mathcal{X}(M).$$

In local coordinates, the condition reads

$$P^h_{(ij,k)} = a_{(ij}\delta^h_{k)} - P^h_{\alpha(i}P^\alpha_{jk)}. \tag{14.25}$$

Here and in what follows, the comma "," denotes covariant derivative with respect to ∇, δ^h_i is the Kronecker delta, the round bracket denote the cyclic sum on indices involved.

Any geodesic mapping is a π_1-mapping (the characterizing condition can be checked), and any π_1-mapping can be written as a composition of a geodesic mapping followed by a $\tilde{\pi}_1$-mapping. So we can consider geodesic mappings as trivial almost geodesic mappings, and we will omit them in further considerations; they were analysed in [297].

Recall that a pseudo-Riemannian space (M, g) is called a *Ricci-symmetric space*[93] when the Ricci tensor is parallel with respect to the corresponding Levi-Civita connection ∇ of the metric, $\nabla \mathrm{Ric} = 0$. It was proven by Sinyukov [170], that the basic partial differential equations (PDE's) of $\tilde{\pi}_1$-mappings of a manifold (M, ∇) onto Ricci-symmetric pseudo-Riemannian manifolds $(\overline{M}, \overline{g})$ (of arbitrary signature) can be transformed into (an equivalent) closed system of PDE's of first order of the Cauchy type. Hence the solution (if it exists) depends on a finite set of parameters. Consequently, for a manifold with a symmetric connection admitting $\tilde{\pi}_1$-mappings onto Ricci-symmetric spaces, the set of all

[93] In analogy to symmetric spaces that are characterized by parallel Riemannian curvature tensor

Ricci-symmetric spaces $(\overline{M}, \overline{g})$ which can serve as images of the given manifold (M, ∇) under $\tilde{\pi}_1$-mappings is finite. The cardinality r of such a set is bounded by the number of free parameters.

On the other hand, geodesic mappings form a subclass among $\tilde{\pi}_1$-mappings (they obey the definition). Basic equations describing geodesic mappings of manifolds with linear connection do not form a closed system of Cauchy type (the general solution depends on n arbitrary functions; if the given manifold admits geodesic mappings, the cardinality of the set of possible images is big). It follows that the conditions (14.25) describing $\tilde{\pi}_1$-mappings of manifolds, in general, cannot be transformed into a closed system of Cauchy type. But if we choose a suitable subclass of images and restrict ourselves (for the given manifold) only onto mappings with co-domain in the apropriate subclass we might succeed to get an equivalent closed system of Cauchy type. If this is the case then the given manifold admits either non (if the system is non-integrable) or a finite number of $\tilde{\pi}_1$-images in the given class.

Our aim is to analyse $\tilde{\pi}_1$-mappings of manifolds onto manifolds with linear connection in general, and to use the reached results for examining $\tilde{\pi}_1$-mappings of manifolds onto (pseudo-)Riemannian spaces (in general, without any restrictive conditions onto the Ricci tensor), which will generalize the above result by Sinyukov. In the rest, we will omit "pseudo".

All $\tilde{\pi}_1$-mappings $f\colon M \to M$ can be described by the following system of differential equations [170, 171]:

$$\begin{gathered} 3(\nabla_Z P(X,Y) + P(Z, P(X,Y))) = \\ \sum_{CS(X,Y)} (R(Y,Z)X - \overline{R}(Y,Z)X) + \sum_{CS(X,Y,Z)} a(X,Y)Z. \end{gathered} \tag{14.26}$$

In what follows, we prefer to express our equalities in local coordinates (with respect to a map (U, φ) on M) since the invariant formulas are rather complicated. The above formula has the local expression

$$3(P^h_{ij,k} + P^h_{k\alpha} P^\alpha_{ij}) = R^h_{(ij)k} - \overline{R}^h_{(ij)k} + a_{(ij}\delta^h_{k)}, \tag{14.27}$$

where P^h_{ij}, a_{ij}, R^h_{ijk}, $\overline{R}^h_{ijk}$ are local components of tensors P, a, R, and $\overline{R}$.

14.2.2 Properties of the fundamental equations of $\tilde{\pi}_1$

Assuming (14.27) as a system of PDE's for functions P^h_{ij} on M, the corresponding integrability conditions read

$$\begin{aligned} \overline{R}^h_{(ij)[k,\ell]} = R^h_{(ij)[k,\ell]} + \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + 3(P^\alpha_{ij}\overline{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}) - \\ P^h_{\alpha k}(R^\alpha_{(ij)\ell} - \overline{R}^\alpha_{(ij)\ell}\delta^\alpha_{(i}a_{j\ell)}) + P^h_{\alpha\ell}(R^\alpha_{(ij)k} - \overline{R}^\alpha_{(ij)k}\delta^\alpha_{(i}a_{jk)})\,. \end{aligned}$$

Passing from $\nabla\overline{R}$ to $\overline{\nabla R}$ on the left hand side we get integrability conditions of the system (14.27) in the form

$$\overline{R}^h_{(ij)[k;\ell]} = \delta^h_{(i}a_{jk),\ell} - \delta^h_{(i}a_{j\ell),k} + \Theta^h_{ijk\ell}\,; \tag{14.28}$$

here we denoted

$$\Theta^h_{ijk\ell} = R^h_{(ij)[k,\ell]} + 3(P^\alpha_{ij}\overline{R}^h_{\alpha k\ell} - P^h_{\alpha(j}R^\alpha_{i)k\ell}) - P^h_{\alpha k}(R^\alpha_{(ij)\ell} + \delta^\alpha_{(i}a_{j\ell)}) +$$
$$P^h_{\alpha\ell}(R^\alpha_{(ij)k} + \delta^\alpha_{(i}a_{jk)}) - P^\alpha_{\ell(i}\overline{R}^h_{|\alpha|j)k} - P^\alpha_{\ell(i}\overline{R}^h_{j)\alpha k} + P^\alpha_{k(i}\overline{R}^h_{|\alpha|j)\ell} + P^\alpha_{k(i}\overline{R}^h_{j)\alpha\ell}$$

where ";" denotes covariant derivative with respect to $\overline{\nabla}$.

If we apply covariant differentiation with respect to $\overline{\nabla}$ to the integrability conditions (14.28) of the system (14.27), and then pass from covariant derivation $\overline{\nabla}$ to ∇, we get

$$\overline{R}^h_{(ij)k;\ell m} - \overline{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jk),\ell m} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk\ell m}\,, \tag{14.29}$$

where we denoted

$$T^h_{ijk\ell m} = \overline{R}^h_{\alpha mk}\overline{R}^\alpha_{(ij)\ell} - \overline{R}^\alpha_{\ell mk}\overline{R}^h_{(ij)\alpha} - \overline{R}^\alpha_{jmk}\overline{R}^h_{(i\alpha)\ell} - \overline{R}^\alpha_{imk}\overline{R}^h_{(j\alpha)\ell} -$$
$$P^h_{m\alpha}\delta^\alpha_{(i}a_{jk),\ell} - P^\alpha_{mj}\delta^h_{(i}a_{\alpha k),\ell} - P^\alpha_{mi}\delta^h_{(\alpha}a_{jk),\ell} - P^\alpha_{mk}\delta^h_{(\alpha}a_{ij),\ell} - P^\alpha_{ml}\delta^h_{(i}a_{jk),\alpha} -$$
$$P^h_{m\alpha}\delta^\alpha_{(i}a_{j\ell),k} + P^\alpha_{mi}\delta^h_{(\alpha}a_{j\ell),k} + P^\alpha_{mj}\delta^h_{(i}a_{\alpha\ell),k} + P^\alpha_{mk}\delta^h_{(i}a_{j\ell),\alpha} - P^\alpha_{ml}\delta^h_{(i}a_{j\alpha),k} -$$
$$\Theta^h_{ijk\ell,m} + P^h_{\alpha m}\Theta^\alpha_{ijk\ell} - P^\alpha_{mi}\Theta^h_{\alpha jk\ell} - P^\alpha_{mj}\Theta^h_{i\alpha k\ell} - P^\alpha_{mk}\Theta^h_{ij\alpha\ell} - P^\alpha_{m\ell}\Theta^h_{ijk\alpha}\,.$$

Alternating (14.29) in ℓ, m we get

$$\begin{aligned}&\overline{R}^h_{(ij)m;\ell k} - \overline{R}^h_{(ij)\ell;mk} = \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{j\ell),km} + T^h_{ijk[lm]} + \\ &\overline{R}^h_{(i|\alpha k|}\overline{R}^\alpha_{j)m\ell} + \overline{R}^h_{(ij)\alpha}\overline{R}^\alpha_{km\ell} - \overline{R}^\alpha_{(ij)k}\overline{R}^h_{\alpha m\ell} + \overline{R}^h_{\alpha(i|k|}\overline{R}^\alpha_{j)m\ell} + \\ &\delta^h_{(\alpha}a_{jk)}R^\alpha_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{j\ell m} + \delta^h_{(i}a_{j\alpha)}R^\alpha_{k\ell m} - \delta^h_{(i}a_{jk)}R^\alpha_{\alpha\ell m}\,.\end{aligned} \tag{14.30}$$

Using properties of the Riemannian tensor, we rewrite (14.30) as

$$\overline{R}^h_{im\ell;jk} + \overline{R}^h_{jm\ell;ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - N^h_{ijk\ell m}\,, \tag{14.31}$$

where the last term is

$$N^h_{ijk\ell m} = T^h_{ijk[\ell m]} + \overline{R}^\alpha_{im\ell}\overline{R}^h_{(\alpha j)k} + \overline{R}^\alpha_{jm\ell}\overline{R}^h_{(\alpha i)k} + \overline{R}^\alpha_{km\ell}\overline{R}^h_{(ij)\alpha} -$$
$$\overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{(ij)k} + \delta^h_{(\alpha}a_{jk)}R^\alpha_{i\ell m} + \delta^h_{(\alpha}a_{ik)}R^\alpha_{j\ell m} + \delta^h_{(\alpha}a_{ij)}R^\alpha_{k\ell m} - a_{(ij}R^h_{k)\ell m}\,.$$

Alternating (14.31) over j, k we get

$$\begin{aligned}&\overline{R}^h_{jm\ell;ik} - \overline{R}^h_{km\ell;ij} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(i}a_{km),j\ell} - \\ &N^h_{i[jk]\ell m} + \overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{ikj} + \overline{R}^h_{i\alpha\ell}\overline{R}^\alpha_{mkj} + \overline{R}^h_{im\alpha}\overline{R}^\alpha_{\ell kj} - \overline{R}^\alpha_{im\ell}\overline{R}^h_{\alpha kj}\,.\end{aligned} \tag{14.32}$$

Let us change mutually i and k in (14.31), and then use (14.32). We evaluate

$$\begin{aligned}&2\overline{R}^h_{jm\ell;ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \\ &\delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(j\ell}a_{k),im} + \Omega^h_{ijk\ell m}\,,\end{aligned} \tag{14.33}$$

where we used the notation

$$\Omega^h_{ijk\ell m} = -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{(kj)i} + \overline{R}^h_{j\alpha\ell}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{\ell ik} - \overline{R}^h_{\alpha i(j}\overline{R}^\alpha_{k)m\ell} + \overline{R}^h_{j\alpha\ell}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{\ell ik} - \overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{ikj} - \overline{R}^h_{i\alpha\ell}\overline{R}^\alpha_{mkj} + \overline{R}^\alpha_{im[\ell}\overline{R}^h_{\alpha]kj}\,.$$

On the left side of (14.33), let us pass from the covariant derivation $\overline{\nabla}$ to ∇:

$$\begin{aligned} 2\overline{R}^h_{jm\ell,ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \\ \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} - \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,, \end{aligned} \tag{14.34}$$

where

$$\begin{aligned} S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\,[&\overline{R}^\alpha_{jm\ell,i}P^h_{\ell k} - \overline{R}^h_{\alpha m\ell,i}P^\alpha_{jk} - \\ &\overline{R}^h_{j\alpha\ell,i}P^\alpha_{mk} - \overline{R}^h_{jm\alpha,i}P^\alpha_{\ell k} - \overline{R}^h_{jm\ell,\alpha}P^\alpha_{ik} + \\ &(\overline{R}^\alpha_{jm\ell}P^\beta_{\alpha i} - \overline{R}^h_{\alpha m\ell}P^\alpha_{ij} - \overline{R}^h_{j\alpha\ell}P^\alpha_{im} - \overline{R}^h_{jm\alpha}P^\alpha_{i\ell})P^h_{\beta k} - \\ &(\overline{R}^\alpha_{jm\ell}P^h_{\alpha\beta} - \overline{R}^h_{\alpha m\ell}P^\alpha_{\beta j} - \overline{R}^h_{j\alpha\ell}P^\alpha_{\beta m} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta\ell})P^\beta_{ik} - \\ &(\overline{R}^\alpha_{\beta m\ell}P^h_{\alpha i} - \overline{R}^h_{\alpha m\ell}P^\alpha_{\beta i} - \overline{R}^h_{\beta\alpha\ell}P^\alpha_{im} - \overline{R}^h_{\beta m\alpha}P^\alpha_{i\ell})P^\beta_{jk} - \\ &(\overline{R}^\alpha_{j\beta\ell}P^h_{\alpha i} - \overline{R}^h_{\alpha\beta\ell}P^\alpha_{ji} - \overline{R}^h_{j\alpha\ell}P^\alpha_{\beta i} - \overline{R}^h_{j\beta\alpha}P^\alpha_{i\ell})P^\beta_{km} - \\ &(\overline{R}^\alpha_{jm\beta}P^h_{\alpha i} - \overline{R}^h_{\alpha m\beta}P^\alpha_{ji} - \overline{R}^h_{j\alpha\beta}P^\alpha_{mi} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta i})P^\beta_{k\ell}]\,. \end{aligned} \tag{14.35}$$

14.2.3 Canonical almost geodesic mappings $\tilde{\pi}_1$ onto Riemannian spaces

Let there exist a $\tilde{\pi}_1$-mapping of a manifold $\mathbb{A}_n = (M, \nabla)$ onto a Riemannian manifold $\overline{\mathbb{V}}_n = (M, \overline{g})$ where $\overline{g} \in T^0_2M$ is a metric tensor with components $\overline{g}_{ij}$. Recall that the Riemannian tensor $\overline{R}_{hijk} = \overline{R}^\alpha_{ijk}\overline{g}_{\alpha h}$ of type $(0,4)$ satisfies

$$\overline{R}_{hijk} + \overline{R}_{ihjk} = 0. \tag{14.36}$$

In (14.33), let us apply the metric tensor $\overline{g}_{h\beta}$ and then use symmetrization with respect to h and j. According to (14.36) we get

$$\begin{aligned} \overline{g}_{ih}(a_{m[k,j]l} + a_{l[j,k]m}) + \overline{g}_{ij}(a_{m[k,h]l} + a_{l[h,k]m}) + \\ \overline{g}_{kh}(a_{m[i,j]l} + a_{l[j,i]}) + \overline{g}_{kj}(a_{m[i,h]l} + a_{l[h,i]m}) + \\ \overline{g}_{mh}(a_{k[i,j]l} - a_{ij,kl}) + \overline{g}_{mj}(a_{k[i,h]l} - a_{ih,kl}) + \overline{g}_{lj}(a_{kh,il} - a_{i(h,k)m}) + \\ +2\overline{g}_{jh}(a_{k(l,i)m} - a_{m(i,k)l}) + \overline{g}_{lh}(a_{k[j,i]m} - a_{ij,km}) = -\Omega^\alpha_{i(j|klm}\overline{g}_{\alpha|h)}. \end{aligned} \tag{14.37}$$

Contraction of the last formula with the dual tensor $\overline{g}^{jh}$ ($\|\overline{g}^{ij}\| = \|\overline{g}_{ij}\|^{-1}$) gives

$$a_{kl,im} - a_{im,kl} - a_{km,il} + a_{il,km} = -\tfrac{2}{n+1}\Omega^\alpha_{i\alpha klm}\,. \tag{14.38}$$

Let us symmetrize the above formula over k and l. From (14.38) we get

$$\begin{aligned} 2a_{kl,im} - 2a_{im,kl} = 2a_{\alpha m}R^\alpha_{lik} + a_{\alpha i}R^\alpha_{mlk} + a_{\alpha k}R^\alpha_{mil} + a_{\alpha l}R^\alpha_{mik} + \\ \tfrac{2}{n+1}(\Omega^\alpha_{l\alpha kim} - \Omega^\alpha_{i\alpha(kl)m}). \end{aligned} \tag{14.39}$$

Using (14.38) and (14.39) the equation (14.37) reads

$$\begin{gathered}2\overline{g}_{ih}(a_{km,jl}-a_{jm,kl})+2\overline{g}_{ij}(a_{km,hl}-a_{hm,kl})+\\ 2\overline{g}_{kh}(a_{im,jl}-a_{jm,il})+2\overline{g}_{kj}(a_{im,hl}-a_{hm,il})+\\ \overline{g}_{mk}(a_{ki,jl}-a_{kj,il}-a_{ij,kl})+\overline{g}_{mj}(a_{ki,hl}-a_{kh,il}-a_{ih,kl})+\\ \overline{g}_{lj}(a_{kh,im}-a_{i(h,k)m})+\overline{g}_{lh}(a_{kj,im}-a_{i(k,j)m})=C_{ijkhl},\end{gathered}\tag{14.40}$$

where

$$\begin{aligned}C_{ijkhl}=&-\Omega^{\alpha}_{i(j|klm}\overline{g}_{\alpha|h)}+\tfrac{2}{n+1}\Omega^{\alpha}_{i\alpha klm}\overline{g}_{jh}-\overline{g}_{kh}a_{\alpha l}R^{\alpha}_{mij}+\\ &\overline{g}_{ih}(\tfrac{2}{n+1}\Omega^{\alpha}_{m\alpha ljk}-a_{\alpha k}R^{\alpha}_{(ml)j}-a_{\alpha j}R^{\alpha}_{(l|k|m)}-a_{\alpha m}R^{\alpha}_{lkj}-a_{\alpha l}R^{\alpha}_{mkj})+\\ &\overline{g}_{ij}(\tfrac{2}{n+1}\Omega^{\alpha}_{m\alpha lhk}-a_{\alpha k}R^{\alpha}_{(ml)h}-a_{\alpha h}R^{\alpha}_{(l|k|m)}-a_{\alpha m}R^{\alpha}_{lkh}-a_{\alpha l}R^{\alpha}_{mkh})+\\ &\overline{g}_{kh}(\tfrac{2}{n+1}\Omega^{\alpha}_{m\alpha lji}-a_{\alpha i}R^{\alpha}_{(ml)j}-a_{\alpha j}R^{\alpha}_{(l|i|m)}-a_{\alpha m}R^{\alpha}_{lij}+a_{\alpha l}R^{\alpha}_{mij})+\\ &\overline{g}_{kj}(\tfrac{2}{n+1}\Omega^{\alpha}_{m\alpha lhi}-a_{\alpha i}R^{\alpha}_{(ml)h}-a_{\alpha h}R^{\alpha}_{(l|i|m)}-a_{\alpha m}R^{\alpha}_{lih}+a_{\alpha l}R^{\alpha}_{mih}).\end{aligned}$$

If we contract (14.40) with the dual $\overline{g}^{ij}$ of the metric tensor, use (14.39) and the Ricci identity we get

$$a_{km,hl}-a_{kl,hm}=\tfrac{1}{2(n+3)}(\overline{g}_{hm}\mu_{kl}-\overline{g}_{hl}\mu_{km})+B_{kmhl},\tag{14.41}$$

where $\mu_{km}=a_{\alpha\beta,km}\overline{g}^{\alpha\beta}$, and

$$\begin{aligned}&B_{kmhl}=C_{\alpha\beta kmhl}\overline{g}^{\alpha\beta}+3a_{m\alpha}R^{\alpha}_{lhk}+\tfrac{3}{2}(a_{h\alpha}R^{\alpha}_{mkl}+a_{k\alpha}R^{\alpha}_{mhl}+a_{l\alpha}R^{\alpha}_{mhk})+\\ &\tfrac{3}{n+1}(\Omega^{\alpha}_{l\alpha khm}-\Omega^{\alpha}_{h\alpha(kl)m})-\tfrac{1}{2}(a_{m\alpha}R^{\alpha}_{lkm}+a_{k\alpha}R^{\alpha}_{mhl}+a_{h\alpha}R^{\alpha}_{mkl}+a_{l\alpha}R^{\alpha}_{mkh})-\\ &\tfrac{1}{n+1}(\Omega^{\alpha}_{l\alpha hkm}-\Omega^{\alpha}_{k\alpha(hl)m})-a_{\alpha(h}R^{\alpha}_{k)lm}+\tfrac{1}{2}(a_{k\alpha}R^{\alpha}_{lmh}+a_{h\alpha}R^{\alpha}_{lkm}+a_{m\alpha}R^{\alpha}_{lkh}).\end{aligned}$$

Now contract (14.40) with $\overline{g}^{ih}$. According to (14.41) we get

$$\overline{g}_{kl}\mu_{jm}-\overline{g}_{jl}\mu_{km}+\overline{g}_{km}\mu_{jl}-\overline{g}_{jkm}\mu_{kl}=\frac{n+3}{n+1}\,C_{kljm},\tag{14.42}$$

where

$$C_{kljm}=C_{\alpha jkl(m|\beta|l)}\overline{g}^{\alpha\beta}-2(n+1)(B_{k(ml)j}-a_{\alpha(l}R^{\alpha}_{m)jk}+a_{j\alpha}R^{\alpha}_{(m|k|l)}+a_{k\alpha}R^{\alpha}_{(lm)j}).$$

Contracting (14.42) with $\overline{g}^{k\ell}$ and using the notation $K=\mu_{\alpha\beta}\overline{g}^{\alpha\beta}$ we obtain components of the tensor μ:

$$\mu_{jm}=\frac{1}{n}K\overline{g}_{jm}+\frac{n+3}{n(n+1)}C_{\alpha\beta jm}\overline{g}^{\alpha\beta}.\tag{14.43}$$

Using (14.43) we can rewrite (14.41) in the form

$$a_{km,hl}-a_{hm,kl}=\frac{K}{2n(n+3)}\,(\overline{g}_{mh}\overline{g}_{kl}-\overline{g}_{lh}\overline{g}_{km})+A_{kmhl},\tag{14.44}$$

where

$$A_{kmhl} = B_{kmhl} + \frac{1}{2n(n+1)}\,(\overline{g}_{hm}C_{\alpha\beta kl}\overline{g}^{\alpha\beta} - \overline{g}_{hl}C_{\alpha\beta km}\overline{g}^{\alpha\beta}).$$

Combining (14.40) and (14.44) we get

$$\begin{gathered}\overline{g}_{jl}a_{ih,km} + \overline{g}_{hl}a_{ij,km} - \overline{g}_{jm}a_{ih,kl} - \overline{g}_{hm}a_{ij,kl} = \\ -\tfrac{K}{n(n+3)}\,(\overline{g}_{ih}\overline{g}_{kl}\overline{g}_{jm} - \overline{g}_{ih}\overline{g}_{km}\overline{g}_{jl} + \overline{g}_{ij}\overline{g}_{kl}\overline{g}_{hm} - \overline{g}_{ij}\overline{g}_{km}\overline{g}_{hl} + \\ 3\overline{g}_{kh}\overline{g}_{il}\overline{g}_{jm} - 3\overline{g}_{kh}\overline{g}_{jl}\overline{g}_{im} + 3\overline{g}_{kj}\overline{g}_{il}\overline{g}_{hm} - 3\overline{g}_{lh}\overline{g}_{jk}\overline{g}_{im}) + A_{ijkmhl},\end{gathered} \tag{14.45}$$

where we have denoted

$$A_{ijkmhl} = C_{ijkmhl} - 2(\overline{g}_{i(h}A_{|km|j)l} + \overline{g}_{k(h}A_{|im|jl)} - \overline{g}_{m(h}A_{|ki|j)l} - \overline{g}_{l(h}A_{|k|j)im)}).$$

Finally, symetrization of (14.45) over the indices i, j, followed by contraction with $\overline{g}^{\ell h}$ anables us to express second covariant derivatives of the tensor a,

$$a_{ij,km} = \frac{K}{n(n+3)}\,(\overline{g}_{ij}\overline{g}_{km} + 3\overline{g}_{k(j}\overline{g}_{i)m}) + A_{(ij)km\alpha\beta}\overline{g}^{\alpha\beta}. \tag{14.46}$$

Now we can consider (14.46) as the first order system of PDE's of Cauchy type relative to the tensor ∇a (i.e. in $a_{ij,k}$), find the integrability conditions and contract them with $\overline{g}^{ij}$ and $\overline{g}^{km}$, respectively. We calculate ∇K,

$$K_{,\beta} = \frac{n(n+3)}{n^2+5n-6}\,A_\beta, \tag{14.47}$$

where we denoted

$$\begin{gathered}A_\varrho = \Big[a_{\alpha(j,|k}R^\alpha_{i)m\varrho} + a_{ij,\alpha}R^\alpha_{km\varrho} - \tfrac{K}{n(n+3)}\,(\overline{g}_{ij,[\varrho}\overline{g}_{m]k} + \overline{g}_{ij}\overline{g}_{k[m,\varrho} + \\ 3\overline{g}_{kj,[\varrho}\overline{g}_{m]i} + 3\overline{g}_{kj}\overline{g}_{i[m,\varrho]} + 3\overline{g}_{ki,[\varrho}\overline{g}_{m]j} + 3\overline{g}_{ki}\overline{g}_{j[m,\varrho]}) + \\ A_{(ij)k[m|\alpha\beta|,\varrho]}\overline{g}^{\alpha\beta} + A_{(ij)k[m|\alpha\beta|}\overline{g}^{\alpha\beta}{}_{,\varrho]}\Big]\,\overline{g}^{ij}\overline{g}^{km}.\end{gathered}$$

We use $\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + P^h_{ij}$ and get

$$\overline{g}_{ij,k} = P^\alpha_{ik}\overline{g}_{\alpha j} + P^\alpha_{jk}\overline{g}_{\alpha i}. \tag{14.48}$$

Assume the tensors ∇a and $\nabla\overline{R}$, and denote their components by $a_{ijk} := a_{ij,k}$ and $\overline{R}^h_{ijk\ell} := \overline{R}^h_{ijk,\ell}$, respectively. Then (14.34) and (14.46) take the form

$$\begin{gathered}2R^h_{jmli,k} = \delta^h_{(i}a_{jl)k,m} - \delta^h_{(i}a_{jm)k,l} + \delta^h_{(k}a_{jl)i,m} - \delta^h_{(k}a_{jm)i,l} + \\ \delta^h_{(i}a_{km)j,l} - \delta^h_{(i}a_{kl)j,m} + S^h_{ijklm},\end{gathered} \tag{14.49}$$

$$a_{ijk,m} = \frac{K}{n(n+3)}(\overline{g}_{ij}\overline{g}_{km} + 3\overline{g}_{k(j}\overline{g}_{i)m}) + A_{(ij)km\alpha\beta}\overline{g}^{\alpha\beta}, \tag{14.50}$$

where covariant derivatives of the tensor a_{ijk} in (14.49) are supposed to be expressed according to (14.50), the tensor S was introduced componentwise in (14.35).

The formulas (14.27), (14.47)–(14.50) represent a closed system of Cauchy type for unknown functions

$$\overline{g}_{ij}(x),\ P^h_{ij}(x),\ a_{ij}(x),\ a_{ijk}(x),\ K(x),\ \overline{R}^h_{ijk}(x),\ R^h_{ijkl}(x), \tag{14.51}$$

which, moreover, must satisfy a finite set of algebraic conditions

$$\overline{g}_{[ij]} = P^h_{[ij]} = a_{[ij]} = a_{[ij]k} = \overline{R}^h_{i(jk)} = R^h_{i(jk)l} = 0,\ det\|\overline{g}_{ij}(x)\| \neq 0. \tag{14.52}$$

So we have proven:

Theorem 14.5 *The given manifold $\mathbb{A}_n = (M, \nabla)$ admits $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings of type π_1) onto Riemannian spaces $\overline{\mathbb{V}}_n = (M, \overline{g})$ if and only if there exists solution of the mixed system of Cauchy type (14.27), (14.47)–(14.50), (14.52) for the functions (14.51).*

As a consequence of the additional algebraic conditions, we get an upper boundary for the number r of possible solutions:

Corollary 14.1 *The family of all Riemannian manifolds $\overline{\mathbb{V}}_n$ which can serve as images of the given manifold $\mathbb{A}_n = (M, \nabla)$, depends on at most*

$$\frac{1}{2}n^2(n^2-1) + n(n+1)^2 + 1$$

parameters.

The above Theorem generalizes the result of Sinyukov [170, 171] already mentioned as well as his results on geodesic mappings of Riemannian spaces.

14.2.4 Ricci-symmetric and generalized Ricci-symmetric spaces

Under a *Ricci-symmetric manifold* we mean a manifold (M, ∇) with linear connection for which the Ricci tensor is parallel (= covariantly constant),

$$\nabla \mathrm{Ric} = 0;$$

Ricci symmetric spaces form a particular subclass.

It was proven in [170] that the family of all $\tilde{\pi}_1$-mappings of a manifold (M, ∇) onto Ricci-symmetric ($\overline{\nabla}\,\overline{\mathrm{Ric}} = 0$) (pseudo-) Riemannian spaces $(\overline{M}, \overline{g})$ is given by the integrable system (of Cauchy type) of partial differentiable equations (in covariant derivatives). Consequently, given a manifold with a symmetric connection, the family of all Ricci-symmetric Riemannian spaces $(\overline{M}, \overline{g})$ which can serve as images of the given manifold (M, ∇) under some $\tilde{\pi}_1$-mapping, depends on a finite set of parameters.

On the other hand, the geodesic mappings form a subset in the set of $\tilde{\pi}_1$-mappings; they obey the definition. But the basic equations describing geodesic

mappings of a manifold with the linear connection do not form an integrable system of Cauchy type, since the general solution depends on n arbitrary functions. It follows that the conditions (14.25) describing $\tilde{\pi}_1$-mappings (i.e. canonical almost geodesic mappings) of manifolds do not, in general, induce an integrable system.

In the following, we consider a particular case when (14.25) can be transformed into an integrable system, generalizing the results of Sinyukov. Namely, we will investigate $\tilde{\pi}_1$-mappings of a manifold (M, ∇) onto the so-called generalized Ricci-symmetric manifolds.

A manifold (M, ∇) will be called a *generalized Ricci-symmetric manifold* if its Ricci tensor satisfies

$$\nabla \mathrm{Ric}\,(Y, Z; X) + \nabla \mathrm{Ric}\,(X, Z; Y) = 0, \tag{14.53}$$

that is, $\nabla_X \mathrm{Ric}\,(Y, Z) = -\nabla_Y \mathrm{Ric}\,(X, Z)$. We do not a priori suppose the Ricci tensor be symmetric. If Ric is symmetric and (14.53) holds then Ric is parallel, $\nabla \mathrm{Ric} = 0$, and (M, ∇) is a Ricci-symmetric manifold. Einstein spaces (Riemannian spaces characterized by the property that the Ricci tensor is proportional to the metric tensor) satisfy (14.53) since they satisfy $\nabla \mathrm{Ric} = 0$, hence are generalized Ricci-symmetric. In this sense, the generalized Ricci-symmetric spaces can be considered as a certain generalization of Einstein spaces.

14.2.5 AG mappings $\tilde{\pi}_1$ onto generalized Ricci-symmetric manifolds

Given the n-dimensional manifolds $\mathbb{A} = (M, \nabla)$ and $\overline{\mathbb{A}} = (\overline{M}, \overline{\nabla})$ with the corresponding curvature tensors R and $\overline{R}$, respectively, all connection-preserving mappings $f\colon M \to \overline{M}$ can be described by the system of differential equations (14.26), [170, 171]. These formulas have the local expression (14.27). As we have already proved, from (14.26) it follows (14.28). Using the Bianci identity we can write (14.28) in local coordinates as

$$\overline{R}^h_{i\ell k;j} + \overline{R}^h_{j\ell k;i} = \delta^h_{(i} a_{jk),\ell} - \delta^h_{(i} a_{j\ell),k} + \Theta^h_{ijk\ell}\,,$$

where ";" denotes the covariant derivative with respect to $\overline{\nabla}$. Contraction in h and k gives the following equality for covariant derivatives of components of the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{\nabla}$:

$$\overline{R}_{i\ell;j} + \overline{R}_{j\ell;i} = (n+1) a_{ij,\ell} - a_{\ell(i,j)} + \Theta^\alpha_{ij\alpha\ell}\,. \tag{14.54}$$

In the following let us suppose that the manifold $(\overline{M}, \overline{\nabla})$ is a generalized Ricci-symmetric space, that is, (14.53) holds. In local coordinates, (14.53) reads

$$\overline{R}_{ij;k} + \overline{R}_{kj;i} = 0.$$

Under this assumption, (14.41) reads

$$(n+1) a_{ij,\ell} - a_{\ell i,j} - a_{\ell j,i} = -\Theta^\alpha_{ij\alpha\ell}\,. \tag{14.55}$$

Using symmetrization in ℓ, i gives

$$a_{\ell i,j} + a_{\ell j,i} = -\frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)} + \frac{2}{n}a_{ij,\ell}\,.$$

Now (14.55) reads

$$\frac{n^2+n-2}{n}\,a_{ij,\ell} = -\,\Theta^{\alpha}_{ij\alpha\ell} - \frac{1}{n}\Theta^{\alpha}_{(i|\ell\alpha|j)}. \tag{14.56}$$

Applying the covariant differentiation with respect to $\overline{\nabla}$ to the integrability conditions (14.39), followed by passing from the covariant derivative $\overline{\nabla}$ to ∇ on the right hand side, we get

$$\overline{R}^{h}_{(ij)k;\ell m} - \overline{R}^{h}_{(ij)\ell;mk} = \delta^{h}_{(i}a_{jk),\ell m} - \delta^{h}_{(i}a_{j\ell),km} + T^{h}_{ijk\ell m}\,, \tag{14.57}$$

where

$$\begin{gathered}T^{h}_{ijk\ell m} = \overline{R}^{h}_{\alpha mk}\overline{R}^{\alpha}_{(ij)\ell} - \overline{R}^{\alpha}_{\ell mk}\overline{R}^{h}_{(ij)\alpha} - \overline{R}^{\alpha}_{jmk}\overline{R}^{h}_{(i\alpha)\ell} - \overline{R}^{\alpha}_{imk}\overline{R}^{h}_{(j\alpha)\ell} - \\ P^{h}_{m\alpha}\delta^{\alpha}_{(i}a_{jk),\ell} - P^{\alpha}_{mj}\delta^{h}_{(i}a_{\alpha k),\ell} - P^{\alpha}_{mi}\delta^{h}_{(\alpha}a_{jk),\ell} - P^{\alpha}_{mk}\delta^{h}_{(\alpha}a_{ij),\ell} - P^{\alpha}_{ml}\delta^{h}_{(i}a_{jk),\alpha} - \\ P^{h}_{m\alpha}\delta^{\alpha}_{(i}a_{j\ell),k} + P^{\alpha}_{mi}\delta^{h}_{(\alpha}a_{j\ell),k} + P^{\alpha}_{mj}\delta^{h}_{(i}a_{\alpha\ell),k} + P^{\alpha}_{mk}\delta^{h}_{(i}a_{j\ell),\alpha} - P^{\alpha}_{ml}\delta^{h}_{(i}a_{j\alpha),k} - \\ \theta^{h}_{ijk\ell,m} + P^{\alpha}_{\alpha m}\theta^{\alpha}_{ijk\ell} - P^{\alpha}_{mi}\theta^{h}_{\alpha jk\ell} - P^{\alpha}_{mj}\theta^{h}_{i\alpha k\ell} - P^{\alpha}_{mk}\theta^{h}_{ij\alpha\ell} - P^{\alpha}_{m\ell}\theta^{h}_{ijk\alpha}\,.\end{gathered}$$

Alternating (14.57) over ℓ, m we obtain

$$\begin{gathered}\overline{R}^{h}_{(ij)m;\ell k} - \overline{R}^{h}_{(ij)\ell;mk} = \delta^{h}_{(i}a_{jm),k\ell} - \delta^{h}_{(i}a_{j\ell),km} + T^{h}_{ijk[lm]} + \\ \overline{R}^{h}_{(i|\alpha k|}\overline{R}^{\alpha}_{j)m\ell} + \overline{R}^{h}_{(ij)\alpha}\overline{R}^{\alpha}_{km\ell} - \overline{R}^{\alpha}_{(ij)k}\overline{R}^{h}_{\alpha m\ell} + \overline{R}^{h}_{\alpha(i|k|}\overline{R}^{\alpha}_{j)m\ell} + \\ \delta^{h}_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} + \delta^{h}_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^{h}_{(i}a_{j\alpha)}R^{\alpha}_{k\ell m} - \delta^{h}_{(i}a_{jk)}R^{\alpha}_{\alpha\ell m}\,.\end{gathered} \tag{14.58}$$

Due to the properties of the Riemannian tensor, (14.58) can be written as

$$\overline{R}^{h}_{im\ell;jk} + \overline{R}^{h}_{jm\ell;ik} = \delta^{h}_{(i}a_{j\ell),km} - \delta^{h}_{(i}a_{jm),k\ell} - N^{h}_{ijk\ell m}\,, \tag{14.59}$$

where

$$\begin{gathered}N^{h}_{ijk\ell m} = T^{h}_{ijk[\ell m]} + \overline{R}^{\alpha}_{im\ell}\overline{R}^{h}_{(\alpha j)k} + \overline{R}^{\alpha}_{jm\ell}\overline{R}^{h}_{(\alpha i)k} + \overline{R}^{\alpha}_{km\ell}\overline{R}^{h}_{(ij)\alpha} - \\ \overline{R}^{h}_{\alpha m\ell}\overline{R}^{\alpha}_{(ij)k} + \delta^{h}_{(\alpha}a_{jk)}R^{\alpha}_{i\ell m} + \delta^{h}_{(\alpha}a_{ik)}R^{\alpha}_{j\ell m} + \delta^{h}_{(\alpha}a_{ij)}R^{\alpha}_{k\ell m} - a_{(ij}R^{h}_{k)\ell m}\,.\end{gathered}$$

Let us alternate (14.59) over j, k. We get

$$\begin{gathered}\overline{R}^{h}_{jm\ell;ik} - \overline{R}^{h}_{km\ell;ij} = \delta^{h}_{(i}a_{j\ell),km} - \delta^{h}_{(i}a_{jm),k\ell} - \delta^{h}_{(i}a_{k\ell),jm} + \delta^{h}_{(i}a_{km),j\ell} - \\ N^{h}_{i[jk]\ell m} + \overline{R}^{h}_{\alpha m\ell}\overline{R}^{\alpha}_{ikj} + \overline{R}^{h}_{i\alpha\ell}\overline{R}^{\alpha}_{mkj} + \overline{R}^{h}_{im\alpha}\overline{R}^{\alpha}_{\ell kj} - \overline{R}^{\alpha}_{im\ell}\overline{R}^{h}_{\alpha kj}\,.\end{gathered} \tag{14.60}$$

Let us change mutually i and k in (14.59), and then use (14.60). We evaluate

$$\begin{gathered}2\overline{R}^{h}_{jm\ell;ik} = \delta^{h}_{(i}a_{j\ell),km} - \delta^{h}_{(i}a_{jm),k\ell} - \delta^{h}_{(k}a_{jm),i\ell} + \\ \delta^{h}_{(i}a_{km),j\ell} - \delta^{h}_{(i}a_{k\ell),jm} + \delta^{h}_{(j\ell}a_{k),im} + \Omega^{h}_{ijk\ell m}\,,\end{gathered} \tag{14.61}$$

where

$$\Omega^h_{ijk\ell m} = -N^h_{ijk\ell m} + N^h_{k[ij]k\ell m} - \overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{(kj)i} + \overline{R}^h_{j\alpha\ell}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{\ell ik} - \overline{R}^h_{\alpha i(j}\overline{R}^\alpha_{k)m\ell} + \overline{R}^h_{j\alpha\ell}\overline{R}^\alpha_{mik} + \overline{R}^h_{jm\alpha}\overline{R}^\alpha_{\ell ik} - \overline{R}^h_{\alpha m\ell}\overline{R}^\alpha_{ikj} - \overline{R}^h_{i\alpha\ell}\overline{R}^\alpha_{mkj} + \overline{R}^\alpha_{im[\ell}\overline{R}^h_{\alpha]kj}\,.$$

On the left hand side of (14.61), let us pass from the covariant derivative with respect to $\overline{\nabla}$ to the covariant derivative with respect to ∇:

$$\begin{aligned} 2\overline{R}^h_{jm\ell,ik} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \\ \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} - \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,, \end{aligned} \tag{14.62}$$

where

$$\begin{aligned} S^h_{ijk\ell m} = \Omega^h_{ijk\ell m} - 2\,[\overline{R}^\alpha_{jm\ell,i}P^h_{\ell k} - \overline{R}^h_{\alpha m\ell,i}P^\alpha_{jk} - \\ \overline{R}^h_{j\alpha\ell,i}P^\alpha_{mk} - \overline{R}^h_{jm\alpha,i}P^\alpha_{\ell k} - \overline{R}^h_{jm\ell,\alpha}P^\alpha_{ik} + \\ (\overline{R}^\alpha_{jm\ell}P^\beta_{\alpha i} - \overline{R}^h_{\alpha m\ell}P^\alpha_{ij} - \overline{R}^h_{j\alpha\ell}P^\alpha_{im} - \overline{R}^h_{jm\alpha}P^\alpha_{i\ell})P^h_{\beta k} - \\ (\overline{R}^\alpha_{jm\ell}P^h_{\alpha\beta} - \overline{R}^h_{\alpha m\ell}P^\alpha_{\beta j} - \overline{R}^h_{j\alpha\ell}P^\alpha_{\beta m} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta\ell})P^\beta_{ik} - \\ (\overline{R}^\alpha_{\beta m\ell}P^h_{\alpha i} - \overline{R}^h_{\alpha m\ell}P^\alpha_{\beta i} - \overline{R}^h_{\beta\alpha\ell}P^\alpha_{im} - \overline{R}^h_{\beta m\alpha}P^\alpha_{i\ell})P^\beta_{jk} - \\ (\overline{R}^\alpha_{j\beta\ell}P^h_{\alpha i} - \overline{R}^h_{\alpha\beta\ell}P^\alpha_{ji} - \overline{R}^h_{j\alpha\ell}P^\alpha_{\beta i} - \overline{R}^h_{j\beta\alpha}P^\alpha_{i\ell})P^\beta_{km} - \\ (\overline{R}^\alpha_{jm\beta}P^h_{\alpha i} - \overline{R}^h_{\alpha m\beta}P^\alpha_{ji} - \overline{R}^h_{j\alpha\beta}P^\alpha_{mi} - \overline{R}^h_{jm\alpha}P^\alpha_{\beta i})P^\beta_{k\ell}]\,. \end{aligned}$$

Let us introduce a $(1,4)$-tensor field $R^h_{jm\ell i} = \overline{R}^h_{jm\ell,i}$. Then we get

$$\overline{R}^h_{jm\ell,i} = R^h_{jm\ell i}\,. \tag{14.63}$$

From (14.62), the covariant derivative of the tensor (14.63) satisfies

$$\begin{aligned} 2R^h_{jm\ell i,k} = \delta^h_{(i}a_{j\ell),km} - \delta^h_{(i}a_{jm),k\ell} - \delta^h_{(k}a_{jm),i\ell} + \\ \delta^h_{(i}a_{km),j\ell} - \delta^h_{(i}a_{k\ell),jm} + \delta^h_{(k}a_{j\ell),im} + S^h_{ijk\ell m}\,, \end{aligned} \tag{14.64}$$

where we used (14.56).

It can be verified that the equations (14.37), (14.56), (14.63) and (14.64) for the functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ on (M,∇) form an integrable system; the above functions must satisfy also additional algebraic conditions

$$\begin{gathered} P^h_{ij}(x) = P^h_{ji}(x), \qquad a_{ij}(x) = a_{ji}(x), \qquad \overline{R}^h_{i(jk)}(x) = \overline{R}^h_{(ijk)}(x) = 0, \\ R^h_{i(jk)\ell}(x) = R^h_{(ijk)\ell}(x) = 0. \end{gathered} \tag{14.65}$$

So we have succeeded to prove the following generalization of the result of Sinyukov [171] (we use the above notation).

Theorem 14.6 *Let (M, ∇) be a manifold with linear connection and $(\overline{M}, \overline{\nabla})$ a generalized Ricci-symmetric manifold. There is a $\tilde{\pi}_1$ mapping $f: M \to \overline{M}$ (i.e. a canonical almost geodesic mapping of type π_1) if and only if there exist functions $P^h_{ij}(x)$, $a_{ij}(x)$, $\overline{R}^h_{ijk}(x)$ and $R^h_{ijkm}(x)$ which satisfy the equations (14.37), (14.56), (14.63), (14.64), and (14.65). The system of equations (14.37), (14.56), (14.63) and (14.64) forms a Cauchy type system of PDE's in covariant derivatives.*

As a consequence we obtain

Corollary 14.2 *The family of all generalized Ricci-symmetric manifolds, which can serve as an image of the given manifold (M, ∇) under some $\tilde{\pi}_1$-mapping, depends on at most*

$$\frac{1}{6}\, n(n+1)(2n^3 - 4n^2 + 5n + 3)$$

parameters.

14.3 On almost geodesic mappings of the type π_1 of Riemannian spaces preserving a system n-orthogonal hypersurfaces

Berezovsky and Mikeš [298] studied almost geodesic mapping of the type π_1 preserving a system n-orthogonal hypersurfaces. We found metrics of Riemannian metrics for some complementary presumptions for which these mappings exist.

14.3.1 Mappings of $\mathbb{V}_n$ preserving a system n-orthogonal hypersurfaces

Suppose that Riemannian spaces have a system n-orthogonal hypersurfaces. Let the mapping f: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ preserves these systems of hypersurfaces. We take the coordinate system $x \equiv (x^1, x^2, \ldots, x^n)$ so as system n-orthogonal hypersurfaces was the system of coordinate surfaces.

We denote $g_{ij}(x)$ and $\overline{g}_{ij}(x)$ components of metric tensors of spaces $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ in coordinate system x. Element of their inverse matrices we denote $g^{ij}(x)$ and $\overline{g}^{ij}(x)$. With respect to system n-orthogonal hypersurfaces in $\mathbb{V}_n$ and $\overline{\mathbb{V}}_n$ we get

$$g_{ij} = \overline{g}_{ij} = g^{ij} = \overline{g}^{ij} = 0 \qquad (i, j = \overline{1, n};\ i \neq j). \tag{14.66}$$

Then following equations hold $\quad g^{ii} = \dfrac{1}{g_{ii}}, \qquad \overline{g}^{ii} = \dfrac{1}{\overline{g}_{ii}}.$

Christoffel symbols Γ^h_{ij} of $\mathbb{V}_n$ have next form:

$$\begin{aligned} &\text{a) } \Gamma^i_{ii} = \tfrac{1}{2}\partial_i \ln g_{ii}\,, \quad \text{b) } \Gamma^i_{ij} = \tfrac{1}{2}\partial_j \ln g_{ii}\,,\ (i \neq j),\\ &\text{c) } \Gamma^h_{ij} = 0\ (h, i, j \neq), \text{ d) } \Gamma^i_{ii} = -\tfrac{1}{2}\partial_j g_{ii}\,/g_{jj}\ \ (i \neq j). \end{aligned} \tag{14.67}$$

For Christoffels symbols $\overline{\Gamma}^h_{ij}$ of $\overline{\mathbb{V}}_n$ similar relationships hold.

Following forms are true for components of deformation tensor $P^h_{ij}(x) \equiv \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ for the mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ preserving system n-orthogonal hypersurfaces:

$$\begin{aligned} &\text{a) } P^h_{ij} = 0 \quad (h,i,j \neq), && \text{b) } P^i_{ii} = \tfrac{1}{2}\partial_i \ln\left(\tfrac{\overline{g}_{ii}}{g_{ii}}\right), \\ &\text{c) } P^i_{ij} = \tfrac{1}{2}\partial_j \ln\left(\tfrac{\overline{g}_{ii}}{g_{ii}}\right) \quad (i \neq j), && \text{d) } P^j_{ii} = \tfrac{1}{2}\left(\tfrac{\partial_j g_{ii}}{g_{jj}} - \tfrac{\partial_j \overline{g}_{ii}}{\overline{g}_{jj}}\right) \quad (i \neq j). \end{aligned} \tag{14.68}$$

14.3.2 Special almost geodesic mappings of the type π_1

When we find nontrivial examples of almost geodesic mapping of the type π_1: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ which preserve system n-orthogonal hypersurfaces we will take some presumtions which simplify equations (14.4). These equations in general we unfortunately did not solve.

We will study mappings of the type π_1: $\mathbb{V}_n \to \overline{\mathbb{V}}_n$ for which the tensor of deformation P^h_{ij} is recurrent, i.e. next condition holds

$$P^h_{ij,k} = \varphi_k P^h_{ij}, \tag{14.69}$$

here comma denotes the covariant derivative of an affine connection in $\mathbb{V}_n$ and φ_k is a covector. Next algebraic equations follow from (14.4) and (14.69)

$$P^h_{\alpha(i}P^\alpha_{jk)} = \delta^h_{(i}a_{jk)}, \tag{14.70}$$

where a_{ij} is a symmetric tensor.

The inverse mapping $(\pi_1)^{-1}$ is also almost geodesic mapping of the type π_1. It follows (14.70) and [170].

For the converse if the mapping $f\colon \mathbb{V}_n \to \overline{\mathbb{V}}_n$ fullfil equations (14.69) and (14.70), then f is the mapping of the type π_1.

Presumption

$$P^h_{hi} = P^k_{ki} = -P^i_{ii} \qquad (h,i,k \neq), \tag{14.71}$$

simplifies the solution of equations (14.69) and (14.70), no sum in latin indices, $h,i,k \neq$ denote that indices h,i,k are different to each other.

If we analyse algebraic condition (14.69) we get

$$\begin{aligned} &\text{a) } 3P^k_{kk}P^h_{ii} = P^k_{ii}P^h_{kk} \qquad (h,i,k \neq), \\ &\text{b) } 9P^j_{jj}P^h_{hh} = P^j_{hh}P^h_{jj} \qquad (h \neq j). \end{aligned} \tag{14.72}$$

It is easy to check that for conditions (14.71) are algebraic equations (14.69) identical.

Using (14.68) we can write conditions $P^h_{hi} = P^j_{ji}$ $(h,i,j \neq)$:

$$\partial_i \ln(\overline{g}_{hh}/g_{hh}) = \partial_i \ln(\overline{g}_{jj}/g_{jj}).$$

From here next form follows

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{jj}/g_{jj}) \cdot F_{hj}(x^h, x^j) \quad (h \neq j). \tag{14.73}$$

where $F_{hj}(\neq 0)$ are function of parameters x^h and x^j.

Similarly

$$\overline{g}_{ii}/g_{ii} = (\overline{g}_{jj}/g_{jj}) \cdot F_{ij}(x^i, x^j) \quad (i \neq j),$$

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{ii}/g_{ii}) \cdot F_{hi}(x^h, x^i) \quad (h \neq i).$$

Using the last three equations we get

$$F_{hi}(x^h, x^i) = \frac{F_{hj}(x^h, x^j)}{F_{ij}(x^i, x^j)} \qquad (h, i, j \neq).$$

and consequently

$$F_{hi}(x^h, x^i) = f_{hj}(x^h) h_{ij}(x^i) \qquad (h, i, j \neq).$$

When we analyse these relationships we obtain

$$F_{hi}(x^h, x^i) = C_{hi} f_h^2(x^h) h_i^2(x^i) \qquad (h \neq i).$$

where $C_{hi} - \text{const} \neq 0$, f_h and h_i are nonzero function of mentioned parameters.

The formulae (14.73) we can write in form

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{jj}/g_{jj}) \cdot C_{hj} \cdot f_h^2(x^h) \cdot h_j^2(x^j) \quad (h \neq j).$$

Now it is easy to see that $h_j(x^j) = f_j^{-1}(x^j)$. Therefore we get

$$\overline{g}_{hh}/g_{hh} = (\overline{g}_{jj}/g_{jj}) \cdot C_{hj} \cdot f_h^2(x^h) \cdot f_j^{-2}(x^j) \quad (h \neq j). \tag{14.74}$$

We use conditions $P_{ii}^i = -P_{hi}^h$ $(h \neq i)$ and (14.68) and we obtain

$$\partial_i \ln \left(\frac{\overline{g}_{hh}}{g_{hh}} \frac{\overline{g}_{ii}}{g_{ii}} \right) = 0 \qquad (h \neq i). \tag{14.75}$$

According to (14.74) for $h \to i$ and $j \to h$ we similarly get

$$\partial_i \ln \left(f_i(x^i) \frac{\overline{g}_{hh}}{g_{hh}} \right) = 0 \qquad (h \neq i).$$

Now we will integrate these equations and we get

$$\frac{\overline{g}_{hh}}{g_{hh}} = \frac{F_h(x^h)}{Q} \tag{14.76}$$

where

$$Q = \prod_{\alpha=\overline{1,n}} f_\alpha(x^\alpha), \tag{14.77}$$

$F_h(x^h)$ are functions.

Substitute (14.77) in (14.75) we get

$$F_h = c_h\, f_h^2(x^h),$$

where $c_h = \text{const} \neq 0$.

In the end we obtain next equations

$$\frac{\overline{g}_{hh}}{g_{hh}} = \frac{c_h f_h^2(x^h)}{Q}. \tag{14.78}$$

And conditions (14.71) and (14.74) follows from these forms.

The components of deformation tensor P_{ij}^h after substitution (14.78) in (14.68) are of the next form

$$\begin{gathered} P_{ii}^i = \tfrac{1}{2}\partial_i \ln f_i, \quad P_{ij}^i = -\tfrac{1}{2}\partial_j \ln f_j \quad (i \neq j), \quad P_{ij}^h = 0 \quad (i,j,k \neq), \\ P_{ii}^j = \frac{1}{2}\left(\frac{\partial_j g_{ii}}{g_{jj}}\left(\frac{c_i f_i^2}{c_j f_j^2} - 1\right) - \frac{c_i f_i^2 g_{ii}}{c_j f_j^2 g_{jj}}\partial_j \ln f_j\right) \quad (i \neq j). \end{gathered} \tag{14.79}$$

Now we will study equations (14.69). For $k = i$, $(h,i,j \neq)$ we get $P_{jj}^h \Gamma_{ii}^j + P_{ii}^h \Gamma_{ij}^i = 0$. From (14.67) for $\partial_j g_{ii} \neq 0$ follows

$$P_{jj}^h / g_{jj} = P_{ii}^h / g_{ii} \quad (h,i,j \neq). \tag{14.80}$$

Using (14.80) and dividing $g_{ii} g_{kk}$ we obtain $P_{ii}^k / g_{ii} = 3P_{kk}^k / g_{kk}$, i.e.

$$P_{ii}^h = 3\, \frac{g_{ii}}{g_{hh}}\, P_{hh}^h \quad (h \neq i). \tag{14.81}$$

Formulae (14.80) hold identically. For $h \neq i$ from (14.79) we get

$$\partial_h \ln g_{ii} = \frac{3c_h f_h^2 - c_i f_i^2}{c_h f_h^2 - c_i f_i^2}\, \partial_h \ln f_h. \tag{14.82}$$

If we analyse equations (14.69) we find that $\varphi_k = -2\partial_k \ln f_k$ and

$$\partial_i \ln g_{ii} = \frac{2\partial_{ii} \ln f_i}{\partial_i \ln f_i} + 4\partial_i \ln f_i + \frac{g_{ii}}{\partial_i \ln f_i} \sum_{\alpha \neq i} \frac{(\partial_\alpha \ln f_\alpha)^2}{g_{\alpha\alpha}} \frac{3c_\alpha f_\alpha^2 - c_i f_i^2}{c_\alpha f_\alpha^2 - c_i f_i^2}. \tag{14.83}$$

Because functions f_i according to assumption are not constant we can find transformation of the coordinate system $x'^h = x'^h(x^h)$, $h = \overline{1,n}$ preserve system n-orthogonal coordinat hypersurfaces so that

$$f_i(x^i) = e^{-x^i}$$

Formulae (14.82) and (14.83) then will have a form

$$\begin{aligned} \partial_i \ln g_{ii} &= -4 - g_{ii} \sum_{\alpha \neq i} \frac{1}{g_{\alpha\alpha}} \frac{3c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}}{c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}}, \\ \partial_h \ln g_{ii} &= \frac{c_i e^{-2x^i} - 3c_h e^{-2x^i}}{c_h e^{-2x^h} - c_i e^{-2x^i}} \qquad (h \neq i). \end{aligned} \tag{14.84}$$

It is clearly that

$$g_{ii} = e^{-x^i} \prod_{\alpha \neq i} (c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}) e^{-x^\alpha}, \qquad c_i = \text{const} \neq 0 \tag{14.85}$$

is the solution of the system (14.84). This solution is obviosly the general solution of this system because this system is of the type Cauchy.

Next theorem follows from this solution and from equations (14.69) and (14.70).

Theorem 14.7 *The Riemannian space $\mathbb{V}_n$ with the metric*

$$g_{ii} = e^{-x^i} \prod_{\alpha \neq i} (c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}) e^{-x^\alpha}, \quad g_{ij} = 0 \ (i \neq j), \quad c_i = const \neq 0,$$

admit almost geodesic mapping of the type π_1 preserving system n-orthogonal hypersurfaces onto Riemannian space $\overline{V}_n$ with metric

$$\overline{g}_{ii} = c_i e^{-2x^i} \prod_{\alpha \neq i} (c_\alpha e^{-2x^\alpha} - c_i e^{-2x^i}), \quad \overline{g}_{ij} = 0 \ (i \neq j).$$

Remark 14.1 This mapping is not geodesic. It means that Riemannian spaces from theorem are one of non trivial examples of mappings of the type π_1 which today we know only minimal number [170].

14.4 On special almost geodesic mappings of type π_1 of spaces with affine connection

14.4.1 Almost geodesic mappings π_1^*

Let a diffeomorphism from A_n onto $\overline{\mathbb{A}}_n$ satisfy

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = a_{ij}\, \delta^h_k, \tag{14.86}$$

where a_{ij} is a symmetric tensor. Diffeomorphisms of this kind are a special case of almost geodesic mappings of type π_1.

We denote them by π_1^*.

Let us derive the integrability condition arising from (14.86). We differentiate (14.86) covariantly by x^m and then alternate with respect to the indices k and m. Next in the integrability condition of (14.86) we contract with respect to the indices h and m. After editing we have

$$(n-1)\, a_{ij,k} = P^\alpha_{ij} R_{\alpha k} - P^\beta_{\alpha(i} R^\alpha_{j)\beta k} - (n-1)\, P^\alpha_{ij} a_{\alpha k}, \tag{14.87}$$

where R^h_{ijk} is the Riemannian tensor in $\mathbb{A}_n$, $R_{ij} \equiv R^\alpha_{ij\alpha}$ is the Ricci tensor, $(i\, j)$ denotes the symmetrization of indices.

Evidently, equations (14.86) and (14.87) represent a system of differential equations of Cauchy type in the space $\mathbb{A}_n$ which is solvable with respect to unknown functions $P^h_{ij}(x)$ and $a_{ij}(x)$, which, naturally, satisfy the algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x). \tag{14.88}$$

We have

Theorem 14.8 *The space $\mathbb{A}_n$ with affine connection admits an almost geodesic mapping π_1^* onto $\bar{\mathbb{A}}_n$ if and only if there exists a solution P^h_{ij} and a_{ij} of system of Cauchy type (14.86) and (14.87) satisfying (14.88).*

Integrability conditions of this system have the form

$$-P^\alpha_{ij}R^h_{\alpha km} + P^h_{\alpha(i}R^\alpha_{j)km} =$$
$$\frac{1}{n-1}\left[(P^\alpha_{ij}R_{\alpha m} - P^\beta_{\alpha(i}R^\alpha_{j)m\beta})\delta^h_k - (P^\alpha_{ij}R_{\alpha k} - P^\beta_{\alpha(i}R^\alpha_{j)k\beta})\delta^h_m\right],$$

$$(n-1)\,a_{\alpha(i}R^\alpha_{j)km} = P^\alpha_{ij}R^h_{\alpha[k,m]} + P^\beta_{\alpha(i}R^\alpha_{j)mk,\beta} +$$
$$+R_{[mk]}a_{ij} + P^\beta_{\gamma[m}R^h_{|i|k]\beta}P^\gamma_{\alpha j} + P^\alpha_{ij}P^\beta_{\alpha\gamma}R^\gamma_{[km]\beta} - P^\alpha_{ij}P^\beta_{\gamma[k}R^\gamma_{|\alpha|m]\beta,}\,.$$

14.4.2 An invariant object of mappings π_1^*

If P^h_{ij} is the deformation tensor ([170]) then Riemannian tensors R^h_{ijk} and $\overline{R}^h_{ijk}$ of spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$ satisfy the following condition

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^\alpha_{i[k}P^h_{j]\alpha}. \tag{14.89}$$

Using formulas (14.86) and (14.89) we obtain

$$\overset{*}{\overline{W}}{}^h_{ijk} = \overset{*}{W}{}^h_{ijk}, \tag{14.90}$$

where

$$\overset{*}{W}{}^h_{ijk} \equiv R^h_{ijk} - \frac{1}{n-1}R_{i[j}\delta^h_{k]}, \quad \text{and} \quad \overset{*}{\overline{W}}{}^h_{ijk} \equiv \overline{R}^h_{ijk} - \frac{1}{n-1}\overline{R}_{i[j}\delta^h_{k]}. \tag{14.91}$$

Clearly, $\overset{*}{W}{}^h_{ijk}$ and $\overset{*}{\overline{W}}{}^h_{ijk}$ is a tensor of type $\binom{1}{3}$ in the space $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

Condition (14.90) shows that this tensor is invariant with respect to almost geodesic mappings π_1^*.

We contract condition (14.90) in indices h and i to obtain the equality

$$W_{ij} = \overline{W}_{ij}, \tag{14.92}$$

where

$$W_{ij} \equiv R_{[ij]} \quad \text{and} \quad \overline{W}_{ij} \equiv \overline{R}_{[ij]}\,. \tag{14.93}$$

Subtract (14.92) from (14.90) to write

$$W^h_{ijk} = \overline{W}^h_{ijk}, \tag{14.94}$$

where W^h_{ijk} and $\overline{W}^h_{ijk}$ are Weyl projective curvature tensors of spaces $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectivelly. We get

Theorem 14.9 *The Weyl projective curvature tensor W^h_{ijk} and tensors $\overset{*}{W}{}^h_{ijk}$ and W_{ij}, which are defined by (14.91) and (14.93), are invariant with respect to almost geodesic mappings π_1^*.*

14.4.3 Mappings π_1^* of affine and projective-euclidean spaces

From Theorem 14.9 it follows

Theorem 14.10 *If a projective-euclidean space or equiaffine space admits an almost geodesic mapping π_1^* onto $\bar{\mathbb{A}}_n$ then $\bar{\mathbb{A}}_n$ is also a projective-euclidean space or an equiaffine space.*

The proof of Theorem 14.10, evidently, follows from the condition $W^h_{ijk} = 0$ in the projective-euclidean space and from the condition $W_{ij} = 0$ in the equiaffine space.

It means that projective-euclidean spaces and equiaffine spaces make up closed classes with respect to mappings π_1^*.

Clearly, the Riemannian tensor is preserved by mappings π_1^* if and only if the tensor a_{ij} vanishes. In this case basic equations have the form

$$P^h_{ij,k} = -P^\alpha_{ij} P^h_{\alpha k}. \tag{14.95}$$

Equations (14.95) are completely integrable in the affine space. Evidently, these equations have a solution for any initial conditions $P^h_{ij}(x_o)$.

If the initial conditions are such that $P^h_{ij}(x_o) \not\equiv \delta^h_{(i}\psi_{j)}(x_o)$ then every solution generates a mapping π_1^* which is not a geodesic mapping of the affine space $\mathbb{A}_n$ onto the affine space $\bar{\mathbb{A}}_n$. Therefore we can write

Theorem 14.11 *Mappings π_1^* of an affine space $\mathbb{A}_n$ onto itself exist. All lines map into planar curves (not necessary lines).*

Moreover, integrability conditions (14.86) and (14.87) in affine space are always true. We obtain

Theorem 14.12 *Riemannian spaces $\mathbb{V}_n$ with non constant curvature admit non geodesic mappings π_1^* which are necessarilly mappings of type π_3 and preserve the quadratic complex of geodesics.*

Proof. Let a Riemannian space $\mathbb{V}_n$ with non constant curvature K admit a non geodesic mapping π_1^*. Integrability conditions (14.86) then have the form

$$K(P^h_{k(i}g_{j)l} - P^h_{l(i}g_{j)k}) + \delta^h_l B_{ijk} - \delta^h_k B_{ijl} = 0, \tag{14.96}$$

where $B_{ijk} \equiv a_{ij,k} + P^{\alpha}_{ij}(a_{\alpha k} + K\, g_{\alpha k})$, g_{ij} is the metric tensor of the space $\mathbb{V}_n$.

From the last formula it follows

$$P^h_{ij} = P^h g_{ij} \tag{14.97}$$

where P^h is a vector. Then the mapping is F-planar [119]. Clearly, on the basis of results in [296], such mappings are almost geodesic mappings of type π_3. It is proved in the paper [296] that mappings $\pi_1 \cap \pi_3$ preserve the quadratic complex of geodesics [338].

After substituting (14.97) in (14.86) we have

$$P^h_{,k} + P^h P_k = \alpha \delta^h_k,$$

where α is a function, P_k is a covector.

These conditions characterize concircular vector fields P^h, which always exist in spaces with constant curvature.

14.4.4 Examples of almost geodesic mappings π_1^*

We present an example of an almost geodesic mapping of type π_1^* of an affine space $\mathbb{A}_n$ onto an affine space $\bar{\mathbb{A}}_n$.

Let $x^1, x^2, \ldots, x^n$ and $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n$ be affine coordinate in $\mathbb{A}_n$ and $\bar{\mathbb{A}}_n$, respectively.

The mapping

$$\overline{x}^h = \frac{1}{2} C^h_{\alpha}(x^{\alpha} - C^{\alpha})^2 + x^h_o, \tag{14.98}$$

where C^h_i, C^h, x^h_o are some constants, $x^h \neq C^h$, and the determinant $\det\left|C^h_i\right| \not\equiv 0$, defines an almost geodesic mapping π_1^* of the space $\mathbb{A}_n$ onto $\bar{\mathbb{A}}_n$.

We can prove directly that the deformation tensor P^h_{ij} in the coordinate system $x^1, x^2, \ldots, x^n$ has the form

$$P^i_{ii} = \frac{1}{x^i - C^i}, \qquad i = \overline{1, n},$$

and the other components are equal to zero.

Evidently, the tensor P^h_{ij} corresponds to equations (14.95). This mapping is not of type π_2 or π_3.

Lines in the space $\mathbb{A}_n$ which are defined by equations $x^h = a^h + b^h\, t$ where t is the parameter, map into parabolas (or lines) of the space $\bar{\mathbb{A}}_n$, which are defined by equations

$$\overline{x}^h = D^h + E^h\, t + F^h\, t^2$$

where

$$D^h = \frac{1}{2} C^h_{\alpha}(a^{\alpha} - C^{\alpha})^2, \quad E^h = C^h_{\alpha}(a^{\alpha} - C^{\alpha}) b^{\alpha}, \quad F^h = \frac{1}{2} C^h_{\alpha}(b^{\alpha})^2$$

in this mapping.

The image is a line if vectors E^h and F^h are collinear.

Finally we remark that formula (13) generates a system of almost geodesic mappings of type π_1 of planar spaces if the coefficients C^h_i, C^h and x^h_o are continuous.

15 RIEMANN-FINSLER SPACES

15.1 Riemann-Finsler spaces

Introduction

The *Riemann-Finsler spaces* are a large family of the metric spaces. (The new name of the Finsler spaces comes from Shiing Shen Chern [335, 336]). This large family of metrics spaces include Euclid's, Riemann's, Berwald's, Landsberg's, etc. spaces.

This chapter aims to define some new special Riemann-Finsler spaces, discussing their properties. These spaces are called *Douglas*, *R-quadratic* and *W-quadratic* spaces. The names of the latter two types of spaces come from Zhongmin Shen [594, 791]. Bácsó and Matsumoto introduced these special spaces also [278].

The projective relation geodesic mappings of Riemann-Finsler spaces has two invariant tensors, the Weyl tensor and Douglas tensor, Riemann-Finsler spaces of scalar curvature is characterized by the vanishing of Weyl tensor [876]. Bácsó and Matsumoto [276] defined the Douglas spaces, where the Douglas tensor vanishes.

First of all, let us give the concept of Riemann-Finsler metric.

Definition 15.1 Let M be an n-dimensional differentiable manifold given with a tangent space T_xM in the point $x = (x^1, x^2, \ldots, x^n)$ of M. The function $L(x, y)$: TM $(= \bigcup T_xM) \to \mathbb{R}$ is *Riemann-Finsler* metric, if the following properties hold:

(1) Regularity: $L(x, y)$ is a function C^∞ on the manifold $TM\backslash O$ of nonzero tangent vectors.

(2) Positive homogenity: $L(x, \lambda y) = \lambda\, L(x, y)$ for all $\lambda > 0$.

(3) Strong covexity: the $n \times n$ matrix $g_{ij}(x, y) = \dfrac{\partial^2 L^2}{\partial y^i \partial y^j}(x, y)$ is positive definite at every $y \neq 0$.

Remark. In some situation, the Riemann-Finsler metric $L(x, y)$ satisfies the criterion of strong covexity. In general, this property is too restrictive.

The above mentioned definition can be found in the doctoral dissertation of Paul Finsler: *Über Kurven und Flächen in allgemeinen Räumen*, 1918, Göttingen. Essentially, some definition was given by Riemann in his famous habilitation dissertation *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, 1854.

Since this definition was considered to be too general in determining the tensor of curvature, Riemann choose a well-known special case

$$L^2(x,y) = g_{ij}(x)y^i y^j,$$

and he started "we will now stick to the case ellipsoids (quadratic forms), because if not, the computation would become very complicated".

Schiing Shen Chern denies Riemannian's statement. He wrote in his latest two papers where he pointed out:

In fact, in general case is just as simple and main pout went unnoticed by Riemann and his success [336].

I believe that a major part of differential geometry in the 21th century should be Riemann-Finsler geometry [335].

It is not difficult to construct a (i.e. non-Riemannian) Riemann-Finsler metric, G. Randers [763] studied the following metric in 1941:

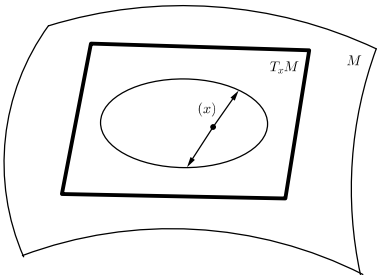
$$L(x,y) = \alpha(x,y) + \beta(x,y),$$

where $\alpha^2(x,y) = a_{ij}(x)y^i y^j$ is a Riemann's metric, $\beta(x,y) = b_i(x)y^i$ is a 1-form.

We can illustrate the metric in two dimensional case in the following way:

In the tangent space T_xM the indicatrix is an ellipse, whose focus in the origin. So we get an original Riemann-Finsler metric, where

$$L(x,y) \neq L(x,-y).$$



Another formulas:

The system of differential equations for *geodesic curves* of Finsler spaces $\mathbb{F}_n$ with respect to the *canonical parameter* t is given by

$$\frac{d^2x^i}{dt^2} = -G^i(x,y), \tag{15.1}$$

where

$$G^i(x,y) = 1/4\ g^{i\alpha}(y^\beta(\partial L^2_{(\alpha)}/\partial x^\beta - \partial L^2/\partial x^\alpha), \tag{15.2}$$

where $(\alpha) = \partial/\partial y^\alpha$. The *Roman* and the *Greek* indices run over the range $1, 2, \ldots, n$, the Roman indices are free but the Greek indices denote summation.

The *Berwald connection* coefficients $G^i_j(x,y)$, $G^i_{jk}(x,y)$ can be derived from the function G^i,

$$G^i_j = G^i_{(j)}, \quad G^i_{jk} = G^i_{j(k)}. \tag{15.3}$$

The *Berwald covariant derivative* can be written as

$$T^i_{j\|k} = \partial_k T^i_j - T^i_{j(\alpha)}G^\alpha_k + T^\alpha_j G^i_{\alpha k} - T^i_\alpha G^\alpha_{jk}, \tag{15.4}$$

where $\partial_k = \partial/\partial x^k$.

Berwald connection is not metrical, than is

$$g_{ij\|k} = -2P_{ijk}, \tag{15.5}$$

$$\text{where} \quad P_{ijk} = C_{ijk\|\alpha}y^{\alpha}, \text{ and } C_{ijk} = 1/2\ g_{ij(k)}. \tag{15.6}$$

$P_{ijk}(x,y)$ is the so-called *Landsberg tensor*, $C_{ijk}(x,y)$ is the *Cartan tensor*.

Using Cartan connection-parameters

$$^{*}\Gamma^{i}_{jk}(x,y) = \Gamma^{i}_{jk}(x,y) + P^{i}_{jk}(x,y), \qquad P^{i}_{jk} = g^{i\alpha}P_{\alpha jk} \tag{15.7}$$

we can give the *Cartan covariant derivative* in following way:

$$T^{i}_{j|k} = \partial_k T^{i}_{j} - T^{i}_{j(\alpha)}G^{\alpha}_{k} + {}^{*}\Gamma^{i}_{\alpha k}T^{\alpha}_{j} - {}^{*}\Gamma^{\alpha}_{jk}T^{i}_{\alpha}. \tag{15.8}$$

From (15.6) we obtain, that

$$T^{i}_{j\|\alpha}y^{\alpha} = T^{i}_{j|\alpha}y^{\alpha} = T^{i}_{j|0}.$$

We need the following *Ricci identities*:

$$T^{i}_{j\|k\|l} - T^{i}_{j\|l\|k} = T^{\alpha}_{j}H^{i}_{\alpha kl} - T^{i}_{\alpha}H^{\alpha}_{jkl} - T^{i}_{j(\alpha)}H^{\alpha}_{kl} \tag{15.9}$$

$$T^{i}_{j\|k(l)} - T^{i}_{j(l)\|k} = T^{\alpha}_{j}G^{i}_{\alpha kl} - T^{i}_{\alpha}G^{\alpha}_{jkl}, \tag{15.10}$$

where

$$G^{i}_{jkl} = G^{i}_{jk(l)}, \tag{15.11}$$

which is the *Berwald tensor*, and

$$H^{i}_{jkl} = \partial_l G^{i}_{jk} - \partial_k G^{i}_{jl} - G^{i}_{\alpha}G^{i}_{jkl} + G^{\alpha}_{k}G^{i}_{\alpha jk} + G^{\alpha}_{jk}G^{i}_{\alpha l} - G^{\alpha}_{jl}G^{i}_{\alpha k} \tag{15.12}$$

in the *Berwald curvature tensor*. From (15.12) we have

$$H^{i}_{kl} = H^{i}_{\alpha kl}y^{\alpha} = 1/3\ (H^{i}_{l(k)} - H^{i}_{k(l)}), \tag{15.13}$$

where

$$H^{i}_{l} = H^{i}_{\alpha l}y^{\alpha}. \tag{15.14}$$

The following equations will be useful for us:

$$L_{\|i} = 0,\ L_{|i} = 0,\ L_{(i)} = l_i,\ y^i/L = l^i; \tag{15.15}$$

$$l^{i}_{\|j} = 0,\ l^{i}_{|j} = 0,\ l^{i}_{(j)} = \frac{1}{L}h^{i}_{j},\ h^{i}_{j} = \delta^{i}_{j} - l^{i}l_j; \tag{15.16}$$

$$l_{i\|j} = 0,\ l_{i|j} = 0,\ l_{i(j)} = \frac{1}{L}h_{ij},\ h^{i}_{j} = g_{ij} - l^{i}l_j; \tag{15.17}$$

$$y^{i}_{\|j} = 0,\ y^{i}_{|j} = 0,\ y_{i\|j} = 0,\ y_{i|j} = 0,\ y_i = g_{i\alpha}y^{\alpha}; \tag{15.18}$$

$$h^{i}_{j\|} = 0,\ h^{i}_{j|k} = 0,\ h_{ij\|k} = -2P_{ijk},\ h_{ij|k} = 0,\ y_i = g_{i\alpha}y^{\alpha}; \tag{15.19}$$

Finally, we give the *Weyl* and *Douglas tensors*. Let $\mathbb{F}_n(M_n, L)$ and $\overline{\mathbb{F}}_n(M_n, \overline{L})$ be two Finsler spaces on the common underlying manifold M_n.

Definition 15.2 [114] L and $\overline{L}$ are said to be *projectively related* if any geodesic of $\mathbb{F}_n$ is geodesic of $\overline{\mathbb{F}}_n$ as a point set and vice versa. The $\mathbb{F}_n$ is *projective* to $\overline{\mathbb{F}}_n$.

It is well known the following [548]:

Theorem 15.1 *Two fundamental functions L and $\overline{L}$ are projectively related if and only if there exists a scalar field $p(x,y)$ on $TM/\{0\}$ satisfying*

$$\overline{G}^i = G^i + p\, y^i. \tag{15.20}$$

The projective factor $p(x,y)$ is a projectively homogeneous function of degree 1 in y.

From (15.20) we obtain the following equations:

$$\overline{G}^i_j = G^i_j + p_j y^i + p\delta^i_j,\ p_j = p_{(j)}; \tag{15.21}$$

$$\overline{G}^i_{jk} = G^i_{jk} + p_j\delta^i_k + p_k\delta^i_j + p^i_{jk},\ p_{jk} = p_{j(k)}; \tag{15.22}$$

$$\overline{G}^i_{jkl} = G^i_{jkl} + p_{jk}\delta^i_l + p_{lj}\delta^i_k + p_{kl}\delta^i_j + p_{jkl}y^i,\ p_{jkl} = p_{jk(l)}; \tag{15.23}$$

$$\overline{H}^i_{jkl} = H^i_{jkl} + y^i A_{kl(j)} + \delta^i_j A_{kl} + \delta^i_k A_{l(j)} - \delta^i_l A_{k(j)}; \tag{15.24}$$

$$\overline{H}^i_{kl} = H^i_{kl} + y^i A_{kl} + \delta^i_k A_l - \delta^i_l A_k; \tag{15.25}$$

$$\overline{H}^i_l = H^i_l + y^i A_{\alpha l} y^\alpha + y^i A_l - \delta^i_l A_{\alpha l} y^\alpha; \tag{15.26}$$

$$A_i = p_{\|i} - p\, p_i;\ A_{ij} = p_{i\|j} - p_{j\|i}.$$

The projective mapping $L \to \overline{L}$ has two invariant tensors, that is Weyl and Douglas tensors [276, 602].

Weyl tensor:

$$W^i_j = H^i_j - H\delta^i_j - \frac{1}{n+1}(H^\alpha_{j(\alpha)} - H_{(j)})y^i;\quad H = \frac{1}{n-1}H^\alpha_\alpha; \tag{15.27}$$

$$W^i_{jk} = 1/3\ (W^i_{k(j)} - W^i_{j(k)}; \tag{15.28}$$

$$W^i_{jkl} = W^i_{jk(l)}. \tag{15.29}$$

Douglas tensor:

$$D^i_{jkl} = (G^i - \frac{1}{n+1}G^\alpha_\alpha)_{(j)(k)(l)}, \tag{15.30}$$

$$G = G^\alpha_\alpha = G_{\alpha\beta}g^{\alpha\beta},\ G^\alpha_{jk\alpha} = G_{jk}.$$

For Weyl tensor, we have

$$W^i_{\alpha kl}y^\alpha = W^i_{kl},\quad W^i_{\alpha l}y^\alpha = W^i_l. \tag{15.31}$$

15.1.1 Douglas spaces, Previous results

Szabó proved the following [876, 877]:

Theorem 15.2 *A Riemann-Finsler space $\mathbb{F}_n$ is of scalar curvature (that is $H^i_j = RL^2\delta^i_j$ and $R(x,y)$ scalar curvature functions) if and only if its Weyl tensor satisfies $W^i_{jkl} = 0$.*

After this result raised an interesting question: *Which properties are satisfied by Riemann-Finsler spaces with vanishing Douglas tensor?*

At first, we investigated a special case [161]:

Definition 15.3 Riemann-Finsler space is called an *affinely connected* (or *Berwald*) if the coefficients G^i_{jk} are functions of the position only, that is the Berwald tensor $G^i_{jkl} = G^i_{jk(l)}$ is zero.

Definition 15.4 A Riemann-Finsler space is called a *Landsberg space* if the condition $y_\alpha G^\alpha_{jkl} = -2P_{jkl} = 0$.

Theorem 15.3 ([273]) *A Landsberg space with vanishing Douglas tensor is a Berwald space if $n > 2$.*

Proof. The Douglas tensor is given by

$$D^h_{ijk} = G^h_{ijk} - (y^h G^h_{ij(k)} + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij})/(n+1). \tag{15.32}$$

If we assume that $D^h_{ijk} = 0$ and $P_{ijk} = 0$, then contracting (15.32) by $h^l_h = (\delta^l_h - l^l l_h)$ we get

$$G^l_{ijk} = \frac{1}{n+1}\,(h^l_i G_{jk} + h^l_j G_{ik} + h^l_k G_{ij}). \tag{15.33}$$

We used here the fact that in any Riemann-Finsler space $\mathbb{F}_n$ $(n > 2)$ condition $D^h_{ijk} = 0$ is equivalent to $h^h_\alpha D^\alpha_{ijk} = 0$ [779].

We consider the identities in Landsberg space

$$G_{ihjk} + G_{hijk} = 2C_{hik\|j}, \tag{15.34}$$

$$G_{ihjk} - G_{hijk} = 0, \tag{15.35}$$

where $G_{ihjk} = g_{h\alpha}G^\alpha_{ijk}$ and $2C_{hik} = g_{hi(k)}$.

Substitute from (15.33) to (15.35) we get

$$G_{ik} = \frac{1}{n-1}\,Gh_{ik},\ G = G_{\alpha\beta}g^{\alpha\beta}. \tag{15.36}$$

So (15.33) can be written in the form

$$G_{ihjk} = \frac{G^2}{n^2-1}\,(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij}). \tag{15.37}$$

If we consider the following tensor

$$S_{ijkh} = C_{ih\alpha}C^{\alpha}_{jk} - C_{ik\alpha}C^{\alpha}_{jh},\ C^{i}_{jk} = C_{\alpha jk}g^{\alpha i}, \tag{15.38}$$

then we get Landsberg space

$$S_{ijkh\|l} = 0. \tag{15.39}$$

From (15.34), (15.35) and (15.37), we obtain

$$G(h_{ih}C_{jkl} + h_{jk}C_{ihl} - h_{ik}C_{jhl} - h_{jh}C_{ikl}) = 0. \tag{15.40}$$

Transvecting (15.40) and after substition $h = l$ and $j = l$, we have

$$(n-2)GC_i = 0,\ C_i = C^{\alpha}_{i\alpha}, \tag{15.41}$$

i.e. if $g_{ij}(x, y)$ is positive definite, then using of Deicke theorem [367] we have, the Landsberg space is a Berwald or a Riemann. □

Corollary 15.1 *If $n > 2$ and the fundamental metric tensor is positive definite, then "in the set of Landsberg spaces only Berwald spaces have common geodetics with some Riemann spaces."*

In [275] the authors showed, that in the above mentioned proof using Deicke Theorem and positive definite of metric tensor g_{ij} do not need. That is, if we consider the following equations

$$G^{l}_{ijk} = \frac{1}{n+1}\ (h^{l}_{i}G_{jk} + h^{l}_{j}G_{ik} + h^{l}_{k}G_{ij}), \tag{15.42}$$

$$G_{ik} = \frac{1}{n-1}\ Gh_{ik}, \tag{15.43}$$

$$(n-2)G\,C_{i\|j} = 0, \tag{15.44}$$

from (15.44), if $G = 0$, then (15.43) and (15.42) we have $G^{l}_{ijk} = 0$, that is the Landsberg space in a Berwald space.

Using $G_{ij} = C_{i\|j}$ [13] and if $C_i = 0$, then from (15.43) and (15.42) we obtain (15.3) Theorem, Berwald prooved the (15.3) Theorem in the case $n = 2$ [305]. In [275] we can find a new proof.

15.1.2 Douglas spaces

Introduction. We consider a geodesic curve $C\colon\ x_i = x_i(t), t_0 \le t \le t_1$, of an n-dimensional Riemann-Finsler space $\mathbb{F}_n = (M_n, L(x, y))$ on a smooth manifold M_n, equipped with the fundamental function $L(x, y)$ is the extremal of the extremal length integral

$$s = \int_{t_0}^{t_1} L(x, \dot{x})\, dt,\quad \dot{x}^i = dx^i/dt,$$

given by the Euler equation

$$\frac{d}{dt}\ L_{(i)} - L_i = 0,$$

where $L_{(i)} = \partial L/\partial y^i$, and $L_i = \partial L/\partial x^i$.

Putting $F = L^2/2$, we get the fundamental tensor

$$g_{ij} = \partial^2 F/\partial y^i \partial y^j$$

and the well-known functions

$$2G_j = (\partial^2 F/y^j \partial x^\alpha) y^\alpha - \partial F/\partial x^j.$$

Then, $(g^{ij}) = (g^{ij})^{-1}$ and $G^i = g^{i\alpha} G_\alpha$, we get

$$Lg^{i\alpha}(\frac{d}{dt} L_{(\alpha)} - L_\alpha) = \ddot{x}^i + 2G^i(x, \dot{x}) - \frac{\ddot{s}}{\dot{s}} \dot{x}^i = 0.$$

Consequently, C is given by the system of differential equations

$$\ddot{x}^i \dot{x}^j - \ddot{x}^j \dot{x}^i + 2D^{ij}(x, \dot{x}) = 0, \tag{15.45}$$

where we put

$$D^{ij}(x, \dot{x}) = G^i(x, y) y^j - G^j(x, y) y^i. \tag{15.46}$$

We are in particular, concerned with a two-dimensional Finsler space F^2 with a local coordinate system $(x^1, x^2) = (x, y)$ and we put $(y^1, y^2) = (p, q)$. Let us take x as a parameter of curves and denote

$$y' = dy/dx$$

$$y'' = d^2 y/d^2 y = (p\dot{q} - q\dot{p}) p^3.$$

Since $D^{ij}(x, y; p, q) = p^3 D^{ij}(x, y; 1, p/q)$ provided $p > 0$. Consequently (15.45) can be written in the form

$$y'' = \{G^1(x, y; 1, y') y' - G^2(x, y; 1, y')\}.$$

We consider the *Berwald connection* $B\Gamma = (G^i_{jk}, G^i_j)$. Then we get $2G^i = G^i_{\alpha\beta} y^\alpha y^\beta$, and the equation above, can be written in the form

$$y'' = X_3 (y')^3 + X_2 (y')^2 + X_1 (y') + X_0, \tag{15.47}$$

where we put

$$X_3 = G^1_{22},\ X_2 = 2G^1_{12} - G^2_{22}, X_1 = G^1_{11} - 2G^1_{12},\ X_0 = -G^2_{11}. \tag{15.48}$$

Suppose that F_2 under consideration is a Berwald space [13, 113], that is function of position (x, y) alone. Then X's of (15.48) is function of (x, y) and, in consequence (15.47) shows that the right-hand side of the equation $y'' = f(x, y, y')$ of a geodesic is a polynomial in y' of degree at most three. If on discussion is restricted to Riemannian space of dimension two, then G^i_{jk} are Christoffel symbols, and hence $f(x, y, y')$ is, of course, a polynomial in y' of degree at most three of all geodesic of any two-dimensional Riemannian space.

The remarkable property of $y'' = f(x, y, y')$ as above given does not depend on the choice of coordinates (x, y) [380].

Corollary 15.2 *The right-hand side of the equation (15.47) is a polynomial in y' of degree at most three, if and only if $D^{12}(x,y;p,q)$ is a homogeneous polynomial in (p,q) of degree three.*

And now it follows the definition of Douglas space.

Definition 15.5 A Riemann-Finsler space is said to be of *Douglas type* or a *Douglas space*, if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

At first, we consider the two-dimensional case.

Theorem 15.4 *A two-dimensional Finsler space is a Douglas space if and only if, in a local coordinate system (x,y), the right-hand side $f(x,y,y')$ of the equation of geodesic $y'' = f(x,y,y')$ is a polynomial in y' of degree at most three.*

We treat a Finsler space F_n with Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$. $\mathbb{F}_n$ is by definition a Douglas space if and only if

$$(G^l y^m - G^m y^l)_{(h)(i)(j)(k)} = 0.$$

We have first for $D^{lm} = G^l y^m - G^m y^l$

$$D^{lm}_{(h)} = G^l_h y^m + G^l \delta^m_h - [l,m],$$

where $[l,m]$ denotes the interchange of indices (l,m) of the preceding term.

Next

$$D^{lm}_{(h)(i)} = G^l_{hi} y^m + G^l_h \delta^m_i + G^l_i \delta^m_h - [l,m],$$

$$D^{lm}_{(h)(i)(j)} = G^l_{hij} y^m + \{G^l_{hi}\delta^m_j + (h,i,j)\} - [l,m],$$

where (h,i,j) denotes the cyclic permutation of the indices (h,i,j) of the preceding terms in the parentheses and $G^l_{hij} = G^l_{hi(j)} = G^l_{(h)(i)(j)}$, are the components of the Berwald tensor. Further, introducing tensor $G^l_{hijk} = G^l_{hij(k)}$, we obtain

$$D^{lm}_{(h)(i)(j)(k)} (= D^{lm}_{hijk}) = G^l_{hijk} y^m + \{G^l_{hij}\delta^m_k + (h,i,j,k)\} - [l,m], \qquad (15.49)$$

where (h,i,j,k) in the symbol analogous to (h,i,j). D^{lm}_{hijk} are components of a tensor and $D^{lm}_{hijk} = 0$ is necessary and sufficient for $\mathbb{F}_n$ to be a Douglas space. By $G^l_{hij\alpha} y^\alpha = -G^l_{hij}$, (15.49) yields.

$$D^{l\alpha}_{hij\alpha} = (n+1) D^l_{hij}, \qquad (15.50)$$

where D^l_{hij} are components of the well-known Douglas tensor ([13]):

$$D^l_{hij} = G^l_{hij} - \frac{1}{n+1} G_{hij} y^l - \frac{1}{n+1}\{G^h_i \delta^l_j + (h,i,j)\}, \qquad (15.51)$$

where $G_{hi} = G^\alpha_{hi\alpha}$ is the hv-Ricci tensor of $B\Gamma$ and $G_{hij} = G_{hi(j)} = G^\alpha_{hi\alpha j}$. Therefore the Douglas tensor must vanish for $\mathbb{F}_n$.

Conversely, if the Douglas tensor of an $\mathbb{F}_n$ vanishes identically, then $\mathbb{F}_n$ is a Douglas space, because it is easy to show the following equality

$$D^{lm}_{hijk} = D^{l}_{hij(k)}y^m + \{D^{l}_{ijk}\delta^m_h + (h,i,j,k)\} - [l,m]\,. \tag{15.52}$$

Therefore we have the following

Theorem 15.5 *A Riemann-Finsler space is of Douglas type, if and only if the Douglas tensor vanishes identically.*

It is well-known, the Douglas tensor is a projective invariant. Hence we have

Corollary 15.3 *If a Riemann-Finsler space projective mapping onto a Douglas space, then it is also a Douglas space.*

Example 15.1 We consider a two-dimensional Riemann-Finsler space with the metric

$$L(x,y;p,q) = q\,\tan^{-1}\frac{p}{q} - p\log\sqrt{1+(p/q)^2} - x\,q.$$

The differential equation of the geodesic of F_2 is given by $y'' = (y')^2 + 1$, which shows that F_2 is a Douglas space [604, Example 4]. The finite equation of geodesic is

$$y = C_1 - \log|\cos(x + C_2)|,$$

where the C's are arbitrary constants. We have

$$2LG^1 = -p^2q + p(p^2+q^2)(\tan^{-1}\frac{p}{q} - x),$$

$$2LG^2 = -p^2q + p(p^2+q^2)\log\sqrt{1+(p/q)^2}.$$

Consequently, F_2 is certainly not a Berwald space, but we have $2(G^1q - G^2p) = p(p^2+q^2)$, which implies again that F_2 is a Douglas space.

If we consider Wagner spaces, then we can give a new important example. The notion of Wagner space was originally defined by V.V. Wagner in 1943 [936], and established strictly from the modern standpoint by M. Hashiguchi in 1975 [114, 461].

Let $s_i(x)$ be components of a covariant vector field on a manifold M_n. The Wagner connection $W\Gamma(s) = (F^i_{jk}, N^i_j, C^i_{jk})$ of a Riemann-Finsler space $\mathbb{F}_n = (M_n, L(x,y))$ in a definition a Riemann-Finsler connection which is uniquely determined by the following five axioms:

(1) h-metrical $g_{ij|k} = 0$,
(2) $h(h)$-torsion tensor $T^i_{jk} = F^i_{jk} - T^i_{kj}$ is given by $T^i_{jk} = \delta^i_j s_k - \delta^i_k s_j$,
(3) deflection tensor: $D^i_j = y^\alpha F^i_{\alpha j} - N^i_j = 0$,
(4) v-metrical $g_{ij|k} = 0$,
(5) $v(v)$ – torsion tensor: $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$.

This Riemann-Finsler space $\mathbb{F}_n$ is called *Wagner space*, if its $W\Gamma(s)$ is linear, that is, F^i_{jk} are functions of position (x^i) alone. Consequently the notion of Wagner space is regarded as a generalization of Berwald space.

Let $C\Gamma = ({}^*\Gamma^i_{jk}, G^i_j, C^i_{jk})$ be the Cartan connection of a Riemann-Finsler space with a Wagner connection $W\Gamma(s)$. The difference $D^i_{jk} = F^i_{jk} - {}^*F^i_{jk}$ of $W\Gamma(s)$ from $C\Gamma$ is given [272] by

$$F^i_{jk} - {}^*F^i_{jk} = V^i_{jk\alpha}s^\alpha,$$

where $s^i = g^{\alpha i}s_\alpha$, $y_i = g_{\alpha i}y^\alpha$,

$$\begin{aligned} V^i_{jkh} = {} & g_{jk}\delta^i_h - g_{jh}\delta^i_k - C^i_{jh}y_k - C^i_{kh}y_j + C^i_{jk}y_h + C_{jkh}y^i \\ & + L^2(S^i_{jka} + C^i_{j\alpha}C^\alpha_{kh}). \end{aligned} \tag{15.53}$$

Consequently, we have

$$V^i_{00h} = L^2h^i_h,\ F^i_{\alpha\beta}y^\alpha y^\beta - {}^*F^i_{\alpha\beta}y^\alpha y^\beta = L^2s^i - s_0y^i,$$

where $s_0 = s_\alpha y^\alpha$.

Thus we get

$$F^i_{\alpha\beta}y^\alpha y^\beta = 2G^i + L^2s^i - s_0y^i,$$

which implies

$$F^i_{\alpha\beta}y^\alpha y^\beta y^j - F^j_{\alpha\beta}y^\alpha y^\beta y^i = 2(G^iy^i - G^jy^i) + L^2(g^{i\alpha}y^j - g^{j\alpha}y^i)s_\alpha.$$

Therefore from the definition of Wagner space, we obtain

Theorem 15.6 *For a Wagner space with* $W\Gamma(s)$

$$2(G^iy^i - G^jy^i) + L^2(g^{i\alpha}y^j - g^{j\alpha}y^i)s_\alpha$$

are homogeneous polynomials in y^i *of degree three.*

From the definition of Douglas space and 15.6 Theorem, it follows

Corollary 15.4 *Let* $\mathbb{F}_n$ *be a Wagner space with a Wagner connection* $W\Gamma(s)$. $\mathbb{F}_n$ *is of Douglas type, if and only if*

$$W^{ij} = L^2(g^{i\alpha}y^j - g^{j\alpha}y^i)s_\alpha.$$

are homogeneous polynomials in y^i *of degree three.*

Using of the Wagner spaces and the results of Matsumoto [604], we can give further examples.

We consider a Kropina space $\mathbb{F}_n = (M_n, L = \alpha^2/\beta)$, $\alpha^2 = a_{ij}(x)y^iy^j$, $\beta = b_i(x)y^i$ [604]. If we give the function

$$W^{ij} = \frac{1}{b^2}(b^2\alpha^2a^{iv} - \alpha^iB^iB^v + 2\beta B^iy^v)s_vy^j - [i,j],$$

where $B^i = a^{i\alpha}b_\alpha$, $b^2 = B^2b_\alpha$, then we can see, that W^{ij} above are homogeneous polynomials in y^i of degree three. Therefore $\mathbb{F}_n$ is of Douglas type, provided that it is a Wagner space. In particular, a Kropina space $\mathbb{F}_2$ is a Wagner space, as shown by Matsumoto in [604]. Therefore we have the

Theorem 15.7 *Let $\mathbb{F}_n$ be a Kropina space.*

(1) If $\mathbb{F}_n (n > 2)$ is a Wagner space, then it is a Douglas space.

(2) $\mathbb{F}_2$ is a Douglas space.

At now, we consider an example for the two-dimensional space. Matsumoto showed in [604], that the family of solutions of a second order linear differential equation

$$y'' + P(x)y' + Q(x)y - R(x) = 0$$

coincides with the family of geodesics of the two-dimensional Riemann-Finsler space $\mathbb{F}_2$ with the metric

$$L(x, y; p, q) = 1/p \; e^{\int P dx}[(2R - Qy)yp^2 + q^2] + E_x p + E_y q,$$

where $E = E(x, y)$ is an arbitrary function.

Consequently this $\mathbb{F}_2$ is a Douglas space. It is observed that this metric is of Kropina type (of Theorem 15.7).

Now we consider the Randers spaces with metric $L(x, y) = \alpha(x, y) + \beta(x, y)$, which have played a central role in the theory of so-called (α, β)-metric, where $\alpha^2 = a_{ij}(x)y^i a y^j$ and $\beta = b_i(x)y^i$, see [13].

For the Riemann-Finsler space $\mathbb{F}_n(M_n, L(\alpha, \beta))$ with metric (α, β)-metric the Riemannian space $\mathbb{V}_n = (M_n, \alpha)$ is said to be associated with $\mathbb{F}_n$. In $\mathbb{V}_n$ we have the Levi-Civita connection $\Gamma^i_{jk}(x)$, in which we have the symbols as follows:

$$v_{ij} = 1/2 \; (b_{i,j} + b_{j,i}); \quad s_{ij} = 1/2 \; (b_{i,j} - b_{j,i}); \quad s^i_j = a^{i\alpha} s_{\alpha j}; \quad s_i = b_\alpha s^\alpha_i).$$

The symbol "," denotes the covariant derivation with respect to Levi-Civita connection.

Now we consider a Randers space $\mathbb{F}_n = (M_n, L = \alpha + \beta)$. On account of the simplicity of its metric, the function G^i of $\mathbb{F}_n$ are easily written [605] as

$$2G^i = r^i_{00} + 2(Ay^i + \alpha s^i_0),$$

where $A = (r_{00} - 2\alpha s_0)/2(\alpha + \beta)$. Then we get $2D^{ij} = (r^i_{00}y^i - r^j_{00}y^i) + 2\alpha(s^i_0 y^j - s^j_0 y^i)$. It is obvious that the terms in the first of the right-hand side are homogeneous polynomials in (y^i). Therefore $\mathbb{F}_n$ is a Douglas space, if and only if $s^i_0 y^j - s^j_0 y^i = 0$, transvection by $Y_j = a_{j\alpha}y^\alpha$ gives $s^i_0 = 0$, that is, $s_{ij} = 0$. Therefore

Theorem 15.8 *A Randers space is of Douglas type if and only if $\partial_j b_i - \partial_i b_j = 0$, that is, β is closed form. Then*

$$2G^i = r^i_{00} + \frac{r_{00}}{\alpha + \beta} y^i,$$

where $r_{ij} = b_{i;j}$

15.1.3 A generalization of Douglas spaces

Definition 15.6 [281] A Riemann-Finsler space is called a *generalized Douglas space* if $h^h_\alpha D^\alpha_{ijk\|\beta} y^\beta = 0$, where $h^h_\alpha = \delta^h_\alpha - l^h l_\alpha$, $l^h = y^h/L$, $l_\alpha = L_{(\alpha)}$.

Theorem 15.9 *The class of generalized Douglas spaces is closed under projective mappings.*

Proof. Let $\overline{\mathbb{F}}(M_n, \overline{L})$ be a generalized Douglas space, and suppose that the Riemann-Finsler space $\mathbb{F}(M_n, L)$ is projective mapped to $\overline{\mathbb{F}}(M_n, \overline{L})$. By the assumption,

$$\overline{h}^h_\alpha \ \overline{D}^\alpha_{ijk\|\gamma} y^\gamma = 0,$$

using of the rule calculation of covariant derivation on $\overline{\mathbb{F}}_n$ we have

$$\overline{h}^h_\alpha(\partial\overline{D}^\alpha_{ijk}/\partial x^\beta - \partial\overline{D}^\alpha_{ijk}/\partial y^\gamma G^\gamma_\beta + \overline{G}^\gamma_{\beta i}\overline{D}^\alpha_{\gamma jk} - \overline{G}^\gamma_{\beta i}\overline{D}^\alpha_{\gamma jk} - \overline{G}^\gamma_{\beta j}\overline{D}^\alpha_{i\gamma k} - \overline{G}^\gamma_{\beta k}\overline{D}^\alpha_{ij\gamma})y^\beta = 0$$

If we substitute (15.21) and (15.22) equations into above equation, we obtain, that

$$\overline{h}^h_\beta\{D^\alpha_{ijk\|\beta} y^\beta + p_\gamma D^\gamma_{ijk} y^\alpha\} = 0,$$

where $D = \overline{D}$ for any projective mapping.

Since $\overline{h}^h_\alpha y^\alpha = 0$ this reduces to

$$\overline{h}^h_\alpha D^\alpha_{ijk\|\beta} y^\beta = 0.$$

Contracting above equation with h^l_h we get $h^l_\alpha D^\alpha_{ijk\|\beta} y^\beta = 0$. □

Definition 15.7 [717] A Riemann-Finsler space $\mathbb{F}_n$ is called a generalized Douglas-Weyl spaces (GDW), if

$$D^i_{ijk\|\alpha} y^\alpha = T_{jkl} y^i,$$

where $T_{jkl}(x, y)$ is an arbitrary tensor.

Weyl space is a such Riemann-Finsler space where the Weyl tensor vanish.

Theorem 15.10 [717] *A Randers's space is GDW space if and only if*

$$\alpha^2 s_{ij\|\beta} y^\beta = s_{i\beta\|\gamma} y^\beta y^\gamma y_j - s_{j\beta\|\gamma} y^\beta y^\gamma y_i.$$

Douglas and Weyl spaces belongs to the GDW spaces [779].

15.2 Riemann-Finsler spaces with h-curvature (Berwald Curvature) tensor dependents on position alone

Introduction

In Riemann-Finsler space almost all tensor fields depend on E. Cartan's supporting element (x^i, y^i), that is, they are functions not only on the underlying manifold but on the tangent bundle. We have obtained the rigorous definition of such a Riemann-Finsler tensor field [13, Section 2.2.3], and it is well-known that it is a singular case for a Riemann-Finsler space to have some tensor fields dependent on position alone.

The main purpose of the present part is to consider Riemann-Finsler spaces, whose h-curvature (Berwald curvature) depends on position alone. We will show, that for a Douglas space, the components W^h_{ijk} of the projective Weyl tensor are function of position (x^i) alone.

15.2.1 Projective invariants

We are concerned with an n-dimensional Riemann-Finsler space $\mathbb{F}_n$ with the Berwald connection $B\Gamma = \{G^i_{jk}, G^i_j, 0\}$, and we define the δ-differentiation as

$$\delta_i = \partial_i - G^\alpha_i \dot\partial_i,\ \partial_i = \partial/\partial x^i,\ \dot\partial_i = \partial/\partial y^i = (i).$$

The surviving torsion and curvature tensors [114] are *h-curvature* (*Berwald curvature*) *tensor*:

$$H^h_{ijk} = \delta_k G^h_{ij} + G^\alpha_{ij} G^h_{\alpha k} - [j,k],$$

hv-curvature (*Berwald*) *tensor*:

$$G^h_{ijk} = G^h_{ij(k)},$$

(v)*h-torsion tensor*

$$R^h_{jk} = \delta_k G^h_j - [j,k],$$

where $[j,k]$ indicates the term(s) obtained from the precendings term(s) by interchanging the indices j,k.

A projective mapping $\mathbb{F}_n = (M_n, L) \to \overline{F}_n = (M_n, \overline{L})$ of the Riemann-Finsler metric gives rise two various projective invariants. First we have

$$Q^0\text{−invariants: } Q^h = G^h - \tfrac{1}{n+1} G y^h,$$

$$Q^1\text{−invariants: } Q^h_i = G^h_i - \tfrac{1}{n+1}(G^i y^h + G\delta^h_i),$$

$$Q^2\text{−invariants: } Q^h_{ij} = G^h_{ij} - \tfrac{1}{n+1}(G^i y^h + G_i \delta^h_j + G_j \delta^h_i),$$

where $Q^h_i = Q^h_{(i)}$, $Q^h_{ij} = Q^h_{i(j)}$, $G = G^\alpha_\alpha$, $G_i = G^\alpha_{\alpha i}$ and $G_{ij} = G^\alpha_{\alpha ij}$ is the hv-Ricci tensor $B\Gamma$.

The Q^2 invariants satisfy the following important identities:

$$\text{(a)}\ \ Q^h_{ij} = Q^h_{ji} \ \text{ and } \ \text{(b)}\ \ Q^\alpha_{\alpha j} = 0. \tag{15.54}$$

So we have the following

$$D^h_{ijk} = Q^h_{ij(k)}. \tag{15.55}$$

This can be written in terms of $B\Gamma$ as

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1} G_{ij(k)} y^h - \frac{1}{n+1}\{G_{ij}\delta^h_k + (i,j,k)\},$$

where (i,j,k) indicates the terms, obtained from the preceding term(s) by cyclic permutation of indices i,j,k.

We have another invariant tensors [13], the Weyl tensor

$$W^h_{ijk} = H^h_{ijk} + \frac{1}{n+1}\{\delta^h_i H_{jk} + y^h H_{jk(i)} + \delta^h_j H_{k(i)} - [j,k]\},$$

where $H_{jk} = H^\alpha_{jk\alpha}$ is the h-Ricci tensor of $B\Gamma$ and H_k is the H-vector $H_k = \frac{1}{n-1}(nH_{ok} + H_{ko})$, with the subscript "o" denoting the transvecting by y^i. Let us remark that the Weyl tensor vanishes identically in any two-dimensional Riemann-Finsler space.

Finally, the H-vector gives the K-tensor

$$K_{ij} = (n-1)\{H_{i\|j} - H_{j\|i}\}.$$

We shall turn attention to the projective connection $P\Gamma = (P^i_{jk}, G^i_j, 0)$, suggested by the Q^2-invariants:

$$P^i_{jk} = G^i_{jk} - \frac{1}{n+1} G_{jk} y^i.$$

The surviving torsion and curvature tensors are:

h-curvature $N^h_{ijk} = \delta_k P^h_{ij} + P^\alpha_{ij} P^h_{\alpha k} - [j,k]$,

hv-curvature $U^h_{ijk} = P^h_{ij(k)}$,

$(v)h$-torsion $N^h_{ij} = H^h_{\alpha ij} y^\alpha$,

$(v)hv$-torsion $U^h_{jk} = U^h_{\alpha jk} y^\alpha$.

We have the following relations among those tensors of $B\Gamma$ and $P\Gamma$:

$N^h_{ijk} = H^h_{ijk} - \frac{1}{n+1} y^h (G_{ij\|k} - G_{ik\|j})$,

$U^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}(G_{ij(k)} y^h + G_{ij}\delta^h_k)$,

$N^h_{jk} = R^h_{jk},\ U^h_{jk} = \frac{1}{n+1} y^h G_{jk}$.

The h-Ricci tensor $N_{ij} = N^{\alpha}_{ij\alpha}$ and the hv-Ricci tensor $U_{ij} = U^{\alpha}_{ij\alpha}$ of $P\Gamma$ are writen as

$$N_{ij} = H_{ij} - \tfrac{1}{n+1} G_{ij\|\alpha} y^{\alpha},$$

$$U_{ij} = \tfrac{2}{n+1} G_{ij},$$

The Douglas tensor is written of $P\Gamma$ as

$$D^{h}_{ijk} = U^{h}_{ijk} - \tfrac{1}{2}(\delta^{h}_{i} U_{jk} + \delta^{h}_{j} U_{ik}).$$

The Weyl tensor is also written in the form

$$W^{h}_{ijk} = N^{h}_{hijk} + \{\delta^{h}_{i} M_{jk} + \delta^{h}_{j} M_{ik} - [j,k]\}, \tag{15.56}$$

where we defined the M^1-tensor $M_{jk} = \frac{1}{n^2-1}(nN_{jk} + N_{kj})$.

15.2.2 Q^3-invariants

Starting from the Q^2 we shall introduce the following quantities in way similar to constructing the h-curvature tensor:

Q^3 invariants: $Q^{h}_{ijk} = \delta_k Q^{h}_{ij} + Q^{\alpha}_{ij} Q^{h}_{\alpha k} - [j,k]$.

If we calculate the following

$$\delta_k Q^{h}_{ij} - \delta_j Q^{h}_{ik} =$$

$$\partial_k Q^{h}_{ij} - Q^{h}_{ij(\alpha)} G^{\alpha}_{k} - [j,k] =$$

$$= \partial_k Q^{h}_{ij} - D^{h}_{ij\alpha}\{Q^{\alpha}_{k} + \frac{1}{n+1}\{G_k y^{\alpha} + G\delta^{\alpha}_{k}\}\} - [j,k] =$$

$$= \partial_k Q^{h}_{ij} - Q^{h}_{ij(\alpha)} Q^{\alpha}_{k} - \frac{1}{n+1} G D^{h}_{ijk} - [j,k],$$

we obtain

$$Q^{h}_{ijk} = \partial_k Q^{h}_{ij} - Q^{h}_{ij(\alpha)} Q^{\alpha}_{k} + Q^{\alpha}_{ij} Q^{h}_{\alpha k} - [j,k].$$

Thus the Q^3-invariants are in fact projective invariants.

Using of projective connection, we have

$$Q^{h}_{ij} = P^{h}_{ijk} - \frac{1}{n+1}(G_i \delta^{h}_{j} + G_j \delta^{h}_{i}). \tag{15.57}$$

Hence the Q^3-invariants can be written in terms of $P\Gamma$ as

$$Q^{h}_{ijk} = N^{h}_{ijk} - \frac{1}{n+1}\{G_i \delta^{h}_{i} F_{jk} + \delta^{h}_{j} F_{ik} - [j,k]\}, \tag{15.58}$$

where we put

$$F_{ij} = \delta_j G_i - G_\alpha P^{\alpha}_{ij} + \frac{1}{n+1} G_i G_j. \tag{15.59}$$

The identities $Q^h_{ij} = Q^h_{ji}$, and $Q^\alpha_{\alpha j} = 0$ yield

$$(a)\ \ Q^h_{ijk} + (i,j,k) = 0,\ \ (b)\ \ Q^\alpha_{\alpha jk} = 0. \tag{15.60}$$

hence, we get the symmetric invariants

$$Q_{ij} = Q^\alpha_{ij\alpha}. \tag{15.61}$$

From (15.58) it follows that

$$Q_{ij} = N_{ij} + \frac{1}{n+1}(nF_{ij} - F_{ij}), \tag{15.62}$$

so

$$F_{ij} = \{(n+1)Q_{ij} + \frac{1}{n-1}\{\delta^h_j Q_{ik} - \delta^h_k Q_{ij}\}\}$$

is a tensor field with the components

$$\Pi^h_{ijk} = N^h_{ijk} + \frac{1}{n+1}\delta^h_i\{N_{jk} - N_{kj}\} + \frac{1}{n^2-1}\{\delta^h_j(nN_{ik} + N_{ki}) - \delta^h_k(nN_{ij} + N_{ji})\}. \tag{15.63}$$

Since M^1-tensor gives $N_{ij} = nM_{ij} - M_{ji}$, it is easy to show the right-hand side of (15.63) can be written in the form of the right-hand side of (15.59). Therefore, we have

Theorem 15.11 *The tensor W coincides with the Π-tensor the components of which are written in terms of Q-invariants as (15.63).*

15.2.3 Behaviour of the Weyl tensor in Douglas spaces

Theorem 15.12 *$\mathbb{F}_n$ is a Douglas space, if and only if*

(1) The Douglas tensor D vanishes identically or, as it was shown by (15.55).

(2) The q^2-invariants Q^h_{ij} are function of position (x^i) alone.

On the basis if Q-invariants

$$2Q^h = Q^h_{\alpha\beta}(x)y^\alpha y^\beta$$

so

$$2D^{ij} = \{Q^i_{\alpha\beta}(x)y^\alpha y^\beta\}y^j - \{Q^j_{\alpha\beta}(x)y^\alpha y^\beta\}y^i.$$

Hence, the equation of geodesic (15.45) is written in the form

$$\ddot{x}^i\dot{x}^j - \ddot{x}^j\dot{x}^i + \{Q^i_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta\dot{x}^j - [i,j]\} = 0. \tag{15.64}$$

We shall, in particular, deal with the two dimensional case. Denote (x^1, x^2) by (x, y) and put $y' = dy/dx$. Then (15.64) is written in the form

$$y'' = Q^1_{22}(y')^3 + (Q^2_{22} - 2Q^1_{22}Q^1_{12})(y')^2 + (2Q^2_{21} - Q^1_{11})y' + Q^2_{11}. \tag{15.65}$$

paying attention to (15.54), the above can be rewritten as

$$y'' = f(x, y, y') = Y_3(y')^3 + Y_2(y')^2 + Y_1 y' + Y_0, \tag{15.66}$$

where functions Y are given by

$$Y_3 = Q^1_{22}(x, y),\ Y_2 = 3Q^1_{12}(x, y) = -3Q^2_{22}(x, y),$$

$$Y_1 = 3Q^1_{11}(x, y) = -3Q^2_{12}(x, y),\ Y_0 = -Q^2_{11}(x, y).$$

Since the Q^2-invariants $Q^h_{ij}(x)$ will pay various essential rule in the theory of Douglas spaces, we state the following:

"The set $\{Q^h_{ij}(x)\}$ is called the characteristic of a Douglas space."

For a Douglas space, the Q^3-invariants are functions of position alone:

Theorem 15.13 *For a Douglas space the components of the Weyl tensor are function of position alone.*

15.2.4 On the rectifiability condition of a second ordinary differential equation

In a famous book of Arnold [15], we can find the folloowing theorem:

An equation $d^2y/dx^2 = \varphi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if and only if the right-hand side is polynomial in the derivative of order not greater than 3 both for the equation and for its dual.

This theorem can be formulated in the following form on the basis of [277]:

An equation $d^2y/dx^2 = \varphi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y} = d\overline{x}^2 = 0$ if and only if the path space P_2 (determined by the equation $d^2y/dx^2 = \varphi(x, y, dy/dx)$) is projectively related to a two-dimensional projectively flat Riemann-Finsler space $\mathbb{F}_2$.

Definition 15.8 [13] A Riemann-Finsler space is called *projectively flat*, if it has covering by coordinate neighbourhoods in which it is projective to a locally Minkowski space.

Definition 15.9 [13] A path space P_n and Riemann-Finsler space $\mathbb{F}_n$ are called *projectively related* to each other, if any path of P_n is a geodesic curve of $\mathbb{F}_n$ and vice versa.

Theorem 15.14 [604] *In two dimensions any path space is projectively related to a two dimensional Riemann-Finsler space.*

Theorem 15.15 [13] *If a path space P_n is projectively mapped onto a Finsler space $\mathbb{F}_n$, then we have two invariant tensors called the Weyl and the Douglas tensor respectively.*

Theorem 15.16 [276] *A two-dimensional Riemann-Finsler space $\mathbb{F}_2$ is a Douglas space if and only if (in a local coordinate system (x, y)) the right-hand side $\varphi(x, y, dx/dy)$ of the equation of geodesics $y'' = \varphi(x, y, dy/dx)$ is a polynomial in $dy/dx = y'$ of degree at most three.*

From the previous Theorem and Definitions, we obtain

Theorem 15.17 *The equation $d^2y/dx^y = \varphi(x, y, y')$ that can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = \varphi(x, y, y')$ is projective mapping to a two-dimensional Douglas space.*

Theorem 15.18 [277] *A two-dimensional Douglas space is projectively flat if and only if $\Pi_{ijk} = 0$, where*

$$\Pi_{ijk} = \partial Q_{ij}/\partial x^k + Q^{\alpha}_{ij}Q_{\alpha k} - [i, j].$$

Theorem 15.19 [277] *A two-dimensional Riemann-Finsler space $\mathbb{F}_2$ is Douglas space if and only if the differential quation of geodesic of $\mathbb{F}_2$ has the form*

$$y'' = k(x, y)(y')^3 + h(x, y)(y')^2 + g(x, y)(y') + f(x, y).$$

Theorem 15.20 [277] *A two-dimensional Douglas space is projectively flat if and only if*

$$\Pi_{112} = 0 \text{ and } \Pi_{212} = 0.$$

Theorem 15.21 *In a Douglas space*

$$\begin{aligned}\Pi_{112} = & -f_{yy} + \tfrac{2}{3}g_{xy} - \tfrac{1}{3}gg_y + fh_y + hf_y - \tfrac{1}{3}h_{xx} + \tfrac{1}{3}h_x g + \\ & kf_x - \tfrac{2}{27}g^2h + \tfrac{2}{3}gkh - \tfrac{2}{3}g_{xy}fhk,\end{aligned}$$

$$\begin{aligned}\Pi_{212} = & -\tfrac{1}{3}g_{yy} + \tfrac{2}{3}h_{xy} - \tfrac{1}{3}g_y h + k_y f + 2kf_y - k_{xx} + \tfrac{2}{3}hh_x + \\ & \tfrac{2}{3}h_x k - \tfrac{4}{27}gh^2 + \tfrac{2}{3}hk_x - -\tfrac{1}{3}gk_x - \tfrac{1}{3}g_x k - \tfrac{2}{9}hgk + \tfrac{2}{9}g^2k,\end{aligned}$$

where $f_x = \partial f/\partial x,\ \partial f/\partial y,\ \ldots.$

Assume that two-dimensional Riemann-Finsler space $\mathbb{F}_2$ on a domain of the (x, y)-plane has geodesics given by the equations:

1) $y'' = f(x, y)$

2) $y'' = x(x, y)y' + f(x, y)$

3) $y'' = k(x, y)(y')^3$

4) $y'' = h(x, y)(y')^2$

5) $y'' = g(x, y)y'$

The components of the tensor Π are the following in the respective cases:

1) $\Pi_{112} = -f_{yy};\ \Pi_{212} = 0,$

2) $\Pi_{112} = -f_{yy} + \frac{2}{3}g_{xy} - \frac{1}{3}h_x g g_y;\ \Pi_{212} = -\frac{1}{3}g_{yy}.$

3) $\Pi_{112} = -f_{yy};\ \Pi_{212} = -k_{xx},$

4) $\Pi_{112} = -\frac{1}{3}h_{xx};\ \Pi_{212} = h_{xy} + hh_x,$

5) $\Pi_{112} = \frac{2}{3}g_{xy} - \frac{1}{3}h_x g g_y;\ \Pi_{212} = -\frac{1}{3}g_{yy}.$

Consequently, $\mathbb{F}^2$ is projectively flat if and only if

1) $f(x,y) = A(x)y + B(x),$

2) $f(x,y) = \sigma_1(x)y^3 + \sigma_2(x)y^2 + \sigma_3(x)y + \sigma_4(x);\ g(x,y) = \alpha(x)y + \beta(x),$

3) $k(x,y) = C(y)x + D(y),$

4) $h(x,y) = E(y)x + F(y)$, where $dE/dy + (Ex + F)E = 0,$

5) $g(x,y) = \gamma(x)y + \delta(x)$, where $\frac{2}{3}dy/dx - \frac{1}{3}(\gamma y + \delta)\gamma = 0.$

15.3 Riemann-Finsler spaces with h-curvature (Berwald curvature) tensor dependent on position alone

Introduction

In a Riemann-Finsler space the components of a tensor field are usually functions of position (x^i) and direction (y^i). The main purpose of the present part is to consider Riemann-Finsler space having h-curvature tensor whose components are functions of positions alone.

Let $\mathbb{F}_n = (M_n, L(x,y))$ be an n-dimensional Riemann-Finsler space on a smooth n-manifold M_n, equipped with the fundamental metric function $L(x,y)$.

h-cuvature tensor of the Berwald connection $B\Gamma(G^i_{jk}, G^i_j, 0)$ has the following form

$$H^h_{ijk} = \delta_k G^h_{ij} + G^\alpha_{ij} G^h_{\alpha k} - [j,k], \tag{15.67}$$

where $\delta_k = \partial_k - G^\alpha_k \dot{\partial}_\alpha$ and the symbol $[j,k]$ denotes the interchange of the indices j,k.

The stretch curvature tensor $\Sigma = (\Sigma_{hijk})$, reflecting the non-metrical property of $B\Gamma$ is written in the form [114, 304]:

$$\Sigma_{hijk} = -y_\alpha H^\alpha_{hjk(i)} = 2(P_{hij\|k} - P_{hik\|j}),$$

where $(i) = \dot{\partial}/\partial y^i$ and $P_{hij} = g_{h\alpha} P^\alpha_{ij}$.

The geometrical meaning of stretch tensor we are concerned with infinitesimal circuit of M which consist of four points

$$P(x),\ Q(x+d_1x),\ R(x+d_1x+d_2(x+d_1x)),\ S(x+d_1x),$$

and vector field $\nu(\nu^i)$ which is given along the circuit and transformed parallel supporting with respect to a parallel supporting element y. Thus the "stretch and shrink" of the lenght of parallel vector field along circuit is given by the tensor Σ [114].

And now, we introduce the following classes of n-dimensional Riemann-Finsler space of a special kind as follows:

$H_x(n)$ – spaces with the H dependent on position alone
$S(n)$ – spaces with vanishing stretch curvature
$D(n)$ – Douglas spaces
$W_x(n)$ – spaces with the W dependent on position alone.

These yield directly

Theorem 15.22 *$H_x(n) \subset W_x(n)$ and $D_n \subset W_x(n)$.*

We consider the following Bianchi identity

$$H^l_{ihk(j)} = G^l_{ijh\|k} - G^l_{ijk\|h}\ .$$

From this, we obtain

Theorem 15.23 *A Riemann-Finsler space belongs to $H_x(n)$ if and only if*

$$G^l_{ijk\|h} - G^l_{ijh\|k} = 0. \tag{15.68}$$

Now, we have an interesting problem, namely to consider the intersection $H_x(n) \cap D(n)$. $D = 0$ gives

$$(n+1)G^h_{ijk} = G_{ijk}y^h + \{G_{ij}\delta^h_k + (i,j,k)\}.$$

Transvecting by the angular metric tensor $h^i_h = \delta^i_h - l^i l_h$ leads to

$$G^l_{ijk} = G^o_{ijk}y^l/L^2 + \{G_{ij}h^l_k + (i,j,k)\}/(n+1).$$

Consequently, we obtain

$$\begin{aligned} G^l_{ijk\|h} - G^l_{ijh\|k} = \{G^o_{ijh\|k} - (k/h)\}y^i/L^2 + [\{G_{ij\|h}h^l_k - (k/h)\}+ \\ \{G_{jk\|h}h^l_i + G_{ik\|h}h^l_j - (k/h)\}]/(n+1). \end{aligned} \tag{15.69}$$

From (15.69) it follows that

$$G_{jk\|h} - G_{jh\|k} = 0, \quad G^o_{ijk\|h} - G^o_{ijh\|k} = 0.$$

From (15.69) we get

$$G_{ij\|h}h^l_k - G_{jh\|k}h^l_h = 0.$$

Consequently, we obtain

$$(n-2)G_{ij\|h} - G_{ij\|o}y_h/L^2 = 0, \quad \text{and} \quad G_{ij\|o} = 0,$$

So

$$G_{ij\|k} = 0 \quad (n > 2).$$

Consequently, if $\mathbb{F}_n \in D(n),\ n > 2$, it satisfies $G_{ij\|k} = 0$ and

$$G^o_{ijk\|h} - G^o_{ijh\|k} = 0. \tag{15.70}$$

Then (15.69) lead to $G^l_{ijk\|h} - G^l_{ijh\|k} = 0$, hence Bianchi identity shows the $\mathbb{F}_n \in H_x(n)$. the condition (15.70) is nothing but $\mathbb{F}_n \in S_n$ because of

$$\Sigma_{hijk} = -y_\alpha H^\alpha_{hjk(i)} = 2(P_{hij\|k} - P_{hik\|j}).$$

Therefore

Theorem 15.24 *A Douglas space $\mathbb{F}_n$ $(n > 2)$, belongs to $H_x(n)$, if and only if the tensor G_{ij} is a covariant constant in $B\Gamma$ and $\mathbb{F}_n \in S(n)$.*

Finally, we proved the following

Theorem 15.25 [282] *Let $\mathbb{F}_n$ be an R-quadratic Einstein-Finsler space with non-zero Ricci scalar. Then $\mathbb{F}_n$ must be Riemannian.*

Then the Weyl tensor has the following form

$$W^i_k = H^i_k - \frac{1}{n-1}Ric(x)\delta^i_k L^2 + \frac{1}{n+1}\{H^\alpha_{k(\alpha)} - \frac{2}{n-1}Ric(x)y_k\}y^i, \tag{15.71}$$

that is

$$\begin{aligned} W^i_{\alpha k\beta}y^\alpha y^\beta = {} & H^i_{\alpha k\beta}y^\alpha y^\beta - \\ & \frac{1}{n+1}\{(H^\gamma_{\alpha k\beta}y^\alpha y^\beta)_\gamma - \frac{2}{n-1}Ric(x)y_k\}y^i - \frac{1}{n-1}Ric(x)\delta^i_k L^2. \end{aligned} \tag{15.72}$$

By the assumption $\mathbb{F}_n$is R-quadratic, i.e. $H^i_{\alpha k\beta}$ depends only on position x, and it is well-known $H_x(n) \subset W_x(n)$. So we can get

$$\frac{\partial^3}{\partial y^a \partial y^b \partial y^c}(H^i_{\alpha k\beta}y^\alpha y^\beta) = 0 \quad \text{and} \quad \frac{\partial^3}{\partial y^a \partial y^b \partial y^c}(W^i_{\alpha k\beta}y^\alpha y^\beta) = 0.$$

By differentiation of (15.72) with respect y^a, y^b and y^c

$$2(n+1)Ric(x)\delta^i_k C_{abc} - Ric(x)\{2C_{kab(c)}y^i + 2C_{kab}\delta^i_c + 2C_{kac}\delta^i_b + 2C_{kbc}\delta^i_a\} = 0.$$

By multiplying g^{ik}, we will have $2(n-1)(n+2)Ric(x)C_{abc} = 0$. By assumption $\mathbb{F}_n$ is not Ricci flat, so it is Riemannian. □

15.4 Projective mappings between Riemann-Finsler spaces with (α, β)-metric

The purpose of the present section is to give a list of classification of projective mappings between Riemann-Finsler spaces with (α, β)-metric for a fixed (α, β). It is shown that we have a certain complicated situation in a two-dimensional case. The projective mapping between a Riemann-Finsler space with (α, β)-metric and its associated Riemannian space with the metric α has been studied by several authors [113, 247, 462, 602, 798].

15.4.1 Preliminaries

Definition 15.10 Let $\mathbb{V}_n = (M_n, \alpha)$ be an n-dimensional Riemannian space with a quasi-Riemannian metric α, $\alpha^2 = a_{ij}(x)dx^i dx^j$ and $\beta = b_i(x)dx^i$ a differential one-form on M_n. The (α, β)-metric is a Riemann-Finsler metric which is constructed from Riemannian metric α, and a differential one-form β.

Through the paper our consideration is restricted to a domain of M_n where the covariant vector field $b_i(x)$ does not vanish. Further we do not treat the direction y^i where $\beta(x, y) = b_i(x)y^i$ vanishes.

An (α, β)-metric is a Riemann-Finsler metric $L(\alpha, \beta)$ on M_n which is a positively homogeneous function of degree one in the argument (α, β). We are concerned with an n-dimensional Riemann-Finsler space $\mathbb{F}_n = (M_n, L(\alpha, \beta))$ equipped with an (α, β)-metric $L(\alpha, \beta)$. The fundamental tensor of $\mathbb{F}_n$ is, however, not necessary for discussions, and raising and lowering of indices are done by the Riemannian fundamental tensor a_{ij} and its inverse a^{ij}. For instance, we have

$$y_i = a_{i\alpha}y^\alpha,\ b^i = a^{i\alpha}b_\alpha,\ b^2 = b_\alpha b^\alpha.$$

The following quantities play a role in the theory of (α, β)-metrics:

$$\triangle = (\beta/\alpha)^2 - b^2,\ z_i = b_i - (\beta/\alpha^2)y_i,\ r_{ij} = \frac{1}{2}(\beta_{i;j} + b_{j;i}),\ s_{ij} = \frac{1}{2}(\beta_{i;j} - b_{j;i}),$$

where the semicolon stands for the covariant differentiation with respect to the Christoffel symbols γ^i_{jk} constructed from a_{ij}.

The index "o" denotes as usual the transvection by y^i. For instance, we have $\gamma^i_{oj} = y^\alpha \gamma^i_{\alpha j}$, and $F_{io} = F_{i\alpha}y^\alpha$. It is obvious that γ^i_{jk} together with the nonlinear connection γ^i_{oj} constitute Riemann-Finsler connection, denoted by $\Gamma(\alpha)$. Thus, with respect to $\Gamma(\alpha)$ we have $y^i_{;j} = 0,\ \alpha_{;j} = 0,\ \beta_{i;j} = b_{\alpha;j}y^\alpha$.

The well-known Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ of a Riemann-Finsler space is constructed from the quantities $G^i(x, y)$ appearing in the equation of geodesic

$$d^2x^i/dt^2 + 2G^i(x, dx/dt) = 0.$$

Let us introduce

$$s^i_j = a^{i\alpha}s_{\alpha j},\ s_i = b^\alpha s_{\alpha i},$$

and denote by subscripts α and β of L the partial differentiations by α and β respectively that is

$$L_\alpha = \partial L/\partial\alpha, \qquad L_\beta = \partial L/\partial\beta.$$

Then the difference

$$D^i = G^i - \gamma^i_{oo}/2 \tag{15.73}$$

is given by

$$D^i = D_1 y^i + D_2 s^i_o + D_3 z^i,$$

$$D_1 = H/\alpha, \quad D_2 = \alpha L_\beta/L_\alpha, \quad D_3 = (\alpha^2 L_{\alpha\alpha}/2\beta^2 L_\alpha)(\alpha r_{oo} + 2\beta G),$$

where H and G are determined by

$$2\beta^2(\beta L_\alpha G + \alpha^2 L_\beta s_o - \alpha^3 \triangle L_{oo}(\alpha r_{oo} + 2\beta G) = 0, \tag{15.74}$$

$$2LH = L_\beta(\alpha r_{oo} + 2\beta G). \tag{15.75}$$

It is easy to show that D_2, D_3 and G are in the mapping

$$\alpha(r_o D_2 + \triangle D_3) + \beta G = 0. \tag{15.76}$$

Example 15.2 *The most useful and well-known (α, β)-metric will be the Randers metric $L(\alpha, \beta) = \alpha + \beta$. For this metric, we have*

$$D_1 = (r_{oo} - 2\alpha s_o)/2(\alpha + \beta),\ D_2 = \alpha,\ D_3 = 0.$$

Example 15.3 *An (α, β)-metric $L = \alpha^2/\beta$ called the Kropina metric, is also well-know. This metric has D_1, D_2 and D_3 as follows:*

$$D_1 = -(\beta/2b^2\alpha^2)(r_{oo} + \alpha^2 s_o/\beta),\ D_2 = -\alpha^2/2\beta,\ D_3 = (\beta r_{oo} + \alpha^2 s_o)/2b^2\beta.$$

Definition 15.11 A pair is called parallel, if $b_{i;j} = 0$ identically.

Definition 15.12 If the pair (α, β) of a Finsler space $\mathbb{F}_n = (M_n, l(\alpha, \beta))$ with (α, β)-metric is parallel then its $B\Gamma$ coincides with $\Gamma(\alpha) = \{\gamma^i_{jk}, \gamma^i_{oj}\}$.

A) General case

Let $\mathbb{F}_n = (M_n, L)$ and $\overline{\mathbb{F}}_n = (M_n, \overline{L})$ be two Riemann-Finsler spaces on the same underlying manifold M_n. The change $\kappa\colon\ L \to \overline{L}$ of metric is called projective and $\mathbb{F}_n$ is projective to $\overline{\mathbb{F}}_n$ if any geodesic of $\mathbb{F}_n$ is projective to $\overline{\mathbb{F}}_n$ as a point set and vice versa.

It is well-known that p is projective if and only if there exists a positively homogeneous function $p(x, y)$ of degree one in y^i, called the projective factor, satisfying

$$\overline{G}^i(x, y) = G^i(x, y) + p(x, y)y^i.$$

Assume the mapping

$$\kappa\colon\ \mathbb{F}_n(M_n L(\alpha, \beta)) \to \overline{\mathbb{F}}_n(M_n, \overline{L}(\alpha, \beta))$$

of (α, β)-metric to be projective. Then (15.73) shows that a necessary and sufficient condition for κ is given by

$$(\overline{D}_1 - D_1)y^i + (\overline{D}_2 - D_2)s_0^i + (\overline{D}_3 - D_3)z^i = py^i. \tag{15.77}$$

Since we have $y_\alpha y^\alpha = \alpha^2$ and $y_\alpha s_o^\alpha = y_\alpha z^\alpha = 0$, the transvection of (15.77) by y_i implies

$$p = \overline{D}_1 - D_1,$$

and (15.77) reduces to

$$(\overline{D}_2 - D_2)s_o^i - (\overline{D}_3 - D_3)z^i = 0. \tag{15.78}$$

Thus the projective factor p is given as the difference $\overline{D}_1 - D_1$.

Next since we have $b_\alpha s_o^\alpha = s_o$ and $b_\alpha z^\alpha = -\triangle$, the transvection of (15.78) by y_i gives

$$(\overline{D}_2 - D_2)s_o - (\overline{D}_3 - D_3)\triangle = 0. \tag{15.79}$$

It is remarked here that $\triangle$ does not vanish. In fact, $\triangle = 0$ implies $b^2\alpha^2 = s^2$. Then $b^2 \neq 0$ gives $a_{ij} = b_i b_j / b^2$ and, on the other hand, $b^2 = 0$ gives $\beta = 0$, so that we have a contradiction in any case.

Consequently, we get $\overline{D}_3 - D_3 = \overline{D}_2 - D_2 s_o \triangle$ from (15.78) and (15.79) is rewritten in the form

$$\overline{D}_2 - D_2(s_o^i - s_o z^i/\triangle) = 0. \tag{15.80}$$

Therefore we obtain (15.79) and (15.80) as a necessary and sufficient condition for the projective mapping. It is obvious that (15.79) is equivalent to $\overline{G} = G$ from (15.76). Thus we get two cases:

$$\overline{D}_2 = D_2, \quad \overline{D}_3 = D_3, \tag{I}$$

$$s_o^i + s_o z^i/\triangle = 0, \quad \overline{G} = G. \tag{II}$$

Here, we shall show two Theorems for later use.

Theorem 15.26 [462] *Assume that β satisfies $s_{ij} = 0$, that it is a locally gradient vector field, but the pair (α, β) is not parallel $(b_{i;j} \neq 0)$. Then the mapping $\kappa\colon\ \mathbb{F}_n = (M_n, L(\alpha, \beta)) \to \overline{\mathbb{F}}_n = (M_n, \overline{L}(\alpha, \beta))$ is projective if and only if it is a Randers mapping:*

$$\overline{L} = c_1 L + c_2 \beta \tag{15.81}$$

where $c_1 (\neq 0)$ and c_2 are constants.

Proof. For $s_{ij} = 0$ our condition (15.78) reads to $\overline{D}_3 = D_3$ only. Then (15.76) implies $\overline{G} = G$. Since (15.74) is now $2\beta^3 G = \alpha^2 \triangle (L_{\alpha\alpha}/L_\alpha)(\alpha r_{oo} + 2\beta G)$, $\overline{G} = G$ is written as

$$(L_{\alpha\alpha}/L_\alpha - \overline{L}_{\alpha\alpha}/\overline{L}_\alpha)(\alpha r_{oo} + 2\beta G) = 0. \tag{15.82}$$

If $\alpha r_{oo} + 2\beta G = 0$, then (15.74) implies $G = 0$, hence we have $r_{oo} = 0$. Then γ_{ij} and σ_{ij} vanish and the pair (α, β) is only parallel. Consequently, we have $L_{\alpha\alpha}/L_\alpha = \overline{L}_{\alpha\alpha}/\overline{L}_\alpha$, which lead us to Randers mapping as easily verified by integration. □

Theorem 15.27 *If $\alpha^2 \equiv 0\ (mod\beta)$, that is the quadratic form $a_{ij}y^iy^j$ contains the linear from $\beta_i y^i$ as a factor, then the n must be equal to two and $b^2 = 0$.*

Proof. From the assumption, we have another differential one-form $\gamma = C_i(x)y^i$ satisfying $\alpha^2 = \beta\gamma$, that is,

$$a_{ij} = (b_ic_j + b_jc_i)/2. \tag{15.83}$$

Thus we have $rank|a_{ij}| < 3$, so that n must be equal to 2. Since $det(a_{ij}) = -(b_1c_2 - b_2c_1)^2/4$, the Riemannian metric α is not positive-definite and γ is not proportional to β. The transvection of (15.83) by b^j gives $(2 - c_jb^j)b_i = b^2c_i$, which implies $b^2 = 0,\ c_jb^j = 2$. □

Now we consider the (I) case above. $\overline{D}_2 = D_2$ implies $L_\beta/L_\alpha = \overline{L}_\alpha/\overline{L}$ and $L_\alpha/L = \overline{L}_\alpha/\overline{L}$. Thus $\overline{L}/L$ must be constant and the mapping κ is homothetic only.

Next, we consider the case (II) above. The first condition is written in the form of a homogeneous polynomial of degree in y^i as

$$\alpha^2(s_0b^i - b^2s^i_o) + \beta(s^i_o - s_oy^i) = 0. \tag{15.84}$$

If the dimension n is more than two, then Theorem 15.27 shows that $\alpha^2 \neq 0(mod\beta)$, hence (15.84) implies that we have function $f^i(x)$ satisfying

$$\begin{array}{ll} (1) & s_ob^i - b^2s^i_o = \beta f^i, \\ (2) & \alpha^2 f^i + \beta s^i_o - s_0y^i = 0. \end{array} \tag{15.85}$$

The form is written in the form $s_jb_i - b^2s_{ij} = b_if_i$, the transvection of which by b^j gives $b^2F_i = b^2f_i$. Then we have two cases as follows:

(A) $b^2 \neq 0$: We get $f_i = s_i$ and (1) of (15.85) implies $s_{ij} = (b_is_j - b_js_i)/b^2$. Then (2) of (15.85) is written in the form

$$2s_i(a_{jk} - b_jb_k/b^2) = s_j(a_{ik} - b_ib_k/b^2) + s_k(a_{ij} - b_ib_j)/b^2.$$

Transvecting j this equation by a^{ik} we get $(n-2)s_i = 0$, so that $s_{ij} = 0$ and we arrive in the case treated in Theorems 15.26 and 15.27.

(B) $b^2 = 0$: (1) of Theorem 15.85 reduces to $s_0b^i = \beta f^i$, the transvection of which by y_i gives $s_0 = f_0$, that is, $s_i = f_i$ and we have $s_jb_i = b_js_i$.

Consequently, we have a function $f(x)$ such that $f_i = fb_i$, hence (2) of (15.85) is written in the form $(s_{i0} - fy_i)\beta + f\alpha^2b_i = 0$. If $n > 2$ and f does not vanish, then the above implies that β must be a factor of α^2, contrary with Theorem 15.27. Thus, we have $f = 0$ and the above implies $s_{ij} = 0$, the case treated in Theorems 15.26 and 15.27.

Summarizing up all the above, we obtain first

Theorem 15.28 *All the projective mappings*

$$k\colon \mathbb{F}_n(M_n, L(\alpha,\beta)) \to \overline{\mathbb{F}}_n(M_n, \overline{L}(\alpha,\beta)),$$

provided that $n > 2$ are divided into three classes as follows:

(1) k is a homothetic mapping: $\overline{L} = c \cdot L$, where $c(\neq 0)$ is a constant.

(2) $s_{ij} = 0$ and (α,β) is ot parallel: k is a Randers map? $\overline{L} = c_1 L + c_2\beta$, where $c_1(\neq 0)$ and c_2 are constant.

(3) (α,β) is a parallel pair: All the Riemann-Finsler space $\mathbb{F}_n$ with (α,β)-metric are projective to each other and to their associated Riemannian space $\mathbb{V}_n = (M_n,\alpha)$. Their $B\Gamma$ coincide with $\Gamma(\alpha)$.

15.4.2 The two-dimensional case

We are concerned with a projective mapping

$$k : F_2 = (M_2, L(\alpha,\beta)) \to \overline{F}_2 = (M_2, \overline{L}(\alpha,\beta)).$$

If is enough for our purpose to consider the case (II) above. Then we have (15.84) as a condition. We shall divide our discussions into three cases as follows:

(A) $b^2 = 0$: It follows from Theorem 15.27 that we have $\alpha^2 \neq 0\ (mod\beta)$.

Theorem 15.29 *Assume that $n = 2$ and $\beta^2 \neq 0$. Then we have $e = \pm 1$ and another differential one-form $\gamma = C_\alpha(x)y^\alpha$ satisfying*

$$a_{ij} = b_i b_j/b^2 + e c_i c_j. \tag{15.86}$$

We have $c_\alpha b^\alpha = 0$ and $c_\alpha c^\alpha = e$.

Proof. Since $h_{ij} = a_{ij} - b_i b_j/b^2$ satisfies $h_{i\alpha}b^\alpha$, we have rank $(h_{ij}) = 1$. Thus we may assume $h_{ij} \neq 0$, for instance, because $h_{11} = h_{22} = 0$ imply $h_{12} = 0$, a condition. Then, we can get e and $c_1(\neq 0)$ from $h_{11} = e(c_1)^2$ and c_2 from $h_{12} = ec_1c_2$. In consequence, we have $h_{22} = e(e_2)^2$, hence $h_{ij} = ec_ic_j$. Two relations $c_\alpha b^\alpha = 0$ and $c_\alpha c^\alpha = e$ will be easily verified.

Therefore, since we get two independent vector fields b_i and c_i, the skew-symmetric tensor field s_{ij} can be written as

$$s_{ij} = s(b_i c_j - b_j c_i), \tag{15.87}$$

where $s(x)$ is a scalar field. Thus, we get

$$y_i = (\beta/b^2) + e\gamma c_i, \qquad \alpha^2 = (\beta^2/b^2) + e\gamma^2,$$
$$s_0^i = s(\gamma b^i - \beta c^i), \qquad s_0 = sb^2\gamma$$

and our condition (15.85) is automatically satisfied. □

It seems to us that another condition $\overline{G} = G$ will be difficult to consider in detail for general $L(\alpha,\beta)$ and $\overline{L}(\alpha,\beta)$, see Corollary 15.6.

(B) $b^2 \neq 0$ and $\alpha^2 \neq 0(mod\beta)$: This case will be discussed in the same way with the case (II) of the previous section.

(C) $b^2 \neq 0$ and $\alpha^2 \equiv 0(mod\beta)$: On account of (15.82) we have also independent two vector fields b_i and c_i, hence s_{ij} is written in the form (15.87). Then we get

$$y_i = (\gamma b_i + \beta c_i)/2, \qquad \alpha^2 = \beta\gamma$$
$$s_0^i = s(\gamma b^i - \beta c^i), \qquad s_0 = -2s\beta,$$

and then the condition (15.85) is also satisfied.

Summarizing up the above, we have

Theorem 15.30 *All the projective mappings*

$$k\colon\ F_2 = (M_2, L(\alpha, \beta)) \to \overline{F}_2 = (M_2, \overline{L}(\alpha, \beta)).$$

are divided into three classes in Theorem 15.28 and additional three classes as follows:

(4) $b^2 \neq 0$*: a_{ij} and s_{ij} are written as (15.86) and (15.87) respectively and we have the condition $\overline{G} = G$.*

(5) $b^2 = 0$ *and* $\alpha^2 \neq 0(mod\beta)$*: We have cases* (1) *and* (2) *of Theorem 15.28.*

(6) $b^2 = 0$ *and* $\alpha^2 \equiv 0(mod\beta)$*: a_{ij} and s_{ij} are written as (15.83) and (15.27) and we have the condition $\overline{G} = G$.*

Finally we show two corollaries:

Corollary 15.5 [269] *Let a Randers space $\mathbb{F}_n = (M_n, L(\alpha+\beta))$ be projective to Kropina space $\overline{\mathbb{F}}_n = (M_n, \overline{L} = \alpha^2/\beta)$. The pair (α, β) is parallel, independently of the dimension n.*

Proof. Example 2 shows that $b^2 \neq 0$ must be assumed, hence the only possible cases are (3) of Theorem 15.28 and (4) of Theorem 15.30. In case (4) the remaining condition $\overline{G} = G$ reads

$$\alpha^2(2b^2\alpha + \beta)s_0 = (b^2\alpha^2 - \beta^2)r_{00}.$$

Since α is irrational in y^i, the above implies $s_0 = 0$ and r_{00} immediately, hence $s_i = 0$ and $r_{ij} = 0$. s_i implies $s_{ij} = 0$ by virtue of (15.85). Consequently, we have $b_{i;j} = 0$. □

Corollary 15.6 [271] *The Shen's problem* [796]: *"Is there any Douglas metric which is not locally projective to a Berwald metric?"*

We give a Douglas space and Berwald space, which have not common geodesics, that is we give and example for a Randers-Douglas metric and Kropina-Berwald metric which have no common geodesics. Assume that this Randers and Kropina metric are induced by the same Riemannian metric and one-form, and $b_\alpha(x)y^\alpha \neq 0$.

Let us consider a Randers metric $L = \alpha + \beta$, which is a Douglas metric and not a Berwald metric (then $s_{ij} = 0$ and $b_{i;j} \neq 0$) and Kropina metric $l = \alpha^2/\beta$ which is a Berwald space (then r_{ij} is proportional to a_{ij}, that is $r_{ij} = u(x)a_{ij}$, for some $u(x)$, and $s_{ij} = (b_i s_j - b_j s_i)/b^2$).

Hence, the equation

$$b_{i;j} - b_{j;i} = 0, \quad b_{i;j} + b_{j;i} = 2u(x)a_{ij}, \quad b_{i;j} - b_{j;i} = \frac{1}{2}(s_i b_j - s_j b_i)$$

hold at the same time.

From the above equation we obtain

$$b_{i;j} = u(x)\, a_{ij}, \tag{15.88}$$

and s_i is proportional to b_i. This equation are completely integrable equations in a Riemannian space of constant curvature.

At first we consider the integrability conditions of (15.88)

$$b_{i;j;k} - b_{i;k;j} = -b_\alpha R^\alpha_{ijk}, \tag{15.89}$$

where R^α_{ijk} is curvature tensor of the associated Riemannian space $\mathbb{V}_n$. Let $\mathbb{V}_n$ be a space of the constant curvature, so

$$R^\alpha_{ijk} = R(a_{ij}\delta^\alpha_k - a_{ik}\delta^\alpha_j), \tag{15.90}$$

where R is the curvature constant. From the equation (15.89) and (15.90) we get

$$-R(a_{ij}B_k - a_{ik}B_j) = u_{;k}a_{ij} - u_{;j}a_{ik}.$$

Transvection of this equation by tensor a^{ij} gives

$$u_{;k} = -R\, b_k. \tag{15.91}$$

We obtain the integrability conditions of the equation (15.91) in the following formula:

$$u_{;k;l} - u_{;l;k} = -R\, b_{k;l} + R\, b_{l;k} = 0.$$

So (15.88) and (15.91) are completely integrable equation in a Riemannian space of constant curvature.

From this calculation it follows that if $b_i \neq 0$ and $R \neq 0$ then $u(x) \neq 0$. So we obtain two Finsler space: a Randers space $\mathbb{F}_n = (M_n, L = \alpha + \beta)$ of Douglas type, and Kropina space $\overline{\mathbb{F}}_n = (M_n, \overline{L} = \alpha^2/\beta)$ of Berwald type, which are induced by the same Riemannian metric α and a one form β.

From Corollary 1. we know that if a Randers space $\mathbb{F}_n = (M_n, L = \alpha + \beta)$ is projective to a Kropina space $\overline{\mathbb{F}}_n = (M_n, \overline{L} = \alpha^2/\beta)$, then $b_{i;j} = 0$ independently of the dimension n.

The above lead us to

Theorem 15.31 *Let a Riemannian space of constant curvature on an n-dimensional manifold M_n be given by a metric function α, and $b_i(x)$ be a gradient vector field on M_n. If the Randers space $\overline{\mathbb{F}}_n = (M_n, \overline{L} = \alpha^2/\beta)$ is a Berwald space, then F_n and $\overline{\mathbb{F}}_n$ have not common geodesics.*

15.5 Geodesic mappings of weakly Berwald spaces and Berwald spaces onto Riemannian spaces

Introduction

1. This part is devoted to the investigation of the geodesic mappings (projective relations) of weakly Berwald spaces and Berwald spaces $\mathbb{F}_n$ into Riemannian spaces $\overline{\mathbb{V}}_n$. We can see that not every Riemann-Finsler space admits non-trivial geodesics mapping onto Riemannian space.

2. We give a summary on geodesic mappings between Riemann-Finsler spaces $\mathbb{F}_n = (M_n, L)$ and $\overline{\mathbb{F}}_n = (M_n, \overline{L})$ on the same underlying manifold M_n. If any geodesic of a Riemann-Finsler space $\mathbb{F}_n$ coincides with a geodesic of $\overline{\mathbb{F}}_n$ as a set of points and vice versa, the change $L \to \overline{L}$ of the metric is called *projective*, and the mapping $\mathbb{F}_n \to \overline{\mathbb{F}}_n$ is a *geodesic mapping* between $\mathbb{F}_n$ and $\overline{\mathbb{F}}_n$.

It is well-known (Knebelmann's Theorem) that if a Riemann-Finsler space $\mathbb{F}_n$ admits a geodesic mapping onto Riemann-Finsler space $\overline{\mathbb{F}}_n$ than in general with respect to a coordinates $x^1, x^2, \dots, x^n, y^1, y^2, \dots, y^n$ the objects of the Berwald connection of the space $G^h(x,y)$ and $\overline{G}^h(x,y)$ have the following relation:

$$\overline{G}^h(x,y) = G^h(x,y) + p(x,y)\, y^h, \tag{15.92}$$

where a function $p(x,y)$ is a positively homogeneous in y of degree one.

The formula (15.92) is equivalent to the formulae

$$\begin{aligned}
\overline{G}^h_i &= G^h_i + \delta^h_i p + p_i\, y^h, && p_i = p_{(i)},\\
\overline{G}^h_{ij} &= G^h_{ij} + \delta^h_i p_j + \delta^h_j p_i + p_{ij}\, y^h, && p_{ij} = p_{i(j)},\\
\overline{G}^h_{ijk} &= G^h_{ijk} + \delta^h_i p_{jk} + \delta^h_j p_{ik} + \delta^h_k p_{ij} + p_{ijk}\, y^h, && p_{ijk} = p_{ij(k)}.
\end{aligned}$$

If $p_i \neq 0$, then the geodesic mapping is called *non-trivial*, otherwise it is said to be *trivial*.

Riemann-Finsler space is a Berwald space if $G^h_{ijk} = 0$, and a Riemann-Finsler space is a weakly Berwald space if $G^\alpha_{ij\alpha} = G_{ij} = 0$, see [161].

It is also well-known that every positive definite Berwald metric has common geodesics with some Riemannian metric [877]. The aim of this part is to show a new method, which is suitable for investigation of geodesic mapping of Berwald spaces onto Riemannian spaces in general.

15.5.1 Geodesic mappings of weakly Berwald spaces onto $\mathbb{V}_n$

We know, that the Douglas tensor is invariant under geodesic mappings, that is $D = \overline{D}$, where

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}\,(G_{ijk}\, y^h + G_{ij}\delta^h_k + G_{ik}\delta^h_j + G_{jk}\delta^h_i), \quad G_{ijk} = G_{ij(k)}.$$

The Douglas tensor vanishes in Riemannian spaces, that is in the weakly Berwald space we get $G^h_{ijk} = 0$. If a weakly Berwald space has common geodesics with a Riemannian space, we obtain the following

Theorem 15.32 *If a weakly Berwald space has common geodesics with a Riemannian space, than the weakly Berwald space is a Berwald space.*

The fundamental equation (15.92) of geodesic mappings between Finsler spaces $\mathbb{F}_n$ and $\overline{\mathbb{F}}_n$ is equivalent to the following formula:

$$\overline{g}_{ij\|k} = 2p\,\overline{g}_{ij} + p_i\overline{g}_{jk} + p_j\overline{g}_{ik} + \overline{g}_{i\alpha}y^\alpha p_{jk} + \overline{g}_{j\alpha}y^\alpha p_{ik} - 2\,\overline{P}_{ijk},$$

where $\overline{g}_{ij\overline{\|}k} = 2\,\overline{P}_{ijk}$.

If the spaces $\mathbb{F}_n$ and $\overline{\mathbb{F}}_n$ are weakly Berwald spaces the above mentioned equation has the following simple form

$$\overline{g}_{ij\|k} = 2p\,\overline{g}_{ij} + p_i\overline{g}_{jk} + p_j\overline{g}_{ik} - 2\,\overline{P}_{ijk}. \tag{15.93}$$

In this case the functions p_i are indenpendent of y, i.e. $p_i = p_i(x)$.

15.5.2 Geodesic mappings of Berwald spaces onto $\mathbb{V}_n$

Theorem 15.33 *The Berwald space $\mathbb{F}_n$ admits non-trivial geodesic mappings onto Riemannian space $\overline{\mathbb{V}}_n$ with the metric tensor $\overline{g}_{ij}(x)$ if and only if the following system of differential equation with covariant derivatives of Cauchy type has a solution with respect to the symmetric tensor $\overline{g}_{ij}$ $(\det\|\overline{g}_{ij}\| \neq 0)$, the non-zero vector $p_i(x)$ and the function $\mu(x)$:*

$$\begin{aligned}
&\text{(a)} \qquad \overline{g}_{ij\|k} = 2p\,\overline{g}_{ij} + p_i\overline{g}_{jk} + p_j\overline{g}_{ik},\\
&\text{(b)} \qquad n\,p_{i\|j} = n\,p_ip_j + \mu\,\overline{g}_{ij} - H_{ij} - \overline{g}_{i\alpha}\overline{g}^{\beta\gamma}H^\alpha_{\beta\gamma j} - \tfrac{2}{n+1}\,H^\alpha_{\alpha ij},\\
&\text{(c)}\ (n-1)\,\mu_{\|i} = 2\,(n-1)\,p_\alpha\overline{g}^{\beta\gamma}H^\alpha_{\beta\gamma i} + p_\alpha\overline{g}^{\alpha\beta}(5H_{\beta i} + \tfrac{6}{n+1}\,H^\gamma_{\gamma\beta i} - H_{i\beta}) +\\
&\qquad\qquad \overline{g}^{\alpha\beta}\,H^\gamma_{\alpha\beta i\|\gamma} - H_{\alpha i\|\beta} - \tfrac{2}{n+1}H^\gamma_{\gamma\alpha i\|\beta},
\end{aligned} \tag{15.94}$$

where the symbol "$\|$ " denotes covariant derivative with respect to the connection of $\mathbb{F}_n$, $\overline{g}^{ij}(x)$ are components of the matrix inverse to $\|\overline{g}_{ij}(x)\|$, H^h_{ijk} and H_{ij} are curvature and Ricci tensors of the space $\mathbb{F}_n$.

Proof. Suppose that $\mathbb{F}_n$ admits a non-trivial geodesic mapping onto $\overline{\mathbb{V}}_n$ with metric tensor $\overline{g}_{ij}$. Then the connections of $\mathbb{F}_n$ and $\overline{\mathbb{V}}_n$ have the relation (15.93) with $\overline{P}_{ijk} = 0$. If we consider a fundamental tensor $\overline{g}_{ij}$ of $\overline{\mathbb{V}}_n$, then we get the conditions which are sufficient for $\mathbb{F}_n$ to admit a non-trivial geodesic mapping onto $\overline{\mathbb{V}}_n$.

Let us consider integrability conditions of the equations (15.94a)

$$\overline{g}_{h\alpha}H^\alpha_{ijk} + \overline{g}_{i\alpha}H^\alpha_{hjk} = 2\,\overline{g}_{hi}p_{[jk]} + \overline{g}_{j(h}p_{i)k} - \overline{g}_{k(h}p_{i)j}, \tag{15.95}$$

where $p_{ij} = p_{i\|j} - p_ip_j$, $[i\,j]$ and $(i\,j)$ denote alternation and symmetrization with respect to i and j, respectively.

Transvecting (15.95) by $\overline{g}^{ij}$, we get $p_{[ij]} = \frac{1}{n+1}\,H^\alpha_{\alpha jk}$. So we obtain

$$\overline{g}_{\alpha(h}H^\alpha_{i)jk} - \frac{2}{n+1}\,\overline{g}_{hi}H^\alpha_{\alpha jk} = \overline{g}_{j(h}p_{i)k} - \overline{g}_{k(h}p_{i)j}. \tag{15.96}$$

After the transvecting (15.96) by g^{ik} we easily obtain the condition (15.94b) with $\mu = p_{\alpha\beta}\overline{g}^{\alpha\beta}$.

If we consider the equations $\overline{g}^{ik}\overline{g}_{kj} = \delta^i_j$, it is not difficult to show that the equations (15.94a) are equivalent to the relations

$$\overline{g}^{ij}{}_{\|k} = -2\,p_k\overline{g}^{ij} - \delta^i_k p^j - \delta^j_k p^i, \tag{15.97}$$

where $p^i = p_\alpha \overline{g}^{\alpha i}$.

We covariantly differentiate the condition (15.94b) and then alternate the result with respect to the indices j and k taking into account (15.94a), (15.94b), (15.97) and transvecting by $\overline{g}^{ik}$, so we finally obtain the equations (15.94c).

The Theorem 15.33 has been proved. □

From Theorem 15.33 we may conclude that the set of all Riemannian spaces $\overline{\mathbb{V}}_n$ (for which the given Berwald space $\mathbb{F}_n$ admits non-trivial geodesic mapping onto $\overline{\mathbb{V}}_n$) is dependent on

$$r \le r_0 = (n+1)(n+2)/2$$

parameters.

The results under discussion are a generalization of analogous theorem by N.S. Sinyukov [170] and J. Mikeš and V. Berezovski [650, 651] for geodesic mappings of Riemannian spaces.

15.5.3 Riemannian metrics having common geodesics with a Berwald metric

V. Matveev reacted quickly onto the Theorem 15.33 in his paper [614].

Let two metrics be given, they are *geodesicaly equivalent* if the geodesics coincide as unparametrized curves, and *affine equivalent*, if their geodesics coincide as parametrized curves.

Let (M, L) be a Berwald space, where $L\colon TM \to \mathbb{R}$ is a Riemann-Finsler metric function. In the paper [614] the following theorem is proved:

Theorem 15.34 *Let L be a Berwald metric, let Γ be its associated connection. Then there exists a Riemannian metric g affine equivalent to L.*

For strictly convex Riemann-Finsler metrics, Theorem 15.34 is due to [877].

Later proofs were suggested in [878]. Matveev's proof is similar to one mentioned in [878], a small modification which allowed use to include metrics that are not strictly convex is based on the construction from [626].

Theorem 15.35 *Let L be an essential Berwald metric on a connected manifold, let Γ be its associated connection. Suppose a Riemannian metric g is geodesically equivalent to L, but not affine equivalent to g. Then there exists a constant μ, a nonzero symmertic $(2,0)$-tensor a^{ij}, and a non-zero vector field λ^i such that the following equations are fulfilled:*

$$a^{ij}{}_{,k} = \lambda^i\delta^j_k + \lambda^j\delta^i_k, \tag{15.98}$$

$$\lambda^i_{,j} = \mu\,\delta^i_j. \tag{15.99}$$

The equations (15.98) and (15.99) are of Cauchy type for the unknown functions a^{ij} and λ^i. Note that this equations much easier than one in [648]: first all, the equations are linear in unknown functions, second, they contain less equations, and, third, the equations are much simpler than those of [648] and, in particular, contain no curvature terms.

In [614] the author studied the metrics with degree of mobility ≥ 3. In Riemannian case by the results of Sinyukov [170] the tensor a_{ij} and the vector λ_i satisfy the equation (8.6), p. 297:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{15.100}$$

The degree of mobility of a metric g is the dimension of the space of solution of the equation (15.100) considered as equation in the unknowns a_{ij} and λ_i. In [614] we can find the following assumptions: if $\overline{a}_{ij} := g_{ij}$, $\overline{\lambda}_i := 0$ and $\hat{a}_{ij} = \tilde{g}_{ij}$, $\hat{\lambda}_i := 0$ are also solution, then are linear independent of the solution a_{ij}, λ_i.

By results of [542], under the above assumption, for every solution a_{ij}, λ_i of the equation (15.100), in an neigbourhood of almost every point there exists a constant B and a function μ such that the following equations hold:

$$\begin{aligned} \lambda_{i,j} &= \mu\, g_{ij} + B\, a_{ij}, \\ \mu_{,i} &= 2B\lambda_i. \end{aligned} \tag{15.101}$$

This equation system we can find in an earlier paper by Kiosak and Mikeš [542] and what is more alredy prowed by Mikeš, that there exists a covariantly constant vector field [226, 626]. This fact in the main point of the Matveev's proof. Unfortunately there is not any reference for [226, 626].

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A U T H O R S

Josef Mikeš
Professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, Czech Republic. During the years 1972 – 1994 he stays at the Department of Geometry and Topology, Faculty of Mechanics and Mathematics, Odessa State University, Ukraine. *e-mail:* josef.mikes@upol.cz

Elena S. Stepanova
Post-doc at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, Czech Republic. *e-mail:* stephelena@list.ru

Alena Vanžurová
Associate Professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, and at the Department of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Czech Republic. *e-mail:* alena.vanzurova@upol.cz

Sándor Bácsó
Private Professor at Debrecen University, Hungary. *e-mail:* bacsos@unudeb.hu

Vladimir E. Berezovski
Head of Mathematics Department. Uman National University of Horticulture, Ukraine. *e-mail:* berez.volod@rambler.ru

Olena Chepurna
Associate Professor at the Department of Mathematical Methods of Analysis in Economics. Odessa National Economics University, Ukraine. *e-mail:* culeshova@ukr.net

Marie Chodorová
Assistant Professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc, Czech Republic. *e-mail:* marie.chodorova@upol.cz

Hana Chudá
Assistant Professor at the Department of Mathematics. Faculty of Applied Informatics, Thomas Bata University Zlín, Czech Republic. *e-mail:* chuda@fai.utb.cz

Michail L. Gavrilchenko
Professor at the Department of Geometry and Topology. Faculty of Mathematics. Odessa National University, Ukraine

Michael Haddad
Professor at the Wadi International University, Syria.
e-mail: akkad@ukr.net; dean.mef@wiu.edu.sy

Irena Hinterleitner
Assistent Professor at the Department of Mathematics. Faculty of Civil Engineering, Brno University of Technology, Czech Republic. *e-mail:* hinterleitner.irena@seznam.cz

Marek Jukl
Associate Professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc, Czech Republic. *e-mail:* marek.jukl@upol.cz

Lenka Juklová
Assistant Professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc, Czech Republic. *e-mail:* lenka.juklova@upol.cz

Dzhanybek Moldobaev
Professor at the Department of Faculty of Mathematics. Kyrgyz State Paed. University, Bishkek, Kyrgyzstan

Patrik Peška
Assistant Professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc, Czech Republic. *e-mail:* patrik_peska@seznam.cz

Igor G. Shandra
Associate Professor at the Department of Mathematics. Finance University under the Government of Russian Federation, Moscow, Russian Federation. *e-mail:* ma-tematika@yandex.ru

Mohsen Shiha
Professor at the Department of Mathematics. Faculty of Science, Al-Baath University, Homs, Syria. *e-mail:* mohsen_sheha@yahoo.com

Dana Smetanová
Assistant Professor at the Department of Informatics and Natural Sciences. Institute of Technology and Business in České Budějovice, Czech Republic. *e-mail:* smetanova@mail.vstecb.cz

Sergey E. Stepanov
Professor at the Department of Mathematics. Finance University under the Government of Russian Federation, Moscow, Russian Federation. *e-mail:* s.e.stepanov@mail.ru

Vasilij S. Sobchuk
Professor at the Algebra and Information Studies. College of Applied Mathematics, Chernivtsi National University, Ukraine

Irina I. Tsyganok
Associate Professor at the Department of Mathematics. Finance University under the Government of Russian Federation, Moscow, Russian Federation. *e-mail:* i.i.tsyganok@mail.ru

prof. RNDr. Josef Mikeš, DrSc.
Elena Stepanova, Ph.D.
doc. RNDr. Alena Vanžurová, CSc.
et al.

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János Bolyai, *1802–1860*

Born in Kolozsvár (Cluj), a Hungarian soldier and mathematician. Between 1820 and 1823 J. Bolyai prepared a treatise on a complete concept of non-Euclidean geometry, independently of the results of N. I. Lobachevski (1792–1856) and K. Gauss (1777–1855). Bolyai's work was published in 1832 as an Appendix to a mathematics textbook written by his father.

I tis of interest to mention that for a short period (1832–1833) of his military service J. Bolyai was soldier of a garrison in Olomouc (Czech Republic, late Olmütz) as evidenced by his memory inscription at his bust set up in the Olomouc Army House.

Ivestigation of geodesic mappings by Beltrami in 1865 marked the beginning of the general acceptance of Bolyai, Gauss and Lobachevski results.

AUTHORS

Josef Mikeš, Professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, Czech Republic, e-mail: josef.mikes@upol.cz

Elena S. Stepanova, Post-doc at the Department of Algebra and Geometry, fakulty of Science, Palacky University Olomouc, Czech Republic, e-mail: stephelena@list.ru

Alena Vanžurová, Associate professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, and the Department of Mathematics, Faculty of Civil Engineering, Brno University of Technology, Czech Republic, e-mail: alena.vanzurova@upol.cz

Sándor Báscó, Private professor at Debrecen University, Hungary, e-mail: bacsos@unideb.hu

Vladimir E. Berezovski, Head of Mathematics Department. Uman National University of Horticulture, Ukraine, e-mail: berez.volod@rambler.ru

Olena Chepurna, Associate professor at the Department of MMAE. Odessa National Economics University, Ukraine, e-mail: culeshova@ukr.net

Marie Chodorová, Assistant professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc, Czech republic, e-mail: marie.chodorova@upol.cz

Hana Chudá, Assistant professor at the Department of Mathematics. Faculty of Applied Informatics, Thomas Bata University Zlín, Czech Republic, e-mail: chuda@fai.utb.cz

Michail L. Gavrilchenko, Professor at the Department of Geometry and Topology, Faculty of Mathematics, Odessa National University, Ukraine

Michael Haddad, Professor at the Wadi International University, Syria, e-mail: dean.mef@wiu.edu.sy

Irena Hinterleitner, Assistent professor at the Department of Mathematics. Faculty of Civil Engineering, Brno University of Technology, Czech Republic, e-mail: hinterleitner.irena@seznam.cz

Marek Jukl, Associate professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, Czech Republic, e-mail: marek.jukl@upol.cz

Lenka Juklová, Assistant professor at the Department of Algebra and Geometry. Faculty of Science, Palacky University Olomouc. Czech Republic, e-mail: lenka.juklova@upol.cz

Dzhanybek Moldobaev, Professor at the Department of Geometry of Faculty of Mathematics, Kyrgyz State Paed. University, Bishkek, Kyrgyzstan

Patrik Peška, Assistant professor at the Department of Algebra and Geometry, Faculty of Science, Palacky University Olomouc, Czech Republic, e-mail: patrik_peska@seznam.cz

Igor G. Shandra, Associate professor at the Department of Mathematics, Finance University under the Government of Russian Federation, Moscow, Russian Federation, e-mail: ma-tematika@yandex.ru

Mohsen Shiha, Professor at the Department of Mathematics, fakulty of Science, Al-Baath University, Homs, Syria, e-mail: mohsen_sheha@yahoo.com

Dana Smetanová, Assistant professor at the Department of Natural Sciences, The Institute of Technology and Business in České Budějovice, Czech Republic, e-mail: smetanova@mail.vstecb.cz

Sergey E. Stepanov, Professor at the Department of Mathematics, Finance University under the Government of Russian Federation, Moscow, Russian Federation, e-mail: s.e.stepanov@mail.ru

Vasilij S. Sobchuk, Professor at the Algebra and Information Studies, College of Applied Mathematics, Chernivtsi National University, Ukraine

Irina I. Tsyganok, Associate professor at the Department of Mathematics, Finance University under the Government of Russian Federation, Moscow, Russian Federation, e-mail: i.i.tsyganok@mail.ru

9 788024 446714